Baroclinic Instability over a Slope. Part I: Linear Theory

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ABSTRACT

We consider the instability properties of a circular current in the upper layer of a two-layer quasi-geostrophic ocean over a unidirectional slope. This particular flow-topography geometry is intended as a crude model of geophysical gyres where the variation of the Coriolis force is negligible. Such currents occur in the Arctic and in Gulf Stream rings. The slope destabilizes the flow; the critical Froude number is lowered as the slope increases. Baroclinic instabilities tend to generate time-independent or mean currents in the upper and lower layers which, because of the slope, are markedly asymmetric across the gyre.

1. Introduction

Oceanographers have long been occupied with the problem of explaining the observed asymmetry in the circulation of the world's ocean gyres. Westward intensification is thought to be related to the variation of the vertical component of the earth's rotation vector with latitude, although the chain of processes over which this variation manifests itself is not clear. In a polar ocean like the Arctic, this variation of basic rotation is more or less negligible. It appears as if bottom topography may here generate the characteristics of observed (southwest intensification in the Arctic) asymmetries. Here again the process by which the surface layers adjust to the topography is unclear. If the polar ocean is stratified into two layers, an upper layer of Arctic water and a lower layer of Atlantic water, the topography cannot exert any influence on the upper or surface layer unless bottom velocities are generated by some mechanism. The upper layer is caused to move by the action of a wind stress or by motion of the ice pack. If this motion is to feel the bottom at all, some coupling across the interface which can drive motion in the whole of the deep layer must exist. Small-scale mixing through turbulent Ekman layers at the interface between the two layers is a possible physical process and serves as the simplest layer-coupling mechanism in frictional models of the Arctic Ocean circulation (Hart, 1975). While a useful starting point in analytical modeling of polar ocean dynamics, the existence of interfacial Ekman layers outside the laboratory remains to be verified. It is likely that other processes exist which can generate quasi-steady currents in the deep water. A candidate which has recently come to prominence in the literature is baroclinic instability (this has been discussed at length in regard to MODE, the middle ocean dynamics experiment). Baroclinic instability acting on a current solely in the upper layer of a two-layer ocean will tend to generate both time-dependent and mean currents in the lower layer.

In a symmetrical basin on an $f$-plane with a flat bottom, we would expect rectified currents due to nonlinear effects acting on the eddy motions generated by baroclinic instability. If $f$ varies significantly over the basin or if the topography is important, the variations in the mean time-averaged currents will depend on how the baroclinic instabilities interact with the topography or with $\beta$. We consider here a fundamental model to look at the question of how the basic geometrical asymmetries of the basin are reflected in the long-term motion fields. We shall study flow in a basin which is entirely symmetric except for topography. There is no interfacial friction. Bottom motions are due entirely to instability. The topography can affect the motion in several ways. It can alter the basic instability process and hence the structure of the harmonics of the fundamental instability, altering in turn the way they interact to produce mean currents in both the top and bottom layers. It can generate asymmetric mean currents by interacting with broad symmetric currents produced by self-rectification of the instabilities.

We study this problem in two parts. We first consider the basic linear instability properties of a circular current over a uniform slope. Because the basic motion field is an easily calculated exact solution of the governing equations, the instability problem is well posed and admits simple solutions when the bottom is flat. Although the model has a geometry and basic state that is easily generated in the laboratory and resembles certain aspects of the Arctic problem, it also should apply to problems involved with the stability of Gulf Stream rings as they migrate over a slope. Since motion
over a uniform north-south slope produces vorticity changes in an analogous manner to flow on a β-plane, the topography problem is related to the action of instabilities in a mid-latitude gyre. However, we have topography only at the bottom, as this makes the basic state simpler, but the basic physical processes occurring here should also be present at mid-latitudes. The linear problem treated in this part shows that the slope has a strong influence on the short-wave cutoff and on the structure of the basic perturbations. How these instabilities interact to generate mean currents is a nonlinear problem and is treated in the weak-amplitude limit in Part II.

Before proceeding it is useful to note the relation of this work to previous studies of baroclinic instability over a slope. Most studies have taken a uniform current over a uniform slope (Robinson and McWilliams, 1974; Blumssack and Gierasch, 1972) or a unidirectional slope oriented perpendicular to the current (Orlanski, 1969). This latter study is not concerned with spatial asymmetry. Our basic current, being circular, changes its orientation to the slope as one moves around the basin. The assumption of local uniformity of the basic current and topography made by Robinson and McWilliams is not made here, although the small-scale instabilities they studied can be studied within the context of the present formulation. The large-scale instabilities may be expected to feel the slope strongly and interact with it while the small-scale instabilities may interact with the slope only through the slow space-time varying currents generated by local rectification. These latter effects are studied in more detail in Part II.

2. Basic equations

Fig. 1 shows the basic geometry and topography for this study. Two immiscible fluids are contained in a rotating cylinder. Their mean depths \( H \) are equal, as are their viscosities \( \nu \). The density ratio \( \Delta \rho/\rho \equiv (\rho_2 - \rho_1)/\rho_2 \) is much less than 1. Motion is driven by a contact lid which rotates with frequency \( \omega \) faster or slower than the cylinder. The equations of motion are made nondimensional by scaling the lengths by the cylinder radius \( L \), depths by \( H \), velocities by \( \omega L \), and time by \( \omega^{-1} \). The remaining nondimensional parameters are the Rossby number,

\[
\mathcal{R}_o = \frac{\omega}{2\nu L} \ll 1,
\]

the Froude number (or the ratio of \( L^{-1} \) to the Rossby radius of deformation squared),

\[
F = \frac{4\Omega^2 L^2}{g(\Delta \rho/\rho) H^2},
\]

and the Ekman number

\[
E = \frac{\nu}{2\Omega H^2}.
\]

We take \( E^* \) to be less than or of order \( \mathcal{R}_o \). Then, neglecting the frictional effects at the sidewalls, the quasi-geostrophic vorticity equations describe the evolution of the slow motions in the cylinder. The dominant velocities are geostrophic and independent of depth in each layer. That is, with \( i = 1 \) or 2 corresponding to the upper and lower layer respectively,

\[
\frac{\partial P_i}{\partial r} = \frac{1}{r} \frac{\partial P_i}{\partial \theta},
\]

\[
\frac{\partial P_i}{\partial z} = 0.
\]

The prognostic equations are the vertical vorticity equations

\[
\frac{d\omega_1}{dt_1} = -\frac{E^1}{:\nu_2 \mathcal{R}_o} \left[ \left( 1 + \frac{Q}{2} \right) \nabla^2 P_1 - \frac{Q}{2} \nabla^2 P_2 + 2 \right],
\]

\[
\frac{d\omega_2}{dt_2} = -\frac{E^1}{\nu_2 \mathcal{R}_o} \left[ \left( 1 + \frac{Q}{2} \right) \nabla^2 P_2 - \frac{Q}{2} \nabla^2 P_1 \right] - \alpha J(P_2, h),
\]

(2.1)

(2.2)
where
\[ \begin{align*}
\omega_1 &= \nabla^2 P_1 + F(P_2 - P_1), \\
\omega_2 &= \nabla^2 P_2 + F(P_1 - P_2), \\
\frac{d}{dt} &= -J(P_1, ) \\
\alpha &= \alpha^* L/(H Ro)
\end{align*} \]

The bottom topography \( h \) is given by
\[ h = r \cos \theta \]
which corresponds to a shallow point at \( \theta = 0, r = 1 \). The slope is uniform making an angle \( \alpha^* \) with the horizontal (see Fig. 1).

These quasi-geostrophic equations will be used as the governing equations in both Parts I and II. They have been used in many studies of baroclinic instability [Pedlosky (1970) gives a derivation] and should be familiar. Further discussion is given in Hart (1972) where these equations were applied to the question of baroclinic instability of a two-layer fluid over axisymmetric topography. The agreement between theory and experiment there suggested that the neglect of sidewall boundary layers and the replacement of the full viscous boundary condition at \( r = 1 \) by one of no normal flow was adequate.

3. The basic state

The terms multiplied by \( Q \) in (2.1) and (2.2) are those which describe the alteration of the interior vorticities which must result when interfacial friction is present. Friction, however, is confined to thin interfacial Ekman layers. The parameter \( Q = \nu_i / \nu \) measures the viscosity at the interface relative to that at a rigid wall. If \( \nu_i \) is an eddy viscosity coefficient we might expect it to be less than \( \nu \), the eddy viscosity at the wall. In the laboratory with laminar Ekman layers \( Q = 1 \).

Consider now the basic steady circulation induced by the Ekman suction below the differentially rotating lid. The case where \( E^1/Ro \) is fairly small is of most interest here since then nonlinear effects will be important. If \( P^0_0 \ll P^0_1 \), where the superscript denotes the order of approximation of the basic state, we find
\[ P^0_1 = \frac{r^2}{2(1+Q/2)} \]

or
\[ v^0_1 = \frac{r}{(1+Q/2)} \]

If \( P^0_1 \) is zero then this is an exact solution of (2.1) since \( J(P^0, \alpha^0) = 0 \) for an axisymmetric flow. The equation for the lower layer flow, however, is
\[ J(P^0_2, \omega^0_2) + \frac{E^1}{\sqrt{2} Ro} \nabla^2 P^0_2 \left( 1 + \frac{Q}{2} \right) + \alpha J(P^0_2, h) = \frac{Q E^1}{\sqrt{2} Ro (1+Q/2)}. \]

Thus the flow forced in the lower layer for
\[ \frac{E^1}{\sqrt{2} Ro} \ll (\alpha F) \]
is given by
\[ F J(P^0_2, P^0_1) + \alpha J(P^0_2, h) = \frac{2 Q E^1}{Ro (2+Q)}. \]

provided \( \alpha \gg F \). This says that the lower layer is in a Sverdrup type balance. If \( F \gg \alpha \) the problem for the flow in the lower layer is much more complicated although, since this means the contours of depth in the lower layer are for the most part closed, one expects \( P^0_2 \) to be order 1 contrary to our hypothesis. A similar problem to this is considered by Hart (1975). Here we really are more interested in the case described by Eq. (3.2) or in the case where \( Q \) is small (little turbulence at the interface). In both cases the velocity in the lower layer is small. With (3.2),
\[ P^0_2 = \frac{2 Q E^1}{\alpha Ro (2+Q)} \times \text{function (r, \theta)}. \]

Since \( Q \ll 1 \) the lower layer velocity becomes negligible as \( E^1/\alpha Ro \) becomes small. When \( Q \) is small as well then \( P^0_2 \) decreases further. If \( F \gg \alpha \) then the velocity \( P^0_2 \) is of order \( Q \) (see Appendix). We are primarily interested in large-scale eddy coupling across the interface and will then study the stability of the basic flow
\[ v^0_1 = r, \]
\[ v^0_2 = 0, \]
which is attained when \( Q \ll 1 \) or when \( \alpha \gg F \) and \( E^1/\alpha Ro \) is small. In terms of oceanic flows these seem to be realistic situations. To obtain reasonable simplicity in the formulations to follow, the use of the basic state (3.3), (3.4) is mandatory. Thus we shall continue our study of stability by setting \( Q = 0 \), and supposing that the only coupling of the basic flows through the interface is via baroclinic waves which grow through instability.

4. The stability problem

We linearize (2.1) and (2.2) about the basic state (3.3) and (3.4). In addition, we further consider the inviscid limit, obtained when \( E^1 = 0 \). If \( E^1/Ro \ll 1 \) friction will only introduce a small correction to the stability.
curves obtained when \( E = 0 \). As the reader may have anticipated the stability problem is two-dimensional. The complexity added to this difficult type of problem when friction is included (complex instead of real matrices) as a practical matter outweighs the desire to include friction in the model. Friction is included in the weak-amplitude nonlinear calculation of Part II.

The inviscid perturbation equations are

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \right) [\nabla^2 P_1 + F(P_2 - P_1)] + \frac{\partial P_1}{\partial \theta} &= 0, \quad (4.1) \\
\frac{\partial}{\partial t} [\nabla^2 P_2 + F(P_1 - P_2)] - \frac{\partial P_2}{\partial \theta} &= -\alpha J(P_2, k), \quad (4.2)
\end{align*}
\]

We write \( h = (\tau/2)(e^{\iota \theta} - e^{-\iota \theta}) \). The equations are separable only in time. We thus take

\[
P_1 = e^{-\iota \theta} \left[ \sum_{n-n=1}^N \sum_{m=1}^M A_{mn} J_n(l_m r) e^{i \iota \theta} \right], \quad (4.3)
\]

\[
P_2 = e^{\iota \theta} \left[ \sum_{n-n=1}^N \sum_{m=1}^M B_{mn} J_n(l_m r) e^{i \iota \theta} \right]. \quad (4.4)
\]

The limit integers \( N \) and \( M \) determine the truncation level of the system. The \( l_m \) of this Fourier-Bessel expansion are determined from \( J_n(l_m r) = 0 \), except for \( n = 0 \) where \( J_n'(l_m r) = 0 \). These expansions along with the formula for \( h \) are substituted into (4.1) and (4.2). The resulting components of each \( e^{i \iota \theta} \) problem are separated out. This task is facilitated through the use of the Bessel function recursion relations in evaluating the right-hand side of (4.2). The set of sums multiplying each \( e^{i \iota \theta} \) are then integrated over \( r \) by

\[
\int_0^1 r J_n(l_m r) dr.
\]

This application of the Galerkin method then produces a matrix eigenvalue problem for \( \sigma \).

The above techniques applied to (4.1) yield a simple expression

\[
-(\frac{3}{4} \bar{n} - \sigma) A_{mn} (l_{m, n}^2 + F) + \frac{3}{4} hFA_{mn} - (\frac{3}{4} \bar{n} - \sigma) FB_{mn} = 0. \quad (4.5)
\]

Application of the method to (4.2) gives

\[
\sigma l_{m, n}^3 B_{mn} - \sigma F A_{mn} = \frac{1}{3} \int_0^1 r J_n(l_m r) dr
\]

\[
\times \sum_{n'} \left[ (IB)_{m, n-1} \int_0^1 r J_n(l_{m, n} r) \cdot J_{n'}(l_{m, n-1} r) dr \\
+ (IB)_{m, n+1} \int_0^1 r J_n(l_{m, n} r) \cdot J_{n'}(l_{m, n+1} r) dr \right]. \quad (4.6)
\]

All the integrals on the right-hand side are easily evaluated. Notice that if \( \alpha \) is zero the matrix of the coefficients is quadrant diagonal. This just corresponds to the solutions for a flat bottom where solutions of the form \( P_1 \sim J_n(l_m r) \) are possible. In this case

\[
\sigma = \frac{1}{3} \left[ 1 \pm \left( \frac{l_m^2 - 2F}{l_m^2 + 2F} \right) \right] n.
\]

The neutral curve for a given mode \((n,m)\) is just given by

\[
F_c = l_m^2 / 2.
\]

When \( \sigma \) is non-zero the sum on the right of (4.2) causes information to propagate through \( n \) and \( m \) space. That is, the equation for \( B_{mn} \) involves all other \( B_{m', n-1} \) and \( B_{m', n+1} \) terms. One simplification is evident. The whole negative \( n \) spectrum feels the plus \( n \) spectrum only through the coefficients \( B_{mn} \). Now the entire set of negative \( n \) equations are identical to the \( +n \) equations. Thus we only need consider \( n > 0 \) provided the \( n = 0 \) equation is adjusted accordingly by setting \( B_{m', -1} = B_{m', +1} \). Since this reduces the order of the matrix by \( 2 \) it is an important simplification. It has been checked by making a few calculations with all terms included to verify this aspect of the problem.

The matrix eigenvalue problem from Eqs. (4.5)–(4.6) is of order \( 2(N+1)M \). Clearly the truncation levels necessary for accurate solutions will pose a severe constraint on how much calculation can be done with a reasonable amount of computer time. If \( M \) and \( N \) have to be 10, the matrices would be \( 220 \times 220 \). If we had included viscosity as well, the complex coefficients would effectively double the order to unmanageable size. Fortunately such large levels are not needed. With a flat bottom the most rapidly growing disturbance is \( m = 1 \) regardless of \( F \). As \( F \) increases the most rapidly growing mode does occur for larger and larger \( n \). Thus we might expect that reasonable accuracy in \( \sigma \) (a few percent, say) could be attained with a small value of \( M \). Table 1 shows how \( \sigma \) varies with truncation level. It is seen that a higher \( N \) is required as \( \sigma \) increases; significant information passes through to the wings of the \( n \) spectrum as \( \sigma \) becomes larger. We have taken \( M = 3, N = 3 \) for most calculations. Some, at \( \alpha > 3 \), have been done with \( M = 2, N = 8 \) or \( M = 1, N \) up to 16. The resulting accuracy is thought to be better than 3% in all cases, except when \( \sigma \) is small (<0.01). Near the neutral curve slight changes in phase between the lower and upper layer have large effects on \( \sigma \). Thus as more terms are added to the series, they may cause small changes in the appearance of the solution but these may have a large influence on \( \sigma \).

5. Results

The growth rates and wave frequencies calculated numerically with truncation level \( M = 3, N = 5 \) are shown in Fig. 2. It is seen that at \( \alpha = 0 \) there is a short-
Table 1. \( \sigma \) as a function of truncation level.

<table>
<thead>
<tr>
<th>( N )</th>
<th>4</th>
<th>7</th>
<th>11</th>
<th>15</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F = 6.0, \alpha = 3.2, M = 1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>0.0138</td>
<td>0.0141</td>
<td>0.0143</td>
<td>0.0145</td>
<td>0.0145</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>0.468</td>
<td>0.468</td>
<td>0.468</td>
<td>0.468</td>
<td>0.468</td>
</tr>
</tbody>
</table>

\( F = 20, \alpha = 1.6 \) (three modes are unstable)

| \( M = 1 \) | \( N = 8 \) |
| \( \sigma_i \) | 0.074 | 0.279 | 0.210 |
| \( \sigma_r \) | 1.04 | 0.681 | 0.332 |

| \( M = 2 \) | \( N = 6 \) |
| \( \sigma_i \) | 0.051 | 0.285 | 0.209 |
| \( \sigma_r \) | 1.04 | 0.678 | 0.337 |

| \( M = 3 \) | \( N = 5 \) |
| \( \sigma_i \) | 0.047 | 0.287 | 0.209 |
| \( \sigma_r \) | 1.03 | 0.677 | 0.339 |

wave cutoff at which all instability ceases if \( F < 7.8 = F_{cn} \). In the flat bottom case if \( F \) is slightly greater than \( F_{cn} \) wavenumber 1 becomes unstable. As \( \alpha \) increases, the short-wave cutoff critical value of \( F \) decreases. In this sense the flow is destabilized; instability can occur at a lower value of \( F \). This may be of importance in certain stability problems involved with flows which are near critical, or equivalently, which have a flow scale of order the internal radius of deformation. Stability of Gulf Stream rings to large-scale perturbations is one relevant example (see Hart, 1974). The destabilization can be thought of another way. If one is near the critical \( F \) for \( \alpha = 0 \), the raising of \( \alpha \) to non-zero values increases the growth rate for smallish \( \alpha \). Then as \( \alpha \) gets large the growth rate starts to decrease again. The former behavior is consistent with the analytic small slope calculation done in Part II, and the latter is perhaps to be expected from physical considerations. That is, as \( \alpha \) gets very large one would expect motions in the lower layer to be constrained to flow along depth contours. Since these are open in the uniform slope geometry considered here, the lower layer velocities must become small. Baroclinic instability relies on coupling between the two layers so this may well damp the growth rates. Ultimately as \( \alpha \to \infty \) we expect the flow to become more stable than the flat bottom limit, but this cannot be checked with the present model because of convergence problems. As \( \alpha \) gets larger the spectrum of the disturbance broadens and more and more terms need to be included. We are limited to \( \alpha = 6 \).

The destabilization results from a coupling between the various components of the spectrum and the slope which can cause constructive interference with the dominant \( n \). This is most easily seen when the spectrum \( A_{nm} \) is fairly sharp as in the weak slope limit studied in Part II. Here we can note that to the east (\( \theta = 0 \), \( r = 1 \)) the potential vorticity gradient is enhanced by the slope and to the west it is decreased. Thus the difference in base potential vorticity gradients between the two layers is bigger in the \( |\theta| < \pi/2 \) half of the basin. If the two parts of the basin are relatively independent as far as instability is concerned, for example if the dominant wavenumber is large, this increase in the potential vorticity gradient will enhance instability.

As mentioned above, the motion consists of a spectrum of disturbances. Table 2 shows the amplitudes of the \( \mathrm{Re}(A_{nm}) \) for various modes of instability, a mode being defined at one particular eigenvalue \( \sigma \) by the dominant \( A_{nm} \) in the eigenvector. The numerical program conveniently normalizes this dominant component to 1. As \( \alpha \) increases the spectrum broadens out; more energy is contained in the side bands.

At large \( F \) many modes are unstable. Table 3 shows how \( \sigma \) changes as \( F \) is increased, for various \( \alpha \). At \( F = 40 \), for example, five modes are unstable. Each mode is identified by its peak spectral coefficient. For a given mode the growth rate \( \sigma \) decreases as \( \alpha \) increases. Thus at moderate \( F \) away from \( F_{cn} \) the uniform slope appears to stabilize things. Note, however, that as \( F \) or \( n \) increases this effect becomes smaller. Waves of smaller scales do not feel the slope as strongly because the change in depth across the wave or the vortex stretching caused by the slope, becomes smaller as \( n \) increases. Similarly, as \( F \) becomes large, the interface slopes dominate over the topographic slope and the \( \alpha \) effect is minimized.

One may wonder how the polar \( \beta \) effect may affect these instabilities. If the basin is centered on the pole

![Fig. 2. Frequency \( \sigma_r \) and growth rates \( \sigma_i \) for baroclinic instability of circular flow over a slope. \( \alpha \) measures the effective strength of the slope and is equal to the actual slope \( \alpha^* \) over Ro. Values of \( \sigma_i \), <0.01 are difficult to compute, but the neutral points may be found by extrapolating the \( \sigma_i \) curves given to \( \sigma_i = 0 \).](image)
Table 2. Spectral amplitudes $A_{nm}$ for $F=20$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tr>
<td></td>
<td></td>
<td>0.009</td>
<td>1.00</td>
<td>0.031</td>
<td>0.0005</td>
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</tr>
<tr>
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<td>0.0003</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.0089</td>
<td>1.00</td>
<td>0.019</td>
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<td>---</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0035</td>
<td>1.00</td>
<td>0.13</td>
<td>0.0048</td>
<td>0.0031</td>
</tr>
<tr>
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<td></td>
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<td>0.0038</td>
<td>0.0016</td>
<td>0.0010</td>
<td>0.00012</td>
</tr>
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</tr>
<tr>
<td></td>
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<td>0.0010</td>
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<td>---</td>
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<td></td>
<td></td>
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<td>0.027</td>
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<td>0.063</td>
<td>1.00</td>
<td>0.15</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>0.0003</td>
<td>0.0067</td>
<td>0.0042</td>
<td>0.0005</td>
<td>0.0013</td>
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</table>

Table 3. $\sigma$ vs $\alpha$ for supercritical $F$.

<table>
<thead>
<tr>
<th>$(\text{dominant } n,m)$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.8</th>
<th>1.6</th>
</tr>
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<tr>
<td>$F=10$</td>
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</tr>
<tr>
<td>1,1</td>
<td>0.130</td>
<td>0.130</td>
<td>0.128</td>
<td>0.122</td>
<td>0.104</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>0.334</td>
<td>0.335</td>
<td>0.342</td>
<td>0.366</td>
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<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>1,1</td>
<td>0.227</td>
<td>0.227</td>
<td>0.226</td>
<td>0.222</td>
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</tr>
<tr>
<td></td>
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* $\sigma$ presented as $\left(\frac{\sigma_i}{\sigma_r}\right)$.

Fig. 3. Frequency and growth rates on a polar beta-plane. The basin is centered on the pole and extends an arc length $L$ away from it. $\beta_p$ measures the effects of variation of Coriolis force with latitude and equals $L^2/a^2$ where $a$ is the radius of the earth, $\alpha=1.6$.

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this can easily be included in the theory. The vertical component of Coriolis force varies as one moves from the pole to the basin boundary. This effect is normally stabilizing; it increases $F_{\alpha\theta}$. How large does this effect have to be to counteract the slope destabilization? The relevant parameter is $\beta_p=L^2/a^2$, where $L$ is the distance from the pole to the basin edge and $a$ the radius of the earth. Fig. 3 shows some results for $\alpha=1.6$ and two values of $\beta_p$.

Evidently $\beta_p$ must be of order $\alpha$ before the effects become comparable (this would be the conclusion from a scale analysis of the basic equations as well).

Let us, lastly, look at the structure of the motion. Fig. 4 shows the streamlines (pressure field) for the perturbations in the lower layer. The plots are oriented so that the shallow point is to the bottom. The perturbation amplitude is perfectly symmetric if $\alpha=0$ and becomes more asymmetric with more intensity in the shallow end as $\alpha$ becomes larger. If the motion is neutral then these plots made at $t=0$ correspond to relative amplitude. But as $\sigma_i$ is slightly positive here, the perturbation will change its shape slightly as it runs around the basin.

What are the effects of these instabilities on the mean time-independent motions? If a neutral $P_1$ is written as

$$P_1 = \sum_{n,m} [A_{nm} \exp(-in\theta - il\sigma_r) + A_{nm}^* \exp(-in\theta + il\sigma_r)]J_0(l_{nm}r),$$

it is seen that $J_0(P_{1,01})$ can generate motions with a complicated $(r,\theta)$ structure. The interaction of the various components of the spectrum with the conjugates of others generates steady vorticity. If $\alpha=0$, $A_{nm}$ has only non-zero values for a particular $n$, $m$. Thus the
\( \alpha = 0 \) instability can only generate time-independent currents which are axisymmetric. These axisymmetric currents correspond to the correction to the mean current required by the loss of available potential energy to the perturbations. Fig. 5a shows the time average value of \(-J(p_1, \alpha_1)\), or the quasi-steady potential vorticity generation in the upper layer. It is calculated from the linear stability eigenfunction and therefore has a determinate shape but not a known amplitude. The problem of calculating the amplitude and of precisely what circulations are driven by this term is the topic of Part II. From Fig. 5 we see that for \( \alpha = 0 \) the generation is axisymmetric but for \( \alpha > 0 \) there is a marked asymmetry, which apparently can dominate over the axisymmetric part, at least for small \( \sigma \). Note that the destabilization properties of the flow, \( F_s \) and \( \sigma_n \), are independent of the sign of \( \alpha \), a fact which is easily seen from symmetry. However, the vorticity generation does depend on this sign. For supercritical basins with \( F \gg F_{\text{cr}} \), very strong asymmetric vorticity generation can occur as in Fig. 5c.

6. Conclusions

A uniform slope lowers the critical value of \( F \) needed to sustain instability of a circular current in a two-layer ocean in a bounded basin. For \( F \sim F_{\text{critical}} \) the growth rates are enhanced as well. These two effects diminish as either \( F \geq F_{\text{critical}} \) or the wave scale becomes small at fixed \( \alpha \). However, even the small-scale eddies can
rectify in time and generate $\theta$- and $r$-dependent mean flows which might be observed as quasi-steady bottom or surface boundary currents. The topography generates a broader spectrum of disturbance energy than that obtained with a flat bottom. This effect probably would be enhanced if the topography was more complicated, i.e., had more scales, than the simple uniform slope model considered here.

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APPENDIX
The Basic State for $F \gg \alpha$

The basic flow in the lower layer when $F \gg 1 \gtrsim \alpha \sim E^1/R_o$ can be obtained by writing (3.2) as a power series in $F$. We take

$$P_2^{(0)} = P_2^{(00)} + F^{-1} P_2^{(01)} + F^{-2} P_2^{(02)}.$$

From Eq. (3.1) the order $F$ problem is

$$J(P_2^{(00)}, P_1^{(00)}) = 0,$$

or, using the fact that $P_1^{(0)} \sim r^2$, verified a posteriori,

$$P_2^{(00)} = \text{function (} r \text{)}.$$

At order 1,

$$J(P_2^{(00)}, \nabla^2 P_2^{(00)}) - J(P_1^{(0)}, P_1^{(01)})$$

$$+ \frac{E^1}{\sqrt{2} R_o} \left[ \nabla^2 P_2^{(00)} \left( 1 + \frac{Q}{2} \right) - \nabla Q \right]$$

$$+ \alpha J(P_2^{(00)}, h) = 0.$$ (A1)

We integrate (A1) over the area inside a contour $r = r_e$. Then since

$$J(P_2^{(00)}, \nabla^2 P_2^{(00)}) = 0$$

and

$$\int_0^{2\pi} \int_0^{r_e} r dr d\theta J(P_2^{(00)}(r), X(r, \theta)) = 0,$$

for any continuous $X$,

$$\int_0^{2\pi} d\theta \int_0^{r_e} r dr \frac{E^1}{\sqrt{2} R_o} \left[ \nabla^2 P_2^{(00)}(r) \left( 1 + \frac{Q}{2} \right) - \nabla Q \right] = 0,$$

or

$$\frac{\partial P_2^{(00)}}{\partial r}(r_e) = \frac{Q r_e}{\sqrt{2(1 + Q/2)}},$$

Thus $\xi_2^{(0)}$ is in solid-body rotation with angular frequency $\sqrt{2Q/(2 + Q)}$.
REFERENCES


