On the Interaction Between Long and Short Surface Waves

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ABSTRACT

Short, dissipative, surface waves superposed on longer waves cause a growth of the long wave momentum M_l at a rate

$$dM_l/dt = k_l a_l \langle -k_l S_s \sin\theta + \tau_s \cos\theta \rangle,$$

where k_l, a_l are the amplitude and wavenumber of the long waves, so that $k_l a_l$ is their steepness; S_s is the radiation stress of the short waves and τ_s the rate of transfer of momentum to the short waves by the wind; and the angle braces denote an average over the long-wave phase $\theta = k_l x - \omega_l t$.

The first term in the above equation is the radiation stress interaction (Phillips, 1963; Hasselmann, 1971) and is generally negligible compared with the second term, neglected by Hasselmann (1971), which shows that long waves can grow if short wave generation (rather than dissipation) is correlated with the long wave orbital velocity.

Even if the modulation of τ_s is only $O(k_l a_l)$ times $\langle \tau_s \rangle$, this mechanism can contribute a significant fraction of long wave momentum. However, even a substantially greater modulation of τ_s, perhaps due to varying exposure of short waves to the wind, is unlikely to account for all the alleged momentum input to long waves, due to the upper bound $k_l a_l$ on the efficiency of the process.

1. Introduction

It has long been realized that short surface gravity waves should have enhanced amplitudes at the crests of long waves, due to the compression of the short waves by the orbital velocity of the long waves, the working of the long wave rate of strain against the radiation stress of the short waves, and the increased ratio, for the short waves, of potential to kinetic energy near long wave crests (Longuet-Higgins and Stewart, 1960). This led to the expectation that short wave dissipation would occur preferentially at long wave crests, and the implications of this have been the subject of considerable debate.

Phillips (1963) argued that the energy dissipated by the short waves had partly been acquired from the long waves, so that the interaction would damp the long waves.

Longuet-Higgins (1969a) pointed out that as the short waves dissipate they give up their momentum, and so effectively exert a horizontal stress. For short waves and long waves propagating in the same direction, this stress is in phase with the orbital velocity of the long waves and so should lead to their growth. Longuet-Higgins (1969a) showed that this could provide a much greater input of energy than the loss due to Phillips' (1963) mechanism, and, assuming that the short waves were regenerated by the wind, he proposed this as a maser-type mechanism for the generation of long waves.

However, Hasselmann (1971) showed that the energy input to the long waves due to the rate of working of the effective surface stress exerted by dissipating short waves is almost exactly cancelled by a potential energy transfer, the residual being just the original damping term discovered by Phillips (1963).

Our purpose in this paper is to point out that long wave growth can result if short wave *generation* (rather than dissipation) is correlated with the orbital velocity of the long waves. Hasselmann's (1971) analysis included this effect, but he assumed the correlation to be zero. It seems very likely, though, that short wave generation will be enhanced near the crests of long waves if the short waves are larger there [as in the theory of Longuet-Higgins and Stewart (1960) and the observations of Cox (1958)] and more exposed to the wind. The potential importance of a correlation between short wave generation and long wave orbital velocity has also been recognized by Keller and Wright (1975) and Valenzuela and Wright (1976).

As with Longuet-Higgins' (1969a) original maser mechanism, the present effect can lead to long wave damping if the short and long waves are propagating in opposite directions, as for wind blowing against a swell.

Before deriving a general equation for the rate of increase of long wave energy, we first summarize some of the basic results on the interaction of short and long waves.

2. The short wave equations

Short surface gravity waves riding on much longer waves in deep water are effectively in a modified gravitational field given by $\mathbf{g}-D\mathbf{U}_l/Dt$, where $\mathbf{g}=(0,0,-g)$ is gravity and $D\mathbf{U}_l/Dt$ the fluid acceleration at the long wave surface. This modified gravity is equal to $\nabla p/\rho$, and is normal to the long wave surface if atmospheric pressure is uniform.

The frequency of the short waves relative to a frame of reference moving with velocity $\mathbf{U}_l$ is then

$$\omega_s'=|\mathbf{g}-D\mathbf{U}_l/Dt|^{\frac{1}{2}}|\mathbf{K}_s|^{\frac{1}{2}}, \qquad (2.1)$$

where $\mathbf{K}_s$ is the vector wavenumber of the short waves, parallel to the long wave surface. Suffixes l, s will be used throughout to refer to properties of the long and short waves respectively.

Relative to an inertial reference frame, the short wave frequency is

$$\omega_s=\omega_s'+\mathbf{U}_l\cdot\mathbf{K}_s. \qquad (2.2)$$

We can think of the long wave surface as a waveguide for the short waves, with a varying normal restoring force, and a Doppler shifting current equal to the component of $\mathbf{U}_l$ parallel to the long wave surface.

We now assume that the long waves are of small amplitude, so that, to first order in the long wave steepness, (2.1) becomes

$$\omega_s'=(g+\ddot{\zeta}_l)^{\frac{1}{2}}|\mathbf{k}_s|^{\frac{1}{2}}, \qquad (2.3)$$

where ζ_l is the elevation of the long wave surface, and we may replace $\mathbf{K}_s$ by a purely horizontal wavenumber $\mathbf{k}_s$. Eq. (2.2) becomes

$$\omega_s=\omega_s'+\mathbf{u}_l\cdot\mathbf{k}_s, \qquad (2.4)$$

where $\mathbf{u}_l$ is the horizontal part of the orbital velocity $\mathbf{U}_l=(\mathbf{u}_l,W_l)$ of the long waves.

We have assumed an adequate separation of time and space scales of long and short waves so that "local" formulas such as (2.1)–(2.4) apply. Formally we require $\omega_l\ll\omega_s'$ and $|\mathbf{k}_l|\ll|\mathbf{k}_s|$ where ω_l, $\mathbf{k}_l$ are the frequency and wavenumber of the long waves. In practice the scale separation does not have to be all that large for WKB-type results to hold (Kulsrud, 1957), and a ratio of about 10 in wavenumber, and hence only about 3 in frequency, is probably adequate. Further work is required to establish the effects of interactions between waves of rather similar scales.

The group velocity of the short waves relative to the long wave surface is

$$c_{s\alpha}=\partial\omega_s'/\partial k_\alpha=\tfrac{1}{2}(g+\ddot{\zeta}_l)^{\frac{1}{2}}|\mathbf{k}_s|^{-\frac{3}{2}}k_{s\alpha}=\tfrac{1}{2}\omega_s'k_{s\alpha}|\mathbf{k}_s|^{-2}, \qquad (2.5)$$

where subscripts α, β will be used to refer to horizontal components. In this same frame of reference, accelerating vertically at a rate $\ddot{\zeta}_l$, the short wave energy is given by

$$E_s'=\tfrac{1}{2}\rho(g+\ddot{\zeta}_l)a_s^2, \qquad (2.6)$$

with equipartition between potential and kinetic energy; a_s is the short-wave amplitude. The mass flux, or horizontal momentum, associated with the short waves, is

$$\mathbf{M}_s=\rho\overline{\zeta_s\mathbf{u}_s}=E_s'\mathbf{k}_s/\omega_s', \qquad (2.7)$$

where $\mathbf{u}_s$ is the horizontal component of the short-wave orbital velocity at the surface.

The wavenumber $\mathbf{k}_s$ of the short waves changes, due to changes in the horizontal component $\mathbf{u}_l$ of the long wave orbital velocity and in the vertical acceleration $\ddot{\zeta}_l$, at a rate (Phillips, 1966, p. 44)

$$\partial k_{s\alpha}/\partial t+(u_{l\beta}+c_{s\beta}')\partial k_{s\alpha}/\partial x_\beta=-k_{s\beta}\partial u_{l\beta}/\partial x_\alpha -\tfrac{1}{2}\omega_s'^{-1}|k_s|\partial\ddot{\zeta}_l/\partial x_\alpha. \qquad (2.8)$$

Only the explicit dependence of ω_s on $\mathbf{x}$, through $\mathbf{u}_l$ and $\ddot{\zeta}_l$, gives terms on the r.h.s. of (2.8). For a single long wave component with frequency ω_l we have $|\ddot{\zeta}_l|=\omega_l|\mathbf{u}_l|$, so that the magnitude ratio of the second term to the first term on the r.h.s. of (2.8) is $\frac{1}{2}(\omega_l/\omega_s')\sec\phi$, where ϕ is the angle between the directions of propagation of short and long waves. Now $\omega_l\ll\omega_s'$ by assumption, so that for ϕ well away from $\pi/2$ the second term on the r.h.s. of (2.8) may be neglected. In other words, changes in $\mathbf{k}_s$ are largely associated with the effect of the Doppler shift $\mathbf{u}_l\cdot\mathbf{k}_s$ of the short wave frequency.

The analysis is readily extended to include the effect of surface tension, which we omit for the sake of simplicity.

Changes in short wave energy, in the absence of generation or dissipation, are given by the wave action conservation equation (Bretherton and Garrett, 1968)

$$\partial(E_s'/\omega_s')/\partial t+\nabla\cdot[(\mathbf{u}_l+\mathbf{c}_s')E_s'/\omega_s']=0, \qquad (2.9)$$

where ∇ has horizontal components only. It is important to realize that this equation applies to the short wave energy evaluated in the vertically accelerating frame of reference, and may, indeed, be applied without long wave linearization, using (2.1, 2) and related formulas. Previous authors (Longuet-Higgins and Stewart, 1960; Phillips, 1966, p. 61) have worked with equations describing the rate of change of short wave energy measured in a frame of reference moving horizontally with velocity $\mathbf{u}_l$ but not accelerating vertically. There is no difference in the final results for the modulation of short wave properties, but E_s' and (2.9) seem more fundamental, especially in view of the equipartition of potential and kinetic energy in the accelerating frame, and the simple connection (2.7) between E_s' and wave momentum.

If we combine (2.7)–(2.9), and allow for short wave generation and dissipation, we find (Garrett, 1976) that the short wave momentum is governed by

$$\partial M_{s\alpha}/\partial t+\partial[M_{s\alpha}(u_{l\beta}+c'_{s\beta})]/\partial x_\beta = -M_{s\beta}\partial u_{l\beta}/\partial x_\alpha - D_{s\alpha}+\tau_{s\alpha}, \quad (2.10)$$

where $D_{s\alpha}$ is the rate of loss of momentum from the short waves by dissipative processes, and $\tau_{s\alpha}$ is the rate of generation of short wave momentum by the wind or other processes.

If, for simplicity, we now take the short waves and long waves to be propagating in the same direction, and consider just a single long wave component, the modulation of the short wave parameters by the long waves is readily evaluated (most simply by considering the steady problem in a frame of reference moving with the phase velocity of the long waves). To lowest order in the long wave steepness k_la_l, and without, in fact, neglecting the last term in (2.8), the dispersion relation gives

$$\left.\begin{aligned} k_s&=k_0(1+k_la_l\cos\theta)\\ \omega'_s&=\omega'_0\\ c'_s&=c'_0(1-k_la_l\cos\theta)\end{aligned}\right\}, \quad (2.11)$$

where θ refers to the long wave phase, $k_lx-\omega_lt$, and $\zeta_l=a_l\cos\theta$. The constant k_0 is just the average wavenumber of the short waves, $k_0=\langle k_s\rangle$, where $\langle\ \rangle$ denotes an average over θ.

For purely conservative interactions (i.e., $\mathbf{D}_s=\boldsymbol{\tau}_s=0$) (2.9) gives, again to lowest order in k_la_l,

$$E'_s=E'_0(1+k_la_l\cos\theta). \quad (2.12)$$

Now $E'_s=\frac{1}{2}\rho(g+\ddot{\zeta}_l)a_s^2$, so that the short wave amplitude a_s is given by

$$a_s=a_0(1+k_la_l\cos\theta), \quad (2.13)$$

$$M_s=M_0(1+2k_la_l\cos\theta), \quad (2.14)$$

where $a_0=\langle a_s\rangle$ and $M_0=\langle M_s\rangle=E'_0k_0/\omega'_0$.

We have assumed the long waves to be propagating in the positive x direction. If the short waves are propagating in the opposite direction we merely change the sign of k_0 and c'_0. The short waves still have a maximum amplitude at the crests of the long waves.

These results, derived here using an approach somewhat different from that of Longuet-Higgins and Stewart (1960) and Phillips (1966, p. 61) in order to illustrate the power of the wave action conservation equation (2.9), are readily extended to the situation where long and short waves are not propagating in the same direction and the water depth is finite (Phillips, 1966, p. 61).

Given short wave generation and dissipation, but such that the short wave field is stationary with respect to the long waves, the modulation of k_s, ω'_s and c'_s is as in (2.11) and the modulation of M_s may be found by solving (2.10) with D_s and τ_s included. To lowest order in k_la_l and ω_l/ω'_s the equation in one dimension is

$$\frac{dM_s}{d\theta}=-2M_sk_la_l\sin\theta-\omega_l^{-1}(\tau_s-D_s). \quad (2.15)$$

3. The long wave energy equation

If we integrate the full horizontal momentum equations vertically over a thin surface layer containing the short waves, we find that they exert an effective surface stress on the total flow (Phillips, 1966, p. 46; Hasselmann, 1971; Garrett, 1976) given by

$$F_\alpha=-\partial(u_{l\alpha}M_{s\beta}+u_{l\beta}M_{s\alpha}+S_{s\alpha\beta})/\partial x_\beta. \quad (3.1)$$

The term $S_{s\alpha\beta}$ is the radiation stress of the short waves, given by

$$S_{s\alpha\beta}=\int_{\zeta_l-h}^{\zeta_l}(\rho\overline{u_{s\alpha}u_{s\beta}}+\delta_{\alpha\beta}\bar{p}_s)dx_3+\overline{\int_{\zeta_l}^{\zeta_l+\zeta_s}\delta_{\alpha\beta}p\,dx_3}. \quad (3.2)$$

Here ζ_l-h is some depth h below the free surface, deep enough so that the vertical integral in (3.2) contains all the short wave Reynolds stress, but shallow enough to be effectively at the surface for the long waves, i.e., we require $|\mathbf{k}_l|\ll|\mathbf{k}_s|$. The overbar indicates an average over several wavelengths or periods of the short waves. The average pressure $\bar{p}$ is split into two parts, $\bar{p}_s=\rho\overline{u_{s3}^2}$ and p_l, so that p_l satisfies the same free surface boundary condition, p_l= average atmosphere pressure, as in the absence of short waves. The last term in (3.2) may be written as $-\overline{\zeta_s^2}\partial\bar{p}/\partial x_3$ evaluated at ζ_l. To lowest order in the short waves, and first order in the long wave steepness, $\partial\bar{p}/\partial x_3$ at $x_3=\zeta_l$ is given by $-\rho(g+\ddot{\zeta}_l)$ from the vertical component of the equation of motion. Hence, using the local properties of the short waves, the last term in (3.2) exactly cancels the second term in the first integral. In terms of energy and momentum, the short wave radiation stress may now be written

$$S_{s\alpha\beta}=\tfrac{1}{2}E'_sk_{s\alpha}k_{s\beta}/|\mathbf{k}_s|^2=M_{s\alpha}c'_{s\beta}. \quad (3.3)$$

Garrett (1976) showed that if one substracts the wave momentum equation (2.10) the effective surface stress exerted on the flow associated with the long waves is

$$\mathbf{F}_l=\mathbf{M}_s\times\nabla\times\mathbf{u}_l-\mathbf{u}_l\nabla\cdot\mathbf{M}_s+\mathbf{D}_s. \quad (3.4)$$

We note that any direct generation $\boldsymbol{\tau}_s$ of short wave momentum by, for example, atmospheric pressure, enters (2.10) and the total momentum equation [in the last term of (3.2), in fact, in the present formulation], so that it cancels in deriving (3.4). The surface stress $\mathbf{F}_l$ supplies energy to the long waves at a rate $\mathbf{F}_l\cdot\mathbf{u}_l$.

Hasselmann (1971) pointed out that there is also a potential energy transfer, as mass is being supplied from long waves to the short waves at a rate $\nabla\cdot\mathbf{M}_s$ and with potential energy $g\zeta_l$ per unit mass. Hence the full energy equation for the long waves, neglecting direct generation, is

$$dE_l/dt = -\langle g\zeta_l \nabla\cdot\mathbf{M}_s\rangle + \langle \mathbf{u}_l\cdot\mathbf{D}_s\rangle + \langle \mathbf{u}_l\cdot(\mathbf{M}_s\times\nabla\times\mathbf{u}_l - \mathbf{u}_l\nabla\cdot\mathbf{M}_s)\rangle, \quad (3.5)$$

where, as before, the angle braces denote an average over the long waves.

The third and fourth terms on the r.h.s. of (3.5) are quadratic in $\mathbf{u}_l$ and so negligible compared with the first term. We now use (2.10) to evaluate $\mathbf{u}_l\cdot\mathbf{D}_s$ in (3.5), again neglecting terms that are quadratic in $\mathbf{u}_l$. Hence,

$$dE_l/dt = -\underset{[1]}{\langle g\zeta_l\nabla\cdot\mathbf{M}_s\rangle} - \underset{[2]}{\langle \mathbf{u}_l\cdot\partial\mathbf{M}_s/\partial t\rangle} - \underset{[3]}{\langle u_{l\alpha}\partial(M_{s\alpha}c'_{s\beta})/\partial x_\beta\rangle} + \underset{[4]}{\langle \mathbf{u}_l\cdot\boldsymbol{\tau}_s\rangle}. \quad (3.6)$$

Terms [1] and [2] in (3.6) may be written

$$[1]+[2] = \langle -\nabla\cdot(g\zeta_l\mathbf{M}_s) - \partial(\mathbf{u}_l\cdot\mathbf{M}_s)/\partial t + \mathbf{M}_s\cdot(\partial\mathbf{u}_l/\partial t + g\nabla\zeta_l)\rangle, \quad (3.7)$$

in which the first two terms vanish if we assume that $\langle g\zeta_l\mathbf{M}_s\rangle$ is homogeneous and $\langle\mathbf{u}_l\cdot\mathbf{M}_s\rangle$ is stationary. [In fact, if the long wave energy is growing, this last assumption cannot be strictly valid. However, using (2.14) to make a rough estimate of $\langle\mathbf{u}_l\cdot\mathbf{M}_s\rangle$, we find it to be $(\omega_l/\omega'_s)^3(k_0a_0)^2$ times as big as E_l, and so totally negligible.] The final terms in (3.7) vanish, to first order in long wave steepness, from the form of the horizontal momentum equation near the surface, which gives (Hasselmann, 1971)

$$\partial\mathbf{u}_l/\partial t + g\nabla\zeta_l = O(|\mathbf{u}_l|^2,\ |\mathbf{u}_l||\mathbf{M}_s|). \quad (3.8)$$

Equivalently, $\partial\mathbf{u}_l/\partial t + g\nabla\zeta_l$ vanishes for each Fourier component of the long wave field provided that it behaves approximately like a free wave. The net result is that [1]+[2] vanishes with errors that are quadratic in $\mathbf{u}_l$ and $\mathbf{M}_s$.

Term [3] in (3.6) may be written $\langle -u_{l\alpha}\partial S_{s\alpha\beta}/\partial x_\beta\rangle$, from (3.3), or, invoking homogeneity, $\langle S_{s\alpha\beta}\partial u_{l\alpha}/\partial x_\beta\rangle$. This is just the damping term investigated by Phillips (1963).

We may now write (3.6) as

$$dE_l/dt = \langle S_{s\alpha\beta}\partial u_{l\alpha}/\partial x_\beta\rangle + \langle\mathbf{u}_l\cdot\boldsymbol{\tau}_s\rangle. \quad (3.9)$$

We have ignored direct generation of long waves through, for example, atmospheric pressure fluctuations in phase with $-\partial\zeta_l/\partial t$. Eq. (3.9) is equivalent to that derived by Hasselmann (1971), and, indeed, many aspects of the present derivation parallel his. However, we have thought it worthwhile to re-derive the energy equation for the long waves in order to emphasize the significance of the term $\langle\mathbf{u}_l\cdot\boldsymbol{\tau}_s\rangle$, where $\boldsymbol{\tau}_s$ is the rate of transfer of momentum to the short waves.

It seems very likely that $\boldsymbol{\tau}_s$ should at least be proportional to short wave amplitude (as for any wave generation theory involving feedback from the waves to the airflow), and hence that $|\boldsymbol{\tau}_s|$ will be largest where the short waves are largest, i.e., near the long wave crests.

Before exploring various models for the variation of short wave amplitude and $\boldsymbol{\tau}_s$ as functions of long wave phase, we point out that in order to retain the first term on the rhs of (3.9) compared with quadratic terms in $\mathbf{u}_l$ which we have neglected, we require $|\mathbf{u}_l|\ll|\mathbf{c}'_s|$. This amounts to assuming that the long wave steepness k_la_l is much less than the frequency ratio ω_l/ω'_s, which has also been assumed small. This may not be true, so that the long wave damping due to the action of the short wave radiation stress may be overwhelmed in practice by nonlinear effects for steep long waves. However, it is also possible that these quadratic terms are small, as triad interactions do not occur for surface gravity waves.

In any event, the dominant term on the r.h.s. of (3.9) is $\langle\mathbf{u}_l\cdot\boldsymbol{\tau}_s\rangle$, and $\boldsymbol{\tau}_s$ here should include any momentum transfer from the air to the water, whether it goes into short gravity waves (as assumed here), or capillary waves, or straight into drift currents (see Longuet-Higgins, 1969b; Stewart, 1967).

4. Long wave momentum

The implications of (3.9) for the growth of long waves are best understood in terms of the rate of generation of long wave momentum. We first simplify (3.9) to the one-dimensional situation, for which

$$dE_l/dt = \langle S_s\partial u_l/\partial x\rangle + \langle u_l\tau_s\rangle, \quad (4.1)$$

and assume a single long wave component, with frequency ω_l and wavenumber k_l. The long wave momentum is $M_l = E_lk_l/\omega_l$, and

$$dM_l/dt = (k_l\omega_l)\langle S_s\partial u_l/\partial x + u_l\tau_s\rangle. \quad (4.2)$$

We now write $\zeta_l = a_l\cos\theta$, $u_l = \omega_la_l\cos\theta$ (assuming the long waves to be propagating in the positive x direction), where $\theta = k_lx - \omega_lt$ is the long wave phase. Hence

$$dM_l/dt = k_la_l\langle -k_lM_sc'_s\sin\theta + \tau_s\cos\theta\rangle. \quad (4.3)$$

The average $\langle\ \rangle$ is taken over the phase θ.

If we reverse the direction of the short waves relative to the long waves, the first term on the r.h.s. of (4.3) is unchanged, whereas the second term changes sign.

5. Long wave growth

We see from (4.3) that at most a fraction k_la_l of the total wind stress τ_s can go into long wave momentum. This requires that τ_s be a series of delta functions at the long wave crests.

A precise estimate of the importance of the radiation stress term in (4.3) requires the knowledge of $M_s(\theta)$, either from observation, or from the solution of (2.15) given τ_s and D_s as functions of θ. In general the radiation stress term is negligible compared with the term $\tau_s \cos\theta$, as is illustrated by the following very simple example.

MODEL A. Assume that τ_s is a series of delta functions at long wave crests ($\theta = 2n\pi$), and $D_s = \omega_l \beta M_s$. Eq. (2.15) then becomes

$$dM_s/d\theta = -2k_l a_l M_s \sin\theta + \beta M_s, \tag{5.1}$$

which leads to

$$M_s = 2\pi\omega_l^{-1}\tau_0(1+2\epsilon)^{-1}(e^{2\beta\pi}-1)^{-1}\exp(\beta\theta + 2\epsilon\cos\theta), \quad 0 \leqslant \theta < 2\pi, \tag{5.2}$$

where $\tau_0 = \langle\tau_s\rangle$, the average stress, and $\epsilon = k_l a_l$. Hence, neglecting terms of $O(\epsilon)$,

$$\langle -k_l M_s c_s' \sin\theta\rangle = (\omega_l/\omega_0')(1+\beta^2)^{-1}\tau_0. \tag{5.3}$$

Even if β is small, this is small compared with τ_0 as (ω_l/ω_0') is small. However it is interesting that this model, and indeed any model which has $\tau_s - D_s$ larger than average at the long wave crests, predicts that the radiation stress term acts as a *source* of energy for the long waves, albeit a very weak one, in contradiction to the usual assumption. This result is associated with greater amplitudes of wind-generated short waves on the rear faces of long waves, rather than the forward faces, as is generally assumed.

More observations on the distribution of $M_s(\theta)$ will be most valuable, not so much for calculating the radiation stress effect, but rather as a constraint on possible distributions of $\tau_s(\theta)$.

MODEL B. A more plausible assumption for τ_s is that it varies linearly between crest and trough like $\tau_s = \tau_0(1 + b\cos\theta)$. This might correspond to a varying exposure to the wind. The radiation stress term is again negligible and

$$dM_l/dt = \tfrac{1}{2}k_l a_l b \tau_0, \tag{5.4}$$

so that, for $b \leqslant 1$, a maximum of about $\frac{1}{2}k_l a_l$ of the wind stress can appear as long wave momentum.

MODEL C. An alternative hypothesis is that τ_s is proportional to M_s. With this assumption, and a dissipation that, for example, limits the short wave steepness to some maximum, the modulation of M_s, and hence of τ_s, is always less than that obtained for conservative interactions in (2.14). The upper bound on dM_l/dt is then $(k_l a_l)^2\tau_0$.

All of these models assume a monochromatic long wave. A calculation of the momentum transfer to a spectrum of long waves requires specification of how τ_s varies with, say, the elevation due to the long waves. We shall not pursue such models in detail here (but see Valenzuela and Wright, 1976; Longuet-Higgins, 1976), other than to remark that plausible extensions of models B and C lead to momentum transfer rates of something like $\sum(k_l^2 a_l^2)\tau_0$, where the summation is over all the long waves. Now this summation, or integral, depends logarithmically, for an ω_l^{-5} energy spectrum, on the high-frequency cutoff assumed. If we arbitrarily assume the cutoff frequency to be 5 times the peak frequency and use the canonical JONSWAP spectrum (Hasselmann *et al.*, 1976), we find $\sum(k_l a_l)^2 \approx 0.02$ to 0.04, depending on the fetch. An extension of model A (Longuet-Higgins, 1976) can lead to much greater fractions of τ_0 appearing as long wave momentum, although one suspects that, in general, the $k_l a_l$ factor appearing in model A will also limit the efficiency of transfer for a complete long wave spectrum.

According to Hasselmann *et al.* (1976), only about 5% of the total wind stress remains in the wave field, though about 3 or 4 times as much momentum must be imparted to the long waves initially to allow for computed transfers, by conservative nonlinear interactions, to dissipative high frequencies.

Thus it seems that the mechanism described in this paper probably contributes a significant proportion of long wave momentum, although it seems unlikely that it can account entirely for long wave growth.

6. Conclusions

The main conclusions to be drawn from this study are:

1) Any variation of momentum transfer from air to water that is correlated with the orbital velocity of long waves can contribute to the growth of long waves (or decay of swell propagating against the wind), even if the momentum transfer goes into shorter waves first.

2) At most only a fraction $k_l a_l$ of the wind stress can go into long wave momentum by this mechanism.

3) The radiation stress term in the energy equation for the long waves is generally negligible compared with the wind stress term, but could conceivably act as a weak source for long wave energy, rather than a sink as usually assumed.

Any particular assumption for the variation of τ_s and D_s as functions of short wave momentum or long wave phase can be used in Eq. (2.12) to give $M_s(\theta)$ and hence dM_l/dt. But appropriate assumptions for τ_s and D_s are almost totally unknown. The real need is for experimental determination of the variation of wind stress and short wave amplitude relative to the phase of longer waves.

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