Predicting Changes in Tidal Regime: The Open Boundary Problem

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ABSTRACT

Attempts to predict the impact on a tidal regime of large engineering structures are generally based on the use of a numerical model which is calibrated to reproduce the natural tidal regime and then rerun with the structures in place. It is usually assumed that the "input" tide at the open boundary is unchanged by the structures, though this is clearly wrong in principle.

We show how errors in this procedure can be corrected for, or at least estimated, using output from the numerical model and estimates of the impedance of the exterior ocean. The ocean impedance can be expressed as an infinite series in terms of the normal modes of the ocean, with some terms allowing for near-resonant enhancement of particular modes, and the infinite tail corresponding to a local source-like behavior which can be estimated independently.

Application of the technique to the problem of predicting the impact of Fundy tidal power suggests that any predicted change may be uncertain to about ±25% of the change in mass flux across the open boundary. This uncertainty could amount to ±4% of the tidal range for a large tidal power development.

It is clear that numerical models used in this type of problem should generally extend to the edge of the continental shelf. The role of side boundaries from the coast to the edge of the shelf is uncertain, although in the Fundy problem there is little mass flux across them so that they appear not to be important.

We also estimate that the impact of Fundy tidal power development on global ocean tides would be a change of a few millimeters in M_2.

1. Introduction

As man's capacity to undertake very large engineering projects in coastal water increases, a problem that arises more and more frequently is that of predicting changes in tidal regime which might be caused by these projects. Examples are the Delta Project in the Netherlands (Dronkers, 1970), the suggested causeway from New Brunswick to Prince Edward Island across the Northumberland Strait of the Gulf of St. Lawrence (Farquharson, 1959), and tidal power projects in the Severn Estuary in Great Britain (Heaps, 1972) and the Bay of Fundy in eastern Canada (Lawton, 1972). It is this last project which has motivated the present study.

The standard method of predicting tidal changes brought about by structures is as follows (Heaps, 1972; Heaps and Greenberg, 1974):

1) Develop a numerical model of the region in question subject to boundary conditions of zero flux across the shoreline, and prescribed tidal elevation (usually just the dominant constituent) at the so-called "open" boundaries where the model region adjoins other expanses of sea. This prescribed input at the open boundary is best derived from direct observation (Liu *et al.*, 1974), but in some cases (Greenberg, 1975), where data is lacking, it must be adjusted until the model correctly reproduces the tides observed at the coast.

2) Rerun the model with the structures in place (as for dams) or operating in some prescribed manner (as for a tidal power plant) but with the same elevation as before at the open boundary.

The size of the region to be modeled is chosen to satisfy the usual constraints imposed by computer speed and capacity, but also so that, in the opinion of the modeler, it is large enough for the effect of the structure not to be felt at the open boundary. This assumption is obviously wrong in principle as the disturbance in the tidal regime introduced by the structure will propagate away from it and cause changes on the open boundaries of the system, however far away they are.

Garrett (1975) extended the harbor theory of Miles (1971) to include tidal forces, and found that if the region being modeled is much shallower than the ocean outside it and is either small or highly dissipative, then the assumption of fixed elevation at the open boundary may be reasonable. However, a quantitative estimate of the change would clearly be useful.

In this paper we shall discuss the coupling between a modeled region and the exterior ocean, and show how a change at the open boundary may be estimated and used to correct predictions of changes within the model. The theory will be illustrated by application to the problem of predicting changes in tidal regime that might be brought about by tidal power development in the Bay of Fundy.

2. Coupling the model region to the exterior ocean

The equations governing the tides are usually taken to be

$$\partial\hat{\mathbf{u}}/\partial t+\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}+\mathbf{f}\times\hat{\mathbf{u}}+g\nabla(\hat{\zeta}-\hat{\zeta}_e)+\hat{\mathbf{F}}_b-A_H\nabla^2\hat{\mathbf{u}}=0, \quad (2.1)$$

$$\partial\hat{\zeta}/\partial t+\nabla\cdot[(h+\hat{\zeta})\hat{\mathbf{u}}]=0, \quad (2.2)$$

or some simplified version. Here $\hat{\mathbf{u}}(\mathbf{x},t)$, $\hat{\zeta}(\mathbf{x},t)$ are the tidal current and elevation in water of depth $h(\mathbf{x})$; $\hat{\zeta}_e(\mathbf{x})$ is the "equilibrium" tide, $\mathbf{f}(\mathbf{x})$ the Coriolis parameter and $\hat{\mathbf{F}}_b(\mathbf{x},t)$ the bottom friction [usually $\gamma\hat{\mathbf{u}}|\hat{\mathbf{u}}|/(h+\hat{\zeta})$ with $\gamma\approx0.002$]; and $A_H\nabla^2\hat{\mathbf{u}}$ parameterizes horizontal mixing processes. Typical coastal tidal models (e.g., Greenberg, 1975) solve these equations in finite-difference form subject to boundary conditions of $\hat{\mathbf{u}}\cdot\mathbf{n}=0$ at the coastline and prescribed $\hat{\zeta}$ at the open boundaries.

To investigate the coupling between the region G being modeled and the exterior ocean O we first linearize (2.1) and (2.2) and investigate a single frequency ω. With $\hat{\mathbf{u}}=\mathrm{Re}\,\mathbf{u}(\mathbf{x})e^{i\omega t}$, $\hat{\zeta}=\mathrm{Re}\,\zeta(\mathbf{x})e^{i\omega t}$, $\hat{\zeta}_e=\mathrm{Re}\,\zeta_e(\mathbf{x})e^{i\omega t}$ and $\hat{\mathbf{F}}_b=\lambda(\mathbf{x})\hat{\mathbf{u}}$, we have

$$i\omega\mathbf{u}+\mathbf{f}\times\mathbf{u}+g\nabla(\zeta-\zeta_e)+\lambda\mathbf{u}-A_H\nabla^2\mathbf{u}=0, \quad (2.3)$$

$$i\omega\zeta+\nabla\cdot(h\mathbf{u})=0. \quad (2.4)$$

Following Garrett (1975) we divide the solution into two parts: $(\mathbf{u}^{(1)},\zeta^{(1)})$ satisfies (2.3) and (2.4) with $\mathbf{u}^{(1)}\cdot\mathbf{n}=0$ on the boundary M between G and O as well as on the coastline (Fig. 1); whereas $(\mathbf{u}^{(2)},\zeta^{(2)})$ satisfies (2.3) and (2.4) with $\zeta_e=0$ and is generated by a mass flux $h\mathbf{u}^{(2)}\cdot\mathbf{n}=F(s)$ at position s of M. $F(s)$ must satisfy the integral equation

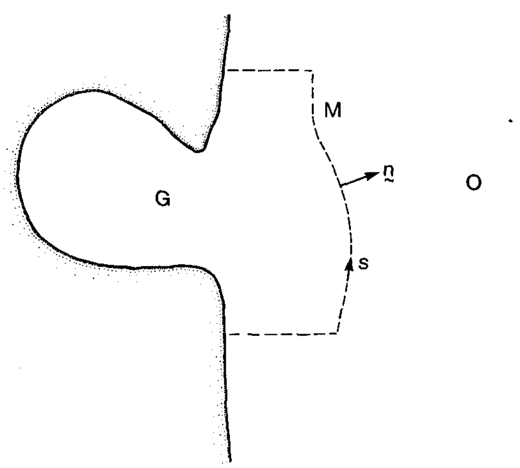

FIG. 1. Schematic of region G separated from the ocean O by a boundary M. **n** is the outward normal from G, and s the position coordinate along M.

$$\zeta_G^{(1)}(s)+\int_M K_G(s,\sigma)F(\sigma)d\sigma=\zeta_O^{(1)}(s)-\int_M K_O(s,\sigma)\times F(\sigma)d\sigma, \quad (2.5)$$

where $K_G(s,\sigma)$ and $-K_O(s,\sigma)$ are the values at s of M of the response in G and O to unit mass flux out of G across M at σ. Both sides of (2.5) equal $\zeta_M(s)$, the resulting elevation on M.

If the model region G is altered in some way, then $\zeta_G^{(1)}(s)$ and $K_G(s,\sigma)$ will change [although $\zeta_G^{(1)}(s)$ and changes in it are typically negligibly small], so that $F(s)$ and $\zeta_M(s)$ are changed. Clearly an estimate of the change in $\zeta_M(s)$ and hence in the tidal regime in G requires some consideration of the external region O, even if only to prove that the change in $\zeta_M(s)$ is unimportant.

From (2.5) we see that changes $\delta F(s)$ in $F(s)$ and $\delta\zeta_M(s)$ in $\zeta_M(s)$ are related by

$$\delta\zeta_M(s)=-\int_M K_O(s,\sigma)\delta F(\sigma)d\sigma. \quad (2.6)$$

This is an integral condition connecting changes in $\zeta_M(s)$ to changes in $F(s)$ at all points of M.

In a numerical model with M divided into N segments (2.6) may be written

$$\delta\zeta_p=-\sum_{q=1}^{N}K_{pq}\delta F_q\Delta s_q, \quad (2.7)$$

where $\delta\zeta_p$ means the value of $\delta\zeta_M(s)$ in segment p, K_{pq} is the response at segment p to unit mass flux through segment q of length Δs_q, and δF_q is the value of $\delta F(s)$ in segment q. $K_O(s,\sigma)$ generally has a logarithmic singularity at $s=\sigma$, but following Lee (1971), in Section 5 of this paper we shall integrate over this to obtain the diagonal elements of K_{pq}. The boundary condition (2.7) is now suitable for application to a numerical solution of Eqs. (2.3) and (2.4) provided that K_{pq} is known.

A similar boundary condition

$$\delta\hat{\zeta}_p=-\sum_{q=1}^{N}K_{pq}\delta\hat{F}_q\Delta s_q \quad (2.8)$$

may be applied to a time-step solution of (2.1) and (2.2) on the assumption that K_{pq} of (2.7) is real and frequency-independent. Greenberg (1975) used a local version $K_{pq}\Delta s_q=-(gh)^{-\frac{1}{2}}\delta_{pq}$ in his time-step model of the Bay of Fundy and Gulf of Maine, assuming that any change from the natural regime propagated seaward as a free one-dimensional gravity wave in water of the local depth h. This clearly has the following shortcomings when compared with the correct boundary condition developed here:

1) It is incorrect to use the local depth if the model boundary is at the edge of the continental shelf. The ocean impedance depends on the depth of the deep ocean beyond the shelf.

2) Assuming K_{pq} to be real rules out the phase difference between current and elevation that we will find occurs with appropriate choices for $K_O(s,\sigma)$.

3) Assuming K_{pq} to be diagonal might be appropriate if the tidal wavelength were very short compared with the length scale of the open boundary so that ray theory applied. In practice the tidal wavelength is typically large compared with the scale of the open boundary, so that the problem involves diffractive effects and $K_O(s,\sigma)$ is not local, i.e., K_{pq} is not diagonal.

These points will become clearer later in the paper.

3. Integral approximation

In a practical problem it is unlikely that $K_O(s,\sigma)$ will be known in detail, so that it is difficult to apply the type of boundary condition suggested above. A simple and physically revealing approximation can be obtained by evaluating a weighted integral of (2.5) over M.

Define $F(s)=If(s)$, where

$$\int_M f(s)ds=1,$$

multiply (2.5) by $f(s)$ and integrate to obtain

$$V_G+Z_GI=V_O-Z_OI=V_M, \tag{3.1}$$

where

$$V_G=\int_M \zeta_G^{(1)}(s)f(s)ds,$$

$$V_O=\int_M \zeta_O^{(1)}(s)f(s)ds,$$

$$V_M=\int_M \zeta_M(s)f(s)ds, \tag{3.2}$$

$$Z_G=\int_M \int_M K_G(s,\sigma)f(s)f(\sigma)dsd\sigma,$$

$$Z_O=\int_M \int_M K_O(s,\sigma)f(s)f(\sigma)dsd\sigma. \tag{3.3}$$

Here Z_G, Z_O are the impedances for G and O (Miles, 1971; Garrett, 1975).

We now consider changes in (3.1) if G is changed. $\zeta_O^{(1)}(s)$ is independent of G but V_O depends on $f(s)$ and so will change if G and hence $f(s)$ is changed. In fact if $\zeta_O^{(1)}(s)$ is fairly smooth over M and if the shape of $F(s)$, as given by $f(s)$, does not change much, then the change in V_O is totally negligible, as will be borne out in Section 6 for our particular application. The changes in V_M and I are then related by

$$\delta V_M=-Z_O\delta I. \tag{3.4}$$

In other words, the change δV_M in input "voltage" is related to the change δI in "current" drawn by the impedance Z_O of the driving system O. This is, of course, a standard result in electronics. Miles (1971) has exploited the analogy further in his discussion of an equivalent circuit analysis of harbors. He averages (2.5) over M with the complex conjugate $f^*(s)$ rather than $f(s)$ as a weighting function. This has the advantage that $-\frac{1}{2}\rho g \,\mathrm{Re} V_M I^*$ may be identified as the energy flux into G, but it does not greatly affect the value of Z_O obtained (see Section 6).

When using the above theory with a finite-difference numerical model, we shall evaluate I, V_M and Z_O from

$$I=\sum_p F_p\Delta s_p,\quad V_M=\sum_p \zeta_{Mp}F_p\Delta s_p/I, \tag{3.5}$$

$$Z_O=\sum_p\sum_q K_{pq}F_pF_q\Delta s_p\Delta s_q/I^2, \tag{3.6}$$

where summation is from 1 to N. As mentioned before, care must be taken in evaluating the diagonal components of K_{pq}.

4. Application

The integral form (3.4) of the open boundary condition is easily applied to a numerical model of a region G. Input data $\zeta_M(s)$ are provided, and $F(s)$ is part of the model output. Hence I and V_M are calculated. Suppose the model is rerun, for a modification of G, with the same input $\zeta_M(s)$, producing a new mass flux $(1+B)I$ instead of I, but a negligible change in the value of V_M as calculated. We assume that the correct solution for the modified regime is the computed solution multiplied by a factor A (this is also a solution as the problem is assumed to be linear). We can determine A from (3.4) which is now

$$(A-1)V_M=-Z_O[A(1+B)-1]I, \tag{4.1}$$

so that

$$A=(V_M+Z_OI)/[V_M+Z_OI(1+B)] \tag{4.2}$$

$$=[1+BR(1+R)^{-1}]^{-1},\quad R=Z_O(I/V_M). \tag{4.3}$$

From (3.1) we see that if V_G is negligible, R is just the impedance ratio Z_O/Z_G. If $|R|\ll 1$, $A=1$ and the assumption of fixed input V_M is a good one. At the other extreme $|R|\gg 1$ gives $A=(1+B)^{-1}$ and implies that fixed mass flux I is the appropriate boundary condition. The general result for finite R does not lie on some linear interpolation between $A=1$ and $A=(1+B)^{-1}$, as B and R are complex. However, if B is small, we may write

$$A=1-BR/(1+R). \tag{4.4}$$

5. Estimates of Z_O

An estimate of the ocean impedance is crucial for a confident prediction of changes in tidal regime. However, a precise calculation of $K_O(s,\sigma)$ or Z_O requires a numerical model of the whole ocean region O and so is unlikely to be possible in view of the major problems that are still associated with global ocean tide models (Hendershott, 1976). We shall restrict our attention here to estimates of Z_O that illustrate the fundamental physics of the problem and that will enable us to place realistic bounds on the actual value of Z_O for the Bay of Fundy problem discussed in Section 6.

a. Semi-infinite ocean of constant depth

If M is a section of length b on the straight boundary of a semi-infinite ocean of depth h_O and constant Coriolis parameter f, we may use the near field of the Green's function of Voit (1958) and Buchwald (1971), i.e.,

$$K_O(s,\sigma)=-\omega(2gh_O)^{-1}\{1-2i\pi^{-1}[\ln(k_O|s-\sigma|/2)+\gamma] -(f/\omega)\,\mathrm{sgn}(s-\sigma)-i(f/\omega)\pi^{-1}\ln[(\omega+f)/(\omega-f)]\}, \tag{5.1}$$

where

$$k_O=(\omega^2-f^2)^{\frac{1}{2}}(gh_O)^{-\frac{1}{2}} \tag{5.2}$$

for $\omega>f$, as is assumed, and $\gamma=0.5772$ is Euler's constant.

In evaluating K_{pq} for use with a finite-difference model we take $s-\sigma=(p-q)\Delta s$ (assuming the grid length Δs_p uniform) for $p\neq q$, and for $p=q$ we follow Lee (1971) in integrating K_O over the element

$$K_{pp}=(\Delta s)^{-1}\int_{-\frac{1}{2}\Delta s}^{\frac{1}{2}\Delta s}K_O(0,\sigma)d\sigma =-\omega(2gh_O)^{-1}\{1-2i\pi^{-1}[\ln(\tfrac{1}{4}k_O\Delta s)+\gamma-1] -i(f/\omega)\pi^{-1}\ln[(\omega+f)/(\omega-f)]\}. \tag{5.3}$$

We shall check the accuracy of this procedure in Section 6, for a particular value of $k_O\Delta s$, by comparing $\sum_p\sum_q K_{pq}$ with the value of Z_O obtained with $f(s)=b^{-1}$, which is [Garrett (1975) with an error in sign corrected]

$$Z_O^{si}=-\omega(2gh_O)^{-1}\{1-2i\pi^{-1}[\ln(\tfrac{1}{2}k_Ob)+\gamma-\tfrac{3}{2}] -i(f/\omega)\pi^{-1}\ln[(\omega+f)/(\omega-f)]\}, \tag{5.4}$$

where the superscript si here refers to the semi-infinite nature of the exterior ocean.

We notice that Z_O^{si} scales more or less inversely with the depth h_O of the exterior ocean. This immediately suggests that if we wish to have $Z_O(I/V_M)$ small to justify holding V_M constant in a numerical model, then we should at least take the boundary M of the model at the edge of the continental shelf if possible.

However, in view of the high Q of ocean basins (Hendershott, 1972) and their proximity to resonance at tidal frequencies (Platzman, 1975), it is hardly suitable to regard them as semi-infinite sinks for energy radiated from coastal seas. A value of Z_O based on $K_O(s,\sigma)$ of (5.1) does embody the near-source response of the exterior ocean, but we should also allow for significant excitation of the normal modes of O.

b. Normal mode expansion

We first assume that the exterior ocean O is free of dissipation and has a well-defined set of normal modes. The linearized equations are then (2.3) and (2.4) with ζ_e, λ, A_H all zero. In the notation of Platzman (1975)

$$(\mathfrak{L}-\omega)a=0, \tag{5.5}$$

where

$$\mathfrak{L}=i\begin{pmatrix}0 & \nabla\cdot h\\ g\nabla & \mathbf{f}\times\end{pmatrix} \tag{5.6}$$

and a is the column vector $(\zeta,\mathbf{u})$. There is an infinite set of normal modes $E_k=(\zeta_k,\mathbf{u}_k)$ with eigenvalues ω_k. Corresponding to a mode with positive frequency ω_k, there is a conjugate mode E_k^* with frequency $-\omega_k$, where the asterisk denotes the complex conjugate. We must also include a mode with frequency $\omega_0=0$ and $E_0=(1,\mathbf{0})$, as in the harbor theory of Miles (1971).

Platzman (1975) defines a scalar product

$$\langle a,a'\rangle=\int_O(g\zeta^*\zeta'+h\mathbf{u}^*\cdot\mathbf{u}')dS, \tag{5.7}$$

where dS is an element of area of O. $\langle a,a\rangle$ is $(4/\rho)$ times the energy associated with a in region O.

A version of Green's formula is

$$\langle a,\mathfrak{L}a'\rangle-\langle\mathfrak{L}a,a'\rangle=i\int_O\nabla\cdot gh(\zeta^*\mathbf{u}'+\zeta'\mathbf{u}^*)dS, \tag{5.8}$$

which can be used to prove that the eigenvalues ω_k are real and the modes E_k orthogonal to each other, in the sense that $\langle E_k,E_l\rangle=0$ for $k\neq l$.

We now seek a normal mode expansion of the solution of (5.5) forced by a prescribed flux $h\mathbf{u}\cdot\mathbf{n}=-F(s)$ on the boundary M, where $\mathbf{n}$ is the outward normal form O. Suppose

$$a=\sum\alpha_lE_l, \tag{5.9}$$

where, of course, the summation includes modes with negative and zero as well as positive eigenvalues. Putting $a'=E_k$ in (5.8) and taking the complex conjugate, we find

$$\alpha_k=-i(\omega-\omega_k)^{-1}\int_M g\zeta_k^*F(s)ds/\langle E_k,E_k\rangle. \tag{5.10}$$

(This assumes that if the modes E_k have open ports at which $\zeta_k=0$, as well as closed boundaries on which $\mathbf{u}_k\cdot\mathbf{n}=0$, then the solution a also satisfies $\zeta=0$ at these ports.)

In the notation of Section 2 the Green's function $K_O(s,\sigma)$ is clearly

$$K_O(s,\sigma)=\sum i(\omega-\omega_k)^{-1}(\langle E_k,E_k\rangle)^{-1}g\zeta_k^*(\sigma)\zeta_k(s). \quad (5.11)$$

This allows for the resonant enhancement of any mode for which $\omega-\omega_k$ is small, but if $|s-\sigma|$ is small compared with a tidal wavelength the infinite sum of high modes in (5.11) will also contribute to $K_O(s,\sigma)$ the logarithmic singularity, appropriate to a source flow, that appears in (5.1).

If we include the dissipative terms in (2.3) the operator $\mathfrak{L}$ becomes

$$\mathfrak{L}=i\begin{pmatrix}0 & \nabla\cdot h\\ g\nabla & \lambda+\mathbf{f}\times-A_H\nabla^2\end{pmatrix} \quad (5.10)$$

As before, there is an infinite set of normal modes $E_k=(\zeta_k,\mathbf{u}_k)$, but the eigenvalues ω_k are now complex. $\mathfrak{L}$ is still pure imaginary, so there are still conjugate modes E_k^* with eigenvalues $-\omega_k^*$. We retain the definition (5.5) of a scalar product but find that the normal modes are no longer orthogonal, thus apparently removing the possibility of the normal mode expansion (5.7).

A normal mode expansion can be obtained, however, if one considers also the eigenfunctions of the adjoint operator $\mathfrak{L}\dagger$. This is well-known in theoretical physics (Morse and Feshbach, 1953) and has been discussed for the tidal problem by Webb (1974). $\mathfrak{L}\dagger$ can be simply defined by changing the sign of any dissipative terms in the governing equations, so that

$$\mathfrak{L}\dagger=i\begin{pmatrix}0 & \nabla\cdot h\\ g\nabla & -\lambda+\mathbf{f}\times+A_H\nabla^2\end{pmatrix}. \quad (5.11)$$

$\mathfrak{L}\dagger$ has an infinite set of eigenfunctions $E_k^\dagger$ which are not simply related to E_k. However, the eigenvalues $\omega_k^\dagger$ of $\mathfrak{L}\dagger$ are related to the ω_k, and with suitable ordering we may take $\omega_k^\dagger=\omega_k^*$. Moreover, the two sets of eigenfunctions are biorthogonal, in that

$$\langle E_k^\dagger,E_l\rangle=0 \quad \text{for} \quad k\neq l. \quad (5.12)$$

[We may also relax Platzman's boundary condition $\mathbf{u}\cdot\mathbf{n}=0$ or $\zeta=0$ on the boundary, except on the part M between G and O, where we retain $\mathbf{u}\cdot\mathbf{n}=0$ for E_k and $E_k^\dagger$. Suppose that the boundary condition is $\mathbf{u}\cdot\mathbf{n}=\beta\zeta$, where the real part of β allows for a flux of energy out of O, then we can retain all the above properties of E_k and $E_k^\dagger$ if we apply a boundary condition $\mathbf{u}\cdot\mathbf{n}=-\beta^*\zeta$ to all solutions of the adjoint problem with operator $\mathfrak{L}\dagger$.]

The coefficients α_k in the normal mode expansion (5.7) are now given by

$$\alpha_k=-i(\omega-\omega_k)^{-1}(\langle E_k^\dagger,E_k\rangle)^{-1}\int_M g\zeta_k^{\dagger*}F(s)ds, \quad (5.13)$$

$$K_O(s,\sigma)=\sum i(\omega-\omega_k)^{-1}(\langle E_k^\dagger,E_k\rangle)^{-1}g\zeta_k^{\dagger*}(\sigma)\zeta_k(s). \quad (5.14)$$

Now the dissipative normal modes of the world oceans and the corresponding solutions of the adjoint problem have not been calculated. We shall assume that dissipation does not greatly alter the inviscid eigenfunctions E_k computed by Platzman (1975), but we will allow for complex eigenvalues by writing $\omega_k(1+\frac{1}{2}iQ_k^{-1})$ for the eigenvalue ω_k in (5.9), where ω_k and Q_k are real.

Of course the normal modes we require are really those of the world oceans without the coastal region G being modeled, rather than the whole world oceans. We assume that this difference is negligible.

In summary, we will estimate the impedance Z_O of a "real" ocean by

$$Z_O=Z_O^{sl}+Z_O^{nm}. \quad (5.15)$$

Z_O^{sl} from (3.6) or (5.4) gives the local logarithmic singularity and

$$Z_O^{nm}=\sum{}' i[\omega-\omega_k(1+\tfrac{1}{2}iQ_k^{-1})]^{-1}(\langle E_k,E_k\rangle)^{-1}g|\zeta_k|^2, \quad (5.16)$$

the normal mode impedance, allows for resonant enhancement of a few modes, the summation $\sum'$ being over just those modes which are significantly excited. $|\zeta_k|$ refers to the average value in M of the elevation of the kth normal mode.

c. Z_O from open boundary data

From (3.1) we have $V_M=V_O-Z_OI$. In this V_M is known either from offshore data or from calibration of the model. I may be obtained from offshore data but is more likely to be obtained from the model. If we can obtain an estimate of V_O we can solve (3.1) for Z_O. Now V_O is the elevation that would be observed at M in the absence of the model region G, and differs from V_M by $Z_O^{sl}I+Z_O^{nm}I$. The first term here, $Z_O^{sl}I$, is largest in the immediate neighborhood of M, whereas Z_O^{nm} varies slowly away from M. Thus if offshore data show a fairly abrupt change of the elevation, from V_B away from M to V_M in the immediate neighborhood of M, then V_B-V_M may be taken as a rough estimate of $Z_O^{sl}I$. (The distance for such a transition should be a fraction of k_O^{-1}, far enough to avoid the singularity of the Green's function, and is typically a few hundred kilometers). It does not appear to be possible to make any estimate of Z_O^{nm} from the data.

The use and limitations of the above estimates for Z_O will become clearer within the context of a particular problem.

6. The tidal regime problem of Fundy tidal power

Fig. 2 shows the region modeled by Greenberg (1975) in his study of the tides of the Bay of Fundy and Gulf of Maine. The tidal elevation is prescribed along the open boundary at the heavy dots. For the moment we disregard completely the two sections of the open

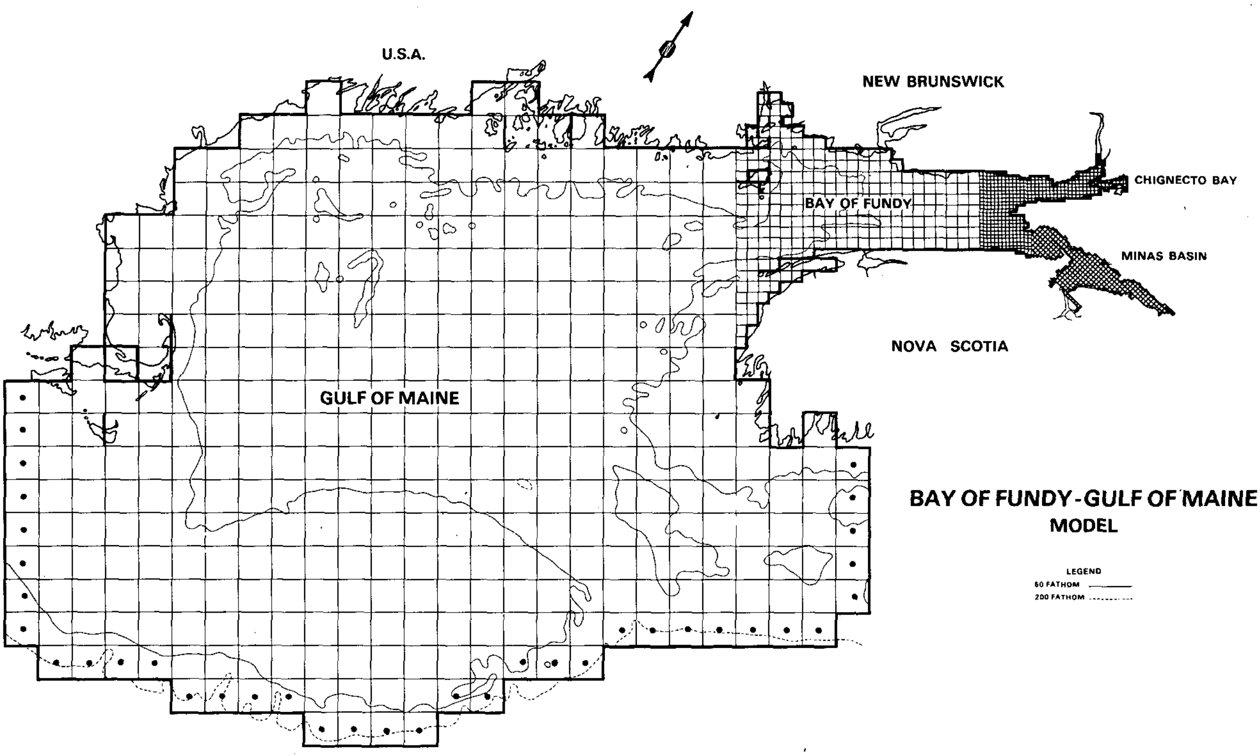

FIG. 2. Domain of Greenberg's (1975) numerical model. The tidal elevation is prescribed in the outermost grid squares, marked with heavy dots. The solid contour is 50 fathoms, the dotted 200 fathoms.

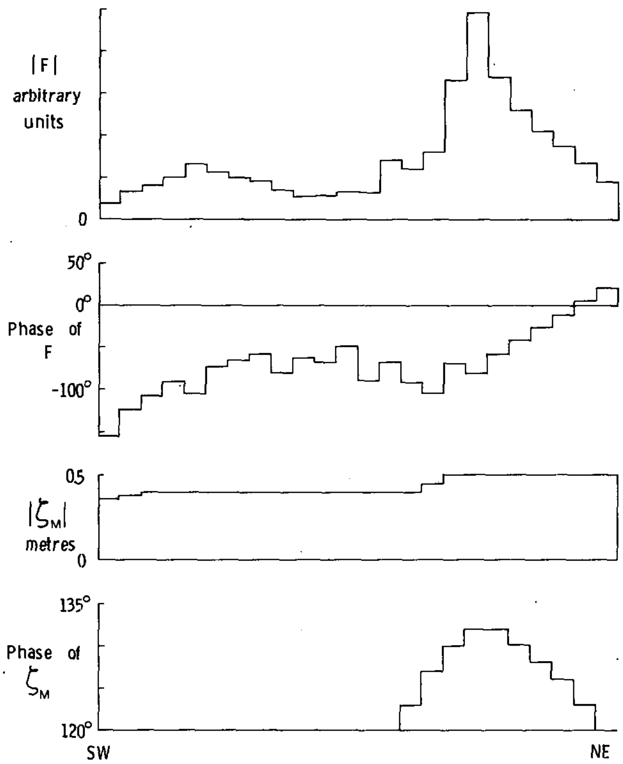

FIG. 3. Prescribed elevation at the open boundary to the Gulf of Maine at the edge of the shelf, and resulting mass flux (outward) across it.

boundary which extend laterally from the coast to the edge of the continental shelf, and create arrays, for the prescribed elevation ζ_{Mp} and resulting normal mass flux F_p, along the open boundary at the edge of the continental shelf. The subscript p runs over 1–24. There are five sections of this open boundary, each of one grid length, that are perpendicular to the general trend of the edge of the shelf. We lump the mass flux across each of these sections with the mass flux across the adjoining grid line that is farthest out toward the Atlantic, so that the fluxes from two elements lumped together go into the same grid square.

The profiles of ζ_{Mp} and F_p for the natural tidal regime are shown in Fig. 3. It should be remembered that ζ_{Mp} is at present chosen to calibrate the model rather than from offshore tidal data. David DeWolfe of Bedford Institute has conducted an offshore tide gaging program in the summer of 1976. Data from this, when available, will be used in a recalibrated numerical model, and in the formulas of this paper.

In evaluating I, V_M and Z_O^{sl} we assume that the grid length Δs is a uniform 22.01 km, ignoring a very slight variation due to the Mercator projection. Other relevant parameters are $f=0.966\times10^{-4}$ s^{-1}, $\omega=1.405\times10^{-4}$ s^{-1}, $g=9.81$ m s^{-2} and we choose $h_O=4\times10^3$ m. Hence from (5.2) $k_O=5.148\times10^{-7}$ m^{-1}, $k_O\Delta s_p=1.133\times10^{-2}$ and $\omega(2gh_O)^{-1}=1.790\times10^{-9}$ m^{-2} s.

With these parameters the matrix K_{pq} is given (from

TABLE 1. Estimates of mass flux, elevation and impedance.

Regime	$I(10^6\ m^3\ s^{-1}, °)$	V_M(m, °)	${}^cV_M$(m, °)	$\omega^{-1}(2gh_O)Z_O^{sl}$	$\omega^{-1}(2gh_O){}^cZ_O^{sl}$
No barriers	24.62, −65.11	0.4563, 128.76	0.4595, 124.92	1.923, −120.84°	2.077, −118.79°
Economy Point*	25.79, −67.28	0.4553, 128.72	0.4591, 124.88	1.922, −120.90°	2.070, −118.88°
Cape Blomidon*	28.71, −63.06	0.4543, 128.43	0.4576, 124.88	1.920, −121.17°	2.047, −119.24°

* Permeable.

Section 5) by

$$K_{pp}=-\omega(2gh_O)^{-1}(1+3.634i), \tag{6.1}$$

$$K_{pq}=-\omega(2gh_O)^{-1}[1-0.688\ \mathrm{sgn}(p-q) +i(2.557-0.637\ \ln|p-q|)], \quad \text{for}\quad p\neq q. \tag{6.2}$$

The impedance estimate

$$Z_O^{sl}=\sum_p\sum_q K_{pq}=\omega(2gh_O)^{-1}(1.791,\ -123.94°)$$

compares very well with the value obtained from (5.4), $Z_O^{sl}=\omega(2gh_O)^{-1}(1.793,\ -123.90°)$, indicating the validity of replacing the continuous integral with a sum.

Estimates of Z_O^{sl} obtained using the actual mass flux distribution F_p in (3.6) are shown in Table 1, which also shows values for the total mass flux I and average elevation V_M from (3.5). Following Miles (1971) we also show the results for V_M and Z_O^{sl} (denoted ${}^cV_M$ and ${}^cZ_O^{sl}$) obtained by using $f^*(s)$ instead of $f(s)$ in a weighted integral of the integral equation (2.5). These are defined by

$${}^cV_M=\sum_p \zeta_{Mp}F_p^*\Delta s_p/I^*, \tag{6.3}$$

$${}^cZ_O^{sl}=\sum_p\sum_q K_{pq}F_pF_q^*\Delta s_p\Delta s_q/(II^*), \tag{6.4}$$

where the asterisk denotes complex conjugate.

All of these quantities are evaluated not only for the natural tidal regime but also with results obtained from the numerical model with a tidal power plant included. The two sites considered here, at Economy Point and Cape Blomidon, are shown in Fig. 4. The word "permeable" refers to the operation of the tidal power plant in a manner similar to that described by Heaps (1972). Runs with an "impermeable" barrier (i.e., a tidal power plant in place but allowing no flow) show a change of about 1° in the phase of I, but negligible changes in the magnitude of I or in the other quantities.

The phases of the mass flux and elevation are in Time Zone 4 and with respect to Greenwich, i.e., the

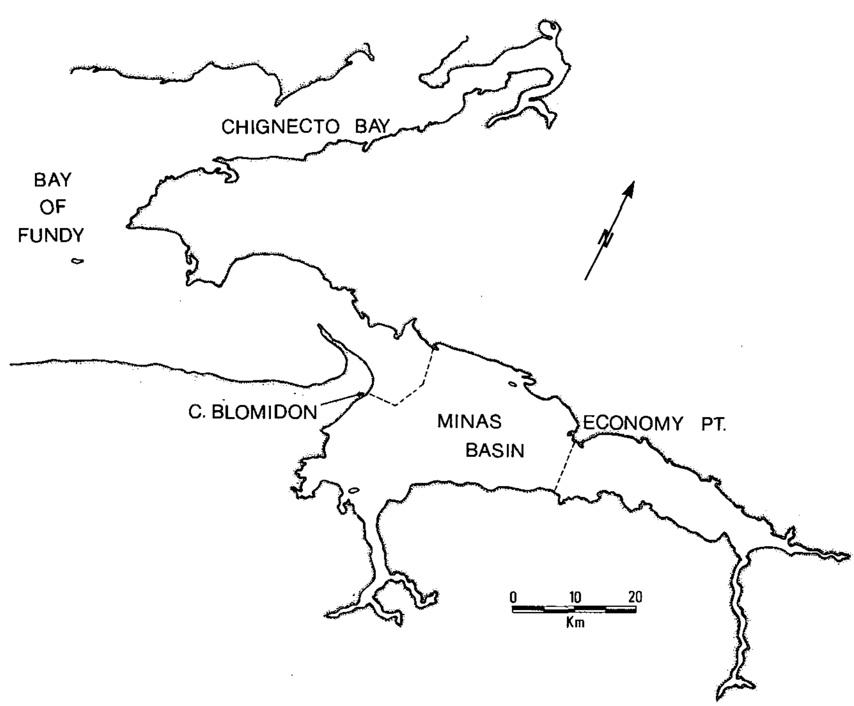

FIG. 4. Proposed sites of tidal power plants at Economy Point and Cape Blomidon.

phase $-65.11°$ of I for the natural regime (no barriers) means that the maximum M_2 mass flux out of the Gulf of Maine lags lunar transit of the Greenwich meridian by 65.11° when the different time zone is allowed for.

We notice that estimates of Z_O^{si} with the appropriate distribution of mass flux do not differ substantially from the value obtained above with a uniform mass flux. From now on we shall take

$$Z_O^{si}=\omega(2gh_O)^{-1}(2.0, -120°) = (3.6\times 10^{-9}\ \mathrm{m^{-2}\ s}, -120°). \quad (6.5)$$

Before calculating the normal mode impedance Z_O^{nm} we first remark that for the natural regime, taking $V_M=(0.46\ \mathrm{m}, 127°)$, we have

$$V_M+Z_O^{si}I=(0.52\ \mathrm{m}, 134°). \quad (6.6)$$

Thus, following the discussion in Section 5c, we might expect the elevation $V_B=V_M+Z_O^{si}I$ on the shelf edge a few hundred kilometers away from the entrance to the Gulf of Maine to be about 0.06 m more than the elevation at the entrance to the Gulf of Maine and have a phase about 7° more. Greenberg's model, in which the open boundary data are adjusted to provide the best fit of the model output to shore data, has M_2 elevation (0.50 m, 120°) at the northeastern end of the open boundary on the edge of the shelf (0.36 m, 120°) at the southwestern end, and an amplitude of 0.4–0.5 m with a phase as high as 132° in the center portion. In other words, Greenberg's model assumes an increase of 12° in the phase lead of the elevation at the center of the open boundary, compared with the prediction of this paper of a 7° decrease. A comparison of different predictions for amplitude is less straightforward due to a general trend along the edge of the shelf.

Preliminary analysis of offshore data (DeWolfe, personal communication) indicates a definite decrease in phase lead at the entrance to the Gulf of Maine, in agreement with the prediction of this paper and contrary to the assumptions of the numerical model. A detailed evaluation of this problem will be presented in due course.

We now evaluate the normal mode impedance Z_O^{nm} from (5.15) using the results of Platzman's (1975) computations of the inviscid normal modes of the combined Atlantic and Indian Oceans. Platzman (1975) finds modes at 9.2, 12.8 and 14.4 h, as well as longer and shorter periods. Moreover, an analysis of admittance curves at Bermuda and Halifax (see the Appendix) suggests that the tides of the western North Atlantic are dominated by a mode with a period of 12.8 h and a Q of about 17, so in calculating Z_O^{nm} we use these values for which $[\omega-\omega_k(1+\frac{1}{2}iQ_k^{-1})]^{-1}=(1.7\times 10^5\ \mathrm{s}, 44°)$ together with Platzman's (1975) computation of E_k. We only include the 12.8 h mode in Z_O^{nm} as the 9.2 and 14.4 h modes change Z_O^{nm} by less than 20%, and the infinite set of higher modes are responsible for the near-source singularity allowed for in Z_O^{si}.

Platzman's (1975) 12.8 h mode has $|\zeta_k|=2.2$ at the entrance to the Gulf of Maine, where ζ_k is normalized to satisfy

$$\int |\zeta_k|^2 dS=S,$$

the total surface area of the oceans. Now this mode has 44.1% of its energy in potential energy, so

$$\langle E_k,E_k\rangle=\int (g|\zeta_k|^2+h|u_k|^2)dS=2.27gS,$$

and with $S=1.56\times 10^{14}\ \mathrm{m^2}$, we finally obtain

$$Z_O^{nm}=(2.3\times 10^{-9}\ \mathrm{m^{-2}\ s}, 134°). \quad (6.7)$$

It is interesting that even with a high Q mode close to resonance, Z_O^{nm} is still less than Z_O^{si}. With perfect resonance ($\omega=\omega_k$) and a doubling of Q, both of which are conceivable but unlikely, Z_O^{nm} increases by a factor of nearly 3.

From (6.5) and (6.7) we now obtain

$$Z_O=Z_O^{si}+Z_O^{nm}=(3.7\times 10^{-9}\ \mathrm{m^{-2}\ s}, -157°).$$

With $I=(24.6\times 10^6\ \mathrm{m^3\ s^{-1}}, -65°)$ and $V_M=(0.46\ \mathrm{m}, 127°)$ we have

$$R=Z_O(I/V_M)=(0.20, 11°) \quad (6.9)$$

$$R(1+R)^{-1}=(0.17, 9°). \quad (6.10)$$

We can now calculate the correction factor A from (4.3) for Greenberg's (1975) predictions for changes in tidal regime due to tidal power plants at Economy Point and Cape Blomidon. The results are shown in Table 2. From the final column we emerge with the result that Greenberg's (1975) predictions for M_2 should be decreased by 0.9% for a tidal power plant at Economy Point and decreased by 2.6% for a plant at Cape Blomidon. There are also small phase changes.

TABLE 2. Correction factors for tidal regime computations.

Site	$1+B$	B	$BR(1+R)^{-1}$	A
Economy Point*	(1.048, −2.17°)	(0.062, −40°)	(0.011, −31°)	(0.991, 0.3°)
Cape Blomidon*	(1.166, 2.05°)	(0.17, 14°)	(0.029, 23°)	(0.974, −0.6°)

* Permeable.

Before taking these numbers too seriously, we should draw attention to some of the uncertainties:

1) Our final result depends critically not only on the rather uncertain magnitude of Z_O, but also on its phase. As an extreme case, if $R=(0.20, -180°)$, i.e., Z_O retains the same magnitude but has its phase reduced by 75°, then $R(1+R)^{-1}=(0.25, -180°)$ and, if the phase of B is small, Eq. (4.4) shows that predictions for M_2 must be increased by 25% of B, which is about 4% for a Cape Blomidon plant.

A more realistic assessment of the sensitivity of our results can perhaps be obtained by evaluating A for some different values of Z_O. If, for example, the world's oceans have a normal mode of exactly M_2 period with $Q=34$, double the value assumed earlier, we obtain $Z_O^{nm}=(6.6\times10^{-9}\ \mathrm{m^{-2}\ s}, 180°)$ and hence $R(1+R)^{-1}=(0.33, 5°)$. This differs substantially from the value in (6.10), and leads to $A=(0.983, 0.7°)$ for an Economy Point plant and $A=(0.949, -1.0°)$ for one at Cape Blomidon. This reduction of 5% for a Cape Blomidon tidal power plant is certainly significant, though based on an extreme assumption for Z_O^{nm}.

Further cases may be evaluated as desired, and as our knowledge of appropriate values for Z_O^{sl} and Z_O^{nm} is refined by further work.

2) Quite apart from problems associated with our present neglect of the two sections of the open boundary from the coastline out to the edge of the continental shelf, which will be discussed in Section 8, we may be erring in assuming a uniform correction factor over the open boundary at the edge of the shelf, and hence a uniform multiplier A for the whole solution. It is possible that the correction to Greenberg's (1975) predictions, while having an average A as calculated here, may be distributed nonuniformly over the Bay of Fundy and Gulf of Maine. However, we do not expect this to be a significant effect and, of course, it is negligible if A itself is not significantly different from 1.

3) In computing the correction factor A, we have assumed the system to be linear. This is certainly inadequate, given the importance of nonlinear bottom friction in determining the tidal response of the Bay of Fundy and Gulf of Maine (Garrett, 1972). Greenberg's (1975) model allows for nonlinear bottom friction, so one could converge on a more accurate correction using an iterative approach. Having calculated A from the present linear theory, we could rerun the numerical model with AV_M as the input, obtain a mass flux different from $A(1+B)I$ because of nonlinear friction, calculate a new value of A from the present theory, and so on. This is hardly worth the effort, given our uncertainties in Z_O and hence A, and it is probably adequate to point out that the effect of nonlinear bottom friction is to moderate any departure of $|A|$ from 1 (Garrett, 1972). This shows up in the response of the present system to the 18.61-year period nodal variation. The astronomical forcing for M_2 varies $\pm3.7\%$, but the response at Saint John (Doodson, 1924) and Bar Harbor and Boston (Garrett, in preparation) is only $\pm2.5\%$, a 30% reduction in the modulation. In fact, as the system approaches resonance, this reduction of any modulation calculated from linear theory approaches 50% (Garrett, 1972).

In summary, $|A|-1$, with A calculated from the present theory, must be reduced by at least 30%, and possibly up to 50%, because of the influence of nonlinear bottom friction on the response of the Bay of Fundy and Gulf of Maine. While the value of A shown in Table 2, corrected as described for the effect of nonlinear friction, is our best guess at present on the correction to Greenberg's (1975) prediction, and is probably negligible, we believe that a more appropriate and conservative interpretation of our calculations is that Greenberg's (1975) predictions are uncertain to about $\pm25\%$ of the predicted change in total mass flux across the edge of the shelf. This amounts to an uncertainty of about 1% for a tidal power plant at Economy Point, but up to 4% for a plant at Cape Blomidon.

7. Effect of tidal power development on global tides

The tidal mass flux I out of the Gulf of Maine excites Platzman's (1975) 12.8 h mode with an amplitude given by $Z_O^{nm}I/\zeta_k$, where the elevation ζ_k of the mode is evaluated on M. The magnitude of this with Z_O^{nm} from (6.7), I from Table 1 and $\zeta_k=2.2$ is 0.03 m. This is then the rms contribution to the elevation, as the mode is normalized to have

$$\int|\zeta_k|^2dS=S.$$

The change in this due to a tidal power plant in the Bay of Fundy is $|B|$ times this, i.e., 2 mm for an Economy Point plant and 5 mm for a Cape Blomidon plant. The maximum effect anywhere in the Atlantic ocean may be 3 or 4 times this rms value and perhaps be detectable, but will certainly not be significant.

8. The shelf problem

We now return to a difficulty of the above theory which has so far been disregarded. Fig. 2 shows that, in our particular application, the boundary M between G and O has two sections on the continental shelf, where the water depth h_G is much less than the value of h_O used in estimating Z_O^{sl}. On these sections the elevation produced by a given mass flux is much greater than it would be at the edge of the shelf, approximately in the ratio h_O/h_G ($\sim$40 in our case) but almost entirely confined to the shelf itself.

We thus contemplate a very simple model in which the parts of M on the continental shelf are lumped together in $0\leqslant s\leqslant\epsilon$, while $\epsilon<s\leqslant1$ represents the part of M along the edge of the continental shelf. Suppose

that a unit mass flux across any part of M on the shelf produces a local elevation only, of magnitude K_2, while a unit mass flux across the boundary at the edge of the shelf produces an elevation K_1 everywhere. If ϵ has already been chosen to allow for the partition of $f(s)$, the double integral of (3.3) gives

$$Z_O = (1-\epsilon)K_1 + \epsilon^2 K_2, \tag{8.1}$$

so that even if K_2 is large compared with K_1 it does not greatly effect the value of Z_O if ϵ is small.

In the case of the Bay of Fundy and Gulf of Maine, the M2 mass fluxes I_{SW} and I_{NE} outward across the southwest and northeast boundaries are given by

$$\left.\begin{aligned} I_{SW} &= (0.92\times10^6\ \mathrm{m^3\ s^{-1}},\ -80°) \\ I_{NE} &= (1.21\times10^6\ \mathrm{m^3\ s^{-1}},\ 81°) \end{aligned}\right\}, \tag{8.2}$$

both of which are small compared with the mass flux $I = (24.6\times10^6\ \mathrm{m^3\ s^{-1}},\ -65°)$ across the boundary at the edge of the shelf. Moreover, I_{SW} and I_{NE} are almost exactly out of phase, so that $|I_{SW}+I_{NE}|/|I| = 0.019$, and a value of $\epsilon = 0.02$ seems appropriate. Z_O is then negligibly affected by the side boundaries even if $K_2/K_1 \approx 40$.

This argument is far from satisfactory, and further work is certainly required, particularly for those situations in which a substantial part of the mass flux into a coastal embayment comes across an open boundary on the shelf rather than the edge of the shelf. One factor we believe to be relevant is that most continental shelves are too narrow to support trapped waves at the M_2 frequency, other than a slightly modified deep-ocean Kelvin wave, which has a deep-ocean impedance associated with it (Munk *et al.*, 1970). Thus any disturbance on the shelf, other than the Kelvin wave, is trapped near the source, so that as the lateral boundaries of a Greenberg-type model are moved along the shelf, the predictions for a changed tidal regime should become insensitive to further movement of the lateral boundaries.

9. Conclusions

In developing a numerical model of tides in coastal waters for prediction of changes in tidal regime that might be brought about by man-made structures, it is important that the open boundary of the model be taken at some point where the impedance of the exterior ocean is small. This probably requires that the model extend to the edge of the continental shelf. An estimate of the validity of imposing an unchanged tidal elevation there, and calculation of a correction factor (see Section 4) can be made using output from the model together with estimates of the impedance of the exterior ocean (see Section 5).

Problems remain in understanding the role of open boundaries extending from the shore to the edge of the shelf. In the present application to the Bay of Fundy and Gulf of Maine there is little mass flux across these boundaries and it is probably reasonable to ignore them.

Greenberg (1975) predicted increases of M_2 mass flux into the Gulf of Maine of about 5% for a tidal power plant at Economy Point, and 17% for one at Cape Blomidon, though the associated increases in M_2 elevation were distributed in a non-uniform way. Our current best estimate of corrections to these predictions amounts to a decrease of less than 1% for a power plant at Economy Point and a decrease of nearly 2% for a plant at Cape Blomidon (so that the net increases in mass flux now become about 4% and 15% for the two projects). However, uncertainty in the value of the deep-ocean impedance leads us to admit that Greenberg's (1975) predictions, and subsequent predictions with the same model, may be incorrect to about ±25% of the change in mass flux across the open boundary. This amounts to an uncertainty of about ±1% in the predicted tidal regime with an Economy Point plant and ±4% for a Cape Blomidon plant.

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APPENDIX

The 12.8 h Mode of the North Atlantic

Our estimate of Z_O^{nm} in Section 6 depends critically on our assumption that the world oceans have a normal mode, dominant in the western North Atlantic, with a period of about 12.8 h. In this Appendix we provide some evidence (from tidal admittances at Bermuda and Halifax) for the importance of this mode and estimate its Q to be about 17.

Garrett and Munk (1971) attempted an interpretation of "the age of the tide" or difference in phase lag between S_2 and M_2 in terms of near-resonant response of high Q normal modes, and Garrett (1972) argued that if one mode is dominant, its frequency and Q may be estimated from the complex ratio of the tidal admittance at two different frequencies within the same band. If $A(\omega)$ represents the admittance to tidal forcing at frequency ω, and this is dominated by the contribution from a mode with natural frequency ω_O and a particular Q, then ω_O and Q may be derived from

$$A(\omega_1)/A(\omega_2) = (1-\omega_2/\omega_O + \tfrac{1}{2}iQ^{-1}) / (1-\omega_1/\omega_O + \tfrac{1}{2}iQ^{-1}) \tag{A1}$$

if the admittances at ω_1 and ω_2 are known.

Zetler *et al.* (1975) have derived the admittances for Bermuda, using the response method and separating the gravitational and radiational parts of the admittance [see also Fig. 9 of Zetler and Munk (1975)]. We shall use just the gravitational admittance. The amplitude and phase of this change rapidly but smoothly across the semi-diurnal band, so in solving (A1) we use the admittances for the furthest separated dominant tidal lines, N_2 and K_2. At Bermuda N_2 has an admittance amplitude 2.34 times larger than that of K_2, and the phase lag of K_2 is 44.2° larger than that of N_2. Thus the left-hand side of (A1) is (2.34, 44.2°) and with $\omega_1=28.44°\ h^{-1}$ and $\omega_2=30.08°\ h^{-1}$, we obtain $\omega_O=28.08°\ h^{-1}$ (corresponding to a period of 12.82 h) and a Q of 16.4.

A full response method analysis separating gravitational and radiational effects has not been done for Halifax, but N_2 is not affected by radiation and K_2 only weakly. Thus we use the harmonic constants obtained by least-squares analysis and obtain $A(\omega_1)/A(\omega_2)=(2.40,\ 47.3°)$. This leads to a resonant period of 12.79 h and a Q of 17.1, much as at Bermuda.

Of course, it is somewhat naïve to think of the response of the North Atlantic in terms of one mode only, and indeed different admittance ratios at different ports do imply the importance of several modes. Nonetheless, it is intriguing that if the data at Bermuda and Halifax are interpreted in terms of a single mode it has the same period as that computed by Platzman (1975). In the absence of more reliable indicators, we thus seize on this 12.8 h mode as real, and assign it a Q of 17.

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