On the Dynamics of the Ocean Surface Mixed Layer

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ABSTRACT

This paper describes a theoretical model of the ocean surface mixed layer. A nonstationary one-dimensional system of equations for an Ekman ocean layer and for heat conduction is closed with a system of dynamic turbulence equations. The latter consists of the turbulent energy equation and an equation for the turbulent energy decay function. The eddy viscosity coefficient is then determined from these equations. The values of the surface and lower boundary temperature and of wind velocity as a function of time are taken from Halpern (1974) data. The computational results show that an increase of wind velocity produces deepening of the mixed layer with some time lag. In the region of the jump in density at the bottom of the layer, the increase in the surface wind generates large velocity gradients across an inner boundary layer. Comparison of the solution with experimental data shows that the model realistically simulates the dynamics of mixed layer deepening both qualitatively and quantitatively.

1. Introduction

Statistical values of oceanic turbulence are determined by its generation and decay mechanisms. Turbulent generation involves the following mechanisms (Monin, 1969, 1973): 1) instability of vertical and horizontal velocity gradients on various scales, 2) buoyancy forces arising from the vertical inhomogeneity of the medium in the earth's gravity field, 3) destruction of surface waves and 4) destruction of internal waves. A turbulent surface layer is formed by these sources. As a result, in the large-scale background of the homogeneous layer of the order of 10 depth, there may develop a step microstructure of the order of decimeters or less (Ozmidov and Belyaev, 1975). The decay mechanisms are the transition of turbulent energy into enthalpy due to molecular viscosity and the transition of turbulent energy into potential energy due to work against buoyancy forces.

The sharply layered structure of the density profile in the upper ocean makes it possible to use integral methods for the parameterization of thermodynamic processes within the quasi-homogeneous layer (e.g., Turner and Kraus, 1967; Kraus and Turner, 1967; Kitaigorodsky and Miropolsky, 1970). With comparatively simple equations, these integral methods allow solutions which can be used in global general circulation models. In the integral methods one must know the vertical distribution of the hydrothermodynamic variables. For special cases, this distribution is determined on the basis of similarity theory as suggested by Monin and Obukhov (1954) and Kazansky and Monin (1960) for the atmospheric surface layer and for the planetary boundary layer. Integral theories do not predict the existence of a mixed layer as a consequence of ocean boundary and initial conditions; its existence must be assumed *a priori*.

In the differential approach the precise patterns of the vertical distribution of hydrothermodynamic and turbulent variables are obtained as a consequence of the solution of a closed system of turbulence and hydrodynamic equations. These can be used to construct integral theories, and, in similarity theory, to establish the form of the universal functions.

Interest in the method of mathematical simulation of turbulence has increased greatly since the evolution of powerful computers and the development of numerical methods for the solutions of nonlinear partial differential equations. In constructing a statistical theory of turbulence, the dynamic equations for correlation functions are derived by averaging the Navier-Stokes equation. For closure of the correlation equations, it is necessary to form some additional hypotheses. In recent years a number of papers by the Los Alamos group (Harlow and Nakayama, 1968; Harlow and Hirt, 1969; Hirt, 1969; Daly and Harlow, 1970; Nakayama, 1970) have closed the system of correlation equations by means of turbulence decay equations. The idea of such closure was suggested earlier by Chou (1945) and Davydov (1958, 1961).

Kochergin *et al.* (1974) employed such a system of dynamic turbulence equations, consisting of the turbulent kinetic energy and turbulent decay equations. These equations account for the buoyancy forces and for stress instability. The effect of surface wave destruction can be taken into account by the surface boundary conditions of the turbulence equations.

The dynamic turbulence equations suggested by Harlow and Nakayama (1968) were considerably simplified and empirical constants were optimized by Kochergin and Sukhorukov (1975). The structure of the turbulent model (Kochergin and Sukhorukov, 1975) is similar to the equations of Jones and Launder (1972) except for the empirical constants. Results of the solution of some oceanic problems involving these equations are presented by Klimok and Sukhorukov (1974), Kochergin *et al.* (1976) and Marchuk *et al.* (1976). In these papers the turbulent equations are solved with a horizontally homogeneous boundary layer approximation. The results of these papers show that the turbulent model reproduces a three-layer structure of the ocean, *viz.*, surface and bottom turbulent layers, and an intermediate quasi-laminar layer with constant transport coefficients (Klimok and Sukhorukov, 1974). The results also realistically estimate the basic values of microscale turbulence in the ocean surface mixed layer (Kochergin *et al.*, 1976; Marchuk *et al.*, 1976).

In their recent paper Mellor and Durbin (1975) studied the quasi-homogeneous layer dynamics with a simpler turbulence model. The stationary turbulent kinetic energy equation, without the diffusion term, is closed with a simple relation for the turbulence scale. The advantage of this model is in the simplicity of its numerical realization. On the other hand, application of the turbulent kinetic energy and turbulent decay equations gives more extensive information of the processes and makes it possible to detail the solution in the zone of the temperature jump.

In the present paper the turbulent model (Kochergin *et al.*, 1976) is applied to simulate the response of the ocean to the onset of a storm. The wind stress and temperature values on the ocean surface, and the temperature values at the depth of 30 m are taken from Halpern (1974). Observations by Halpern (1974) and Anisimova and Speranskaya (1975) under the ice of the Lake Baikal, and during laboratory experiments by Ivasaki and Abe (1975) reveal the existence of an internal boundary layer near the bottom of the mixed layer. In order to resolve this boundary layer, calculations were made with very fine spatial resolution, $\Delta z = 3$ cm. An analysis of results confirms the existence of this internal boundary layer at the same level as the jump in density. Properties of this layer resemble the sublayer near a rigid wall.

2. Statement of the problem

In studying the dynamics of the nonstationary evolution of turbulence in a horizontally uniform Ekman boundary layer, we will solve the following system of equations (Kochergin *et al.*, 1976):

$$\frac{\partial u}{\partial t} - fv = \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = \frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right), \tag{2}$$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(K_T\frac{\partial T}{\partial z}\right), \tag{3}$$

$$\frac{\partial b}{\partial t} = K\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] + \frac{\partial}{\partial z}\left(K\frac{\partial b}{\partial z}\right) - 2\nu D + g\alpha K_T\frac{\partial T}{\partial z}, \tag{4}$$

$$\frac{\partial D}{\partial t} = 1.38\frac{D}{b}K\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] + \frac{\partial}{\partial z}\left(K\frac{\partial D}{\partial z}\right) - 2\nu\beta\frac{D^2}{b} + g\beta\frac{D}{b}\alpha K_T\frac{\partial T}{\partial z}, \tag{5}$$

$$K = 0.08\frac{b^2}{2\nu D}, \quad \beta = 1.4. \tag{6}$$

The basis for (4), (5) and (6) is presented in the Appendix. The boundary and initial conditions are as follows:

$$z=0\begin{cases} K\dfrac{\partial u}{\partial z} = -\dfrac{\tau_x}{\rho_0}, \quad K\dfrac{\partial v}{\partial z} = -\dfrac{\tau_y}{\rho_0}, \quad T=T & (7)\\[2ex] K\dfrac{\partial b}{\partial z} = 0, \quad K\dfrac{\partial D}{\partial z} = 0. & (8)\end{cases}$$

$$z=H\begin{cases} u=v=0, \quad T=T_H \\ b=D=0.\end{cases} \tag{9}$$

$$t=0\colon\ K=K^0, b=b^0, D=D^0, u=u^0, v=v^0, \quad T=T^0. \tag{10}$$

In the above equations t is time; z the vertical (positive *downward*) space coordinate; u, v the mean velocity components; f the Coriolis parameter; T the mean temperature; ν the molecular kinematic viscosity; g the gravitational acceleration; b the turbulent kinetic energy per unit mass; $2\nu D$ the rate of turbulent decay; α the thermal expansion coefficient; and ρ_0 the mean density. The turbulent kinematic viscosity coefficient K is defined from the solution of the dynamic turbulence equations (4), (5) and (6). The relationship between the coefficient of turbulent viscosity K and the turbulent heat conductivity K_T is defined by means of the Munk-Anderson formula (1948), i.e.,

$$K_T=K\left[\frac{1+10\ \mathrm{Ri}}{[1+(10/3)\ \mathrm{Ri}]^3}\right]^{\frac{1}{2}}, \tag{11}$$

where $\mathrm{Ri}=-(g\alpha\partial T/\partial z)/[(\partial u/\partial z)^2+(\partial v/\partial z)^2]$ is the Richardson number.

Note that in the asymptotic case of a very large Rayleigh number the stratification terms in (4) and (5) are balanced by the decay terms. It follows that the empirical coefficients β are the same in the stratification and the decay terms of (5). This condition is not violated by any relation between the turbulent transport coefficients K and K_T.

The wind stresses τ_x and τ_y are defined as follows:

$$\frac{\tau_x}{\rho_a}=u_\tau^2=C_{-z}W_{-z}^2,\quad \tau_y=0, \tag{12}$$

where ρ_a is the density of air (1.25×10^{-3} g cm^{-3}); u_τ the friction velocity; and C_{-z},W_{-z} are the drag coefficient and the mean wind speed at the $(-z)$ level, respectively. The final relation, assuming the logarithmic wind profile in the constant stress layer and formula (12), is (Halpern, 1974)

$$\tau_x=1.625\times10^{-6}(1.17\ W_{-z})^2. \tag{13}$$

The Halpern (1974) data show that the wind speed (m s^{-1}) during the storm can be approximated by the linear function of time

$$W_{-z}=\begin{cases}4+(10/36)t, & 0\leqslant t\leqslant 36\ \mathrm{h}\\ 14-(9/24)(t-36), & 36\ \mathrm{h}\leqslant t\leqslant 60\ \mathrm{h}\\ 5, & t\geqslant 60\ \mathrm{h}.\end{cases}$$

A stationary solution of (1)–(6) with the boundary conditions (7) and (8) and with the wind speed equal to 4 m s^{-1} is used as an initial state. This initial state is obtained numerically with time steps in (1)–(3) much larger than the inertial period ($\Delta t>2\pi/f$). As a result, the inertial oscillations have been filtered out and a stable state of the mixed layer is obtained.

The Halpern observations show that the storm produced an increase in the depth of the homogeneous layer from 18 to 25 m. In the present calculations the lower boundary H was taken equal to 30 m. This is justified by the fact that the Ekman spiral abruptly dies down below the base of the jump in density at the bottom of the mixed layer. Therefore, the boundary condition $u=v=0$ is correct provided $H=30$ m. In our model turbulence will be absent below the homogeneous layer; therefore, a minimum value of the turbulent transport coefficient is required, i.e., $K\geqslant K_0$. The value of K_0 is set equal to 0.1 cm^2 s^{-1} as in Kochergin *et al.* (1976).[1] This value determines the average turbulent transport in the layer of the seasonal thermocline.

[1] A similar constraint is used by Mellor and Durbin (1975) with $K_0\approx0.1$ cm^2 s^{-1}.

3. Method of solution

We now discuss the method of solution. Vector functions $\mathbf{u}$, $\boldsymbol{\tau}$, $\mathbf{u}^0$ and a matrix $\mathbf{G}$ are defined as follows:

$$\mathbf{u}=\begin{bmatrix}u\\ v\end{bmatrix};\quad \boldsymbol{\tau}=\frac{1}{\rho_0}\begin{bmatrix}\tau_x\\ \tau_y\end{bmatrix};\quad \mathbf{u}^0=\begin{bmatrix}u^0\\ v^0\end{bmatrix};\quad \mathbf{G}=\begin{bmatrix}1 & f\Delta t\\ -f\Delta t & 1\end{bmatrix}.$$

We write the equations of motion (1) and (2) on the time interval $k\Delta t\leqslant t\leqslant(k+1)\Delta t$ in a matrix form, using an implicit approximation in time, i.e.,

$$\mathbf{G}\mathbf{u}-\frac{\partial}{\partial z}\left(\Delta tK\frac{\partial\mathbf{u}}{\partial z}\right)=\mathbf{u}^0,\quad \mathbf{u}^0=u(k\Delta t), \tag{14}$$

with the boundary conditions

$$\begin{aligned} z=0:&\quad K\frac{\partial\mathbf{u}}{\partial z}=-\boldsymbol{\tau}\\ z=H:&\quad \mathbf{u}=0.\end{aligned} \tag{15}$$

Eq. (14) is approximated in the space coordinate using the balance method (Marchuk, 1958). The resulting system of algebraic equations is solved using matrix factorization.

The turbulence equations are approximated in time as follows:

$$\frac{D-D^0}{\Delta t}=1.38\frac{D^0}{b^0}K^0\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\left(+\frac{\partial}{\partial z}\right)K^0\frac{\partial D}{\partial z}\right] -2\nu\beta\frac{D^0}{b^0}D+g\beta\frac{D^0}{b^0}\alpha K_T\frac{\partial T}{\partial z}, \tag{16}$$

$$\frac{b-b^0}{\Delta t}=K^0\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right] +\frac{\partial}{\partial z}\left(K^0\frac{\partial b}{\partial z}\right)-2\nu D+g\alpha K_T\frac{\partial T}{\partial z}. \tag{17}$$

Eqs. (16) and (17) are also approximated by finite-difference equations in the space coordinate z using the balance method. The finite-difference equations are of second-order accuracy in space. The resulting algebraic equations are solved by ordinary factorization giving solutions which are absolutely stable (Kochergin *et al.*, 1974) for every finite-difference equation. However, the iterative method of solution of the equations can be unstable if the time steps in the turbulence equations (16) and (17) are greater than some critical value. The characteristic time scale for turbulent vortices is estimated on the basis of the physical assumptions that $t_V \approx L/b^{\frac{1}{2}}$, where L is the scale of turbulent vortices determined by the well-known law (Rotta, 1951) $L=0.2b^{\frac{3}{2}}(2\nu D)^{-1}$. To avoid filtering of physical processes with time scales of order t_V, the time step Δt must satisfy the condition $\Delta t/t_V \ll 1$, i.e., $(2\nu D/0.2b)\Delta t \ll 1$. Many numerical experiments have shown that the above estimate can be a criterion of stability for the iterative method. According to this, for oceanographic problems $\Delta t \ll 10^3$ s.

The complete system of equations is solved successively at each time step. We performed the calculations with a BESM-6 computer: 10^3 mesh points were taken in the space coordinate, the time step Δt was set to be 2 min and $f=1.066\times10^{-4}$ s^{-1} (corresponding to ~latitude 47°). During computations the solution was checked by means of balance of the terms in the turbulence kinetic energy equations in each z_Kth point. The largest error in balance was observed at the temperature jump.

In the region of the jump in density, the restrictions which were required in the computations are important: $b \geqslant 10^{-4}$ cm^2 s^{-2}, $D \geqslant 10^{-6}$ s^{-2}, $K \geqslant 0.1$ cm^2 s^{-1}. The restriction of the turbulent viscosity coefficient becomes minimal at the lower boundary of the homogeneous layer. At this point the temperature profile has an abrupt inflection. The Richardson number was made equal to zero below this point, and, according to the Munk-Anderson (1948) formula $K=K_T=0.1$ cm^2 s^{-1}, below the homogeneous layer. The turbulence values b and D were taken equal to their restrictions somewhat deeper than the lower boundary of the jump in density.

4. Results

The deepening of the homogeneous layer is shown in Fig. 1. The sequence of numbers 1, 2, 3, ... characterizes the solution every 6 h. Zero denotes the stationary solution for $t=0$. The density jump began to form more than 6 h after the sudden increase of the wind. Solution number 6 corresponds to the time when the wind speed reached its maximum of 14 m s^{-1}; at the time of number 10 the wind speed had dropped again to 5 m s^{-1}. The homogeneous layer continued to deepen for about 24 h after a maximum wind speed had been reached.

During 60 h the depth of the mixed layer increased from 19 to 25.5 m. Further calculations for the constant wind speed of 5 m s^{-1} do not result in further deepening of the mixed layer. The Halpern (1974) data show that the storm deepened the mixed layer from 18 to 25 m. For these data an approximate 6 h time lag of mixed layer deepening with respect to the onset of the storm has also been established.

Halpern noted that the onset of the storm generated a large current shear across the transition zone at the bottom of the layer. Evolution of the current

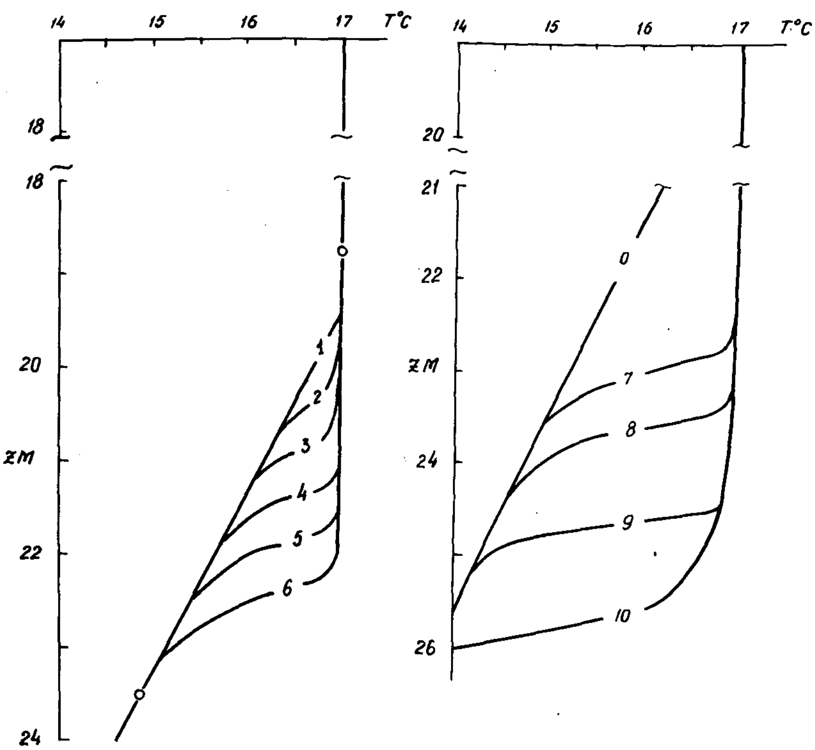

FIG. 1. Evolution of temperature profiles for successive 6 h intervals.

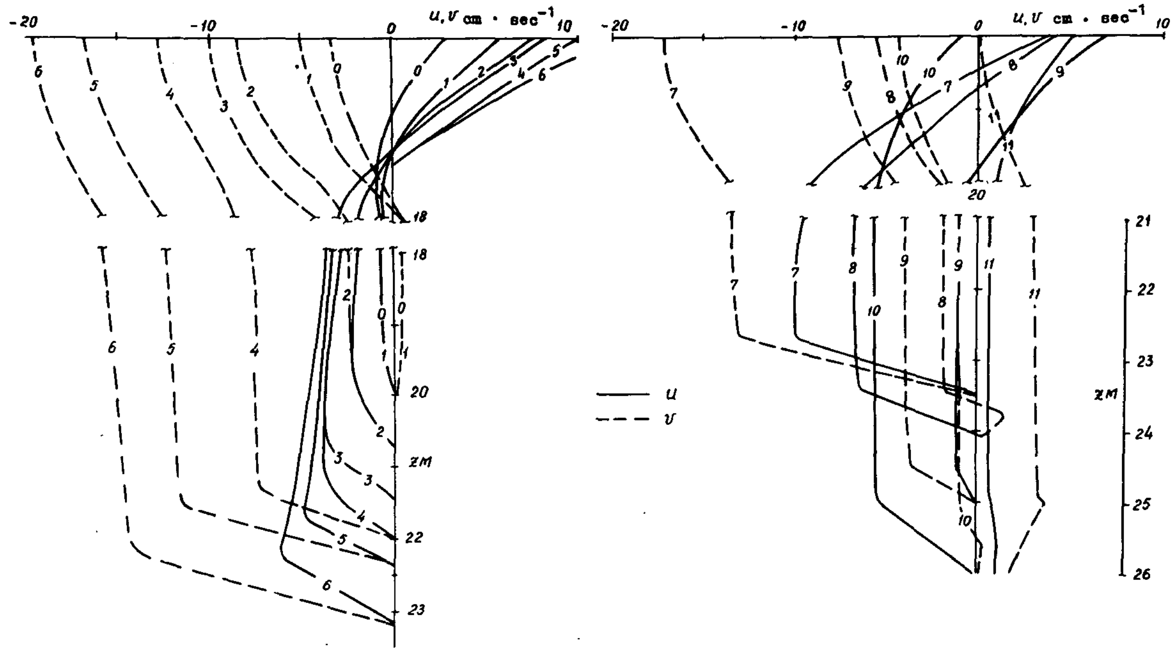

Fig. 2. Evolution of velocity components for successive 6 h intervals (full curves u, dashed curves v).

(Fig. 2) is in agreement with these observational data. The onset of the storm caused the appearance of large velocity gradients across the transition zone with some lag in time. In the transition zone the velocity component profiles have a linear form. This resembles the region near the wall of a viscous sublayer where the velocity profile also has a linear form. In the model calculations, the magnitude of the current was about 300% larger after the onset of the storm than before the arrival of the storm. This is also in agreement with the observational data.

The storm generated a local maximum of the decay function in the transition zone (Fig. 3). This maximum increased with the intensity of the wind speed.

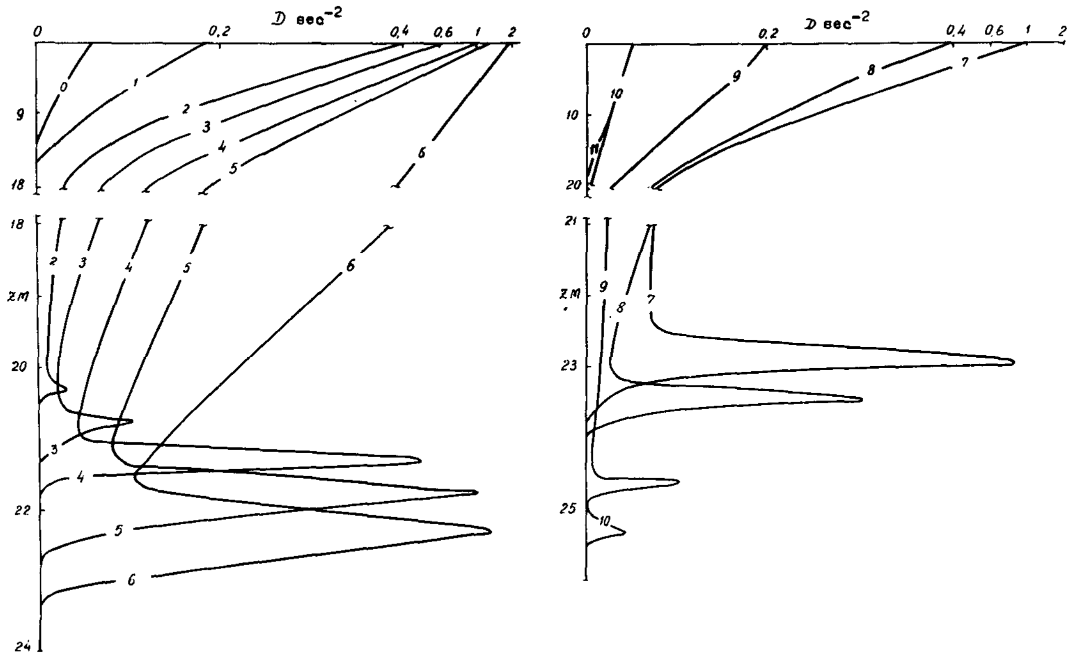

Fig. 3. Evolution of turbulent energy decay profiles of z for successive 6 h intervals.

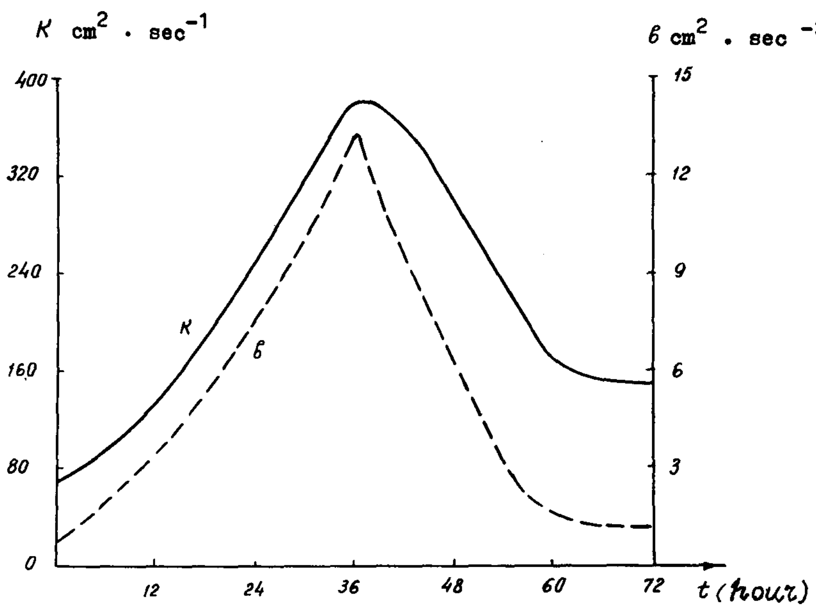

FIG. 4. Time variation of the eddy viscosity coefficient K and turbulent kinetic energy b per unit mass on the ocean surface.

The increase lags in time after the wind, but is in phase with the deepening of the jump in density. After the passage of the storm the local maximum of the decay function disappears. Local extrema in the turbulent energy profile do not arise. The occurrence of the local maximum turbulent decay ($2\nu D$) in the transition zone, where the transport coefficients K and K_T are constant, is due to the response of this turbulence variable to the local growth of the current shear. As noted above in Kochergin *et al.* (1976), the decay function is sensitive to the hydrodynamic background. During the storm its maximum value changed by about two orders of magnitude; therefore, the abscissa in Fig. 3 has a nonuniform gradation from 0.4 to 0.2 s^{-2}.

The evolution of the turbulent viscosity coefficient and the turbulent energy on the ocean surface are shown in Fig. 4. These values decrease monotonically with increasing depth. Estimates of K given by Halpern (1974) are in agreement with the values obtained in our model.

Halpern's measurements reveal a decrease of the Richardson number in the mixed layer, especially so beneath the jump in density (in the transition zone), where large current shear was fixed. Fig. 5 shows the evolution of the Richardson number. It can be seen that with the increase of the wind, Ri decreases throughout the mixed layer. The Richardson number is largest slightly above the jump in density, at the level where the current shear has a minimum. As the wind speed decreases, the value of Ri in the mixed layer increases; e.g., by the time when the wind speeds are 5 m s^{-1}, Ri$=$50. The critical value of the dynamic Richardson number Rf$=(K_T/K)$Ri at the lower boundary of the mixed layer is not constant in view of the fact that this boundary was determined by the $K \geqslant K_0$ condition. At the beginning of the storm, the dynamic Richardson number was $\sim$0.18; at the culmination it reached 0.05 and then it began to grow again.

Evolution of the density flux is shown in Fig. 6. The increase of the wind speed causes an increase of the density flux on the ocean surface and at the jump

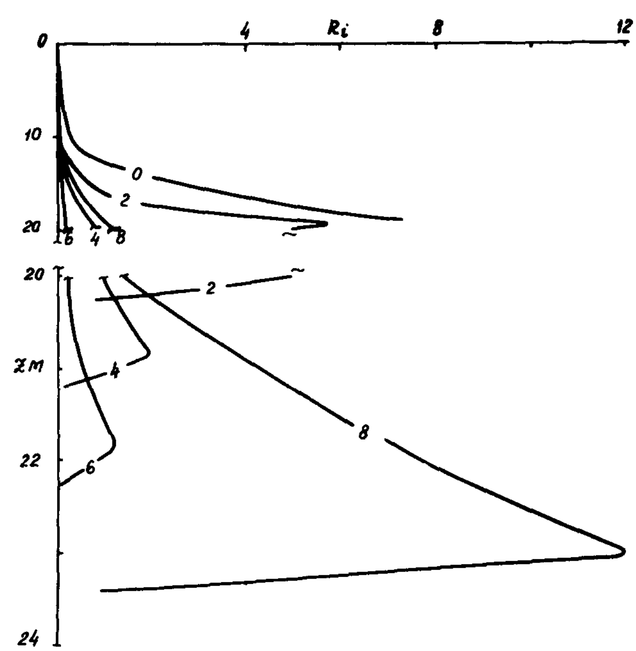

FIG. 5. Evolution of Richardson number profiles for successive 6 h intervals.

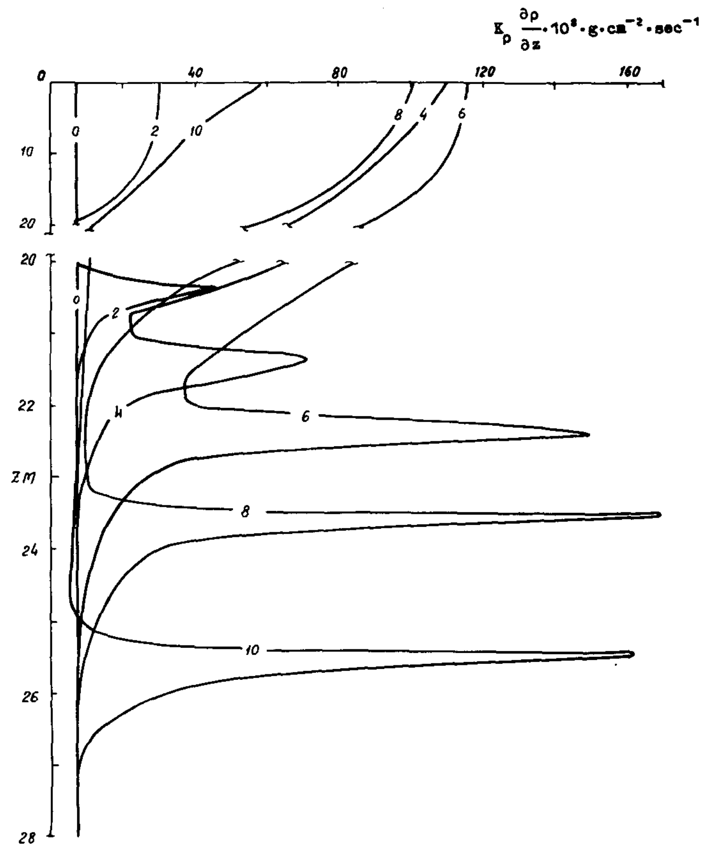

FIG. 6. Evolution of turbulent density flux profiles for successive 6 h intervals.

in density. The decrease of the wind speed results in the decrease of the density flux on the ocean surface and at the jump in density. According to Halpern (1974), at the surface the condition $K_T\partial T/\partial z=0$ was satisfied. We also made calculations with this boundary condition and found that the surface temperature lowered less than 1°C. Both calculations yielded practically the same values for all variables (except for the density flux shown in Fig. 6). This is natural because the stratification term in the turbulence equations is not essential during a storm, which is in agreement with Halpern (1974). According to Halpern (1974), 0.3–0.4% of the kinetic energy input of the wind field to the water was available for vertical mixing.

5. Summary and conclusion

The solutions discussed here show that the turbulence model simulates rather well the basic properties of the deepening of the ocean surface mixed layer both qualitatively and quantitatively within the framework of shear motions produced by winds. The growing wind speeds on the ocean surface produce a distinct internal boundary layer in the zone of the jump in density. Components of the currents have a linear form in this zone. Their gradients intensify as the wind speed decreases. This creates favorable conditions for the development of turbulence. Turbulent decay has a local maximum in this transition zone. Its value is then comparable to that on the ocean surface. This circumstance indicates the analogy with the viscous sublayer near the wall, where the turbulent decay also takes maximum values (Laufer, 1954). The turbulent energy and turbulent decay equations confirm this peculiarity in the viscous sublayer near the rigid wall (Sukhorukov, 1974).

Halpern (1974) presumes that "the mixing was produced by small-amplitude internal perturbations (e.g., ubiquitous internal gravity shear wave motions in the seasonal thermocline duct) that grew in size by extracting energy from the wind-induced current shear." In our model the deepening of the jump in density is produced by the wind-induced current shear. The internal gravity shear wave motions, as a source of turbulence, are not present explicitly. But the turbulent transport coefficients beneath the mixed layer, including those at the jump of density, can be regarded as being associated with turbulent mixing which results from unstable internal perturbations. Our calculations show that an increase of K_0 yields an increase in the velocity of deepening at the jump in density.

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clarifications of many parts of the original version of the manuscript.

APPENDIX

A Two-Parameter Model of Free Turbulence

The concept of the eddy viscosity coefficient is taken as a base for consideration of some turbulent processes. The eddy viscosity coefficient is defined by the Kolmogoroff-Prandtl formula

$$K=c_0 l b^{\frac{1}{2}}, \tag{A1}$$

where l is a characteristic turbulent scale and $b\equiv\overline{u_i' u_i'}/2$. For the variable b we construct a correlation equation. This equation is derived from free convection equations by a splitting of field variables into the average and fluctuating components $u_i=\bar{u}_i+u_i'$, where $\overline{u_i'}=0$, and by the conventional procedures of the ensemble average:

$$\frac{\partial b}{\partial t}+\bar{u}_\alpha\frac{\partial b}{\partial x_\alpha}=-\overline{u_i' u_\alpha'}\frac{\partial \bar{u}_i}{\partial x_\alpha}-\frac{\partial}{\partial x_\alpha}\left(\overline{u_\alpha'\frac{u_i' u_i'}{2}}+\frac{\overline{P_1' u_\alpha'}}{\rho_0}-\nu\frac{\partial b}{\partial x_\alpha}\right)-\nu\overline{\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_\alpha}}-g_i\alpha\overline{u_i' T_1'}, \tag{A2}$$

where $\bar{P}_1$, $\bar{T}_1$ are deviations of pressure and temperature from some mean values and g_i is the gravity constant. The following approximations are used to close the five right-hand terms in (A2) by means of known variables:

$$\overline{u_i' u_\alpha'}=\tfrac{2}{3}b\delta_{i\alpha}-K\left(\frac{\partial \bar{u}_i}{\partial x_\alpha}+\frac{\partial \bar{u}_\alpha}{\partial x_i}\right), \tag{A3}$$

$$\overline{u_\alpha' P_1'}=-K_P\frac{\partial \bar{P}_1}{\partial x_\alpha}, \tag{A4}$$

$$\nu\overline{\left(\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_\alpha}\right)}=2c_1\nu(b/l^2)+2c_2(b^{\frac{3}{2}}/l)=2\nu(\Delta/l^2)b, \tag{A5}$$

$$\Delta=c_1+\frac{c_2}{c_0}\frac{K}{\nu},\quad c_{0\text{-}2}\text{—constant,}$$

$$\delta_{i\alpha}=\begin{cases}1, & i=\alpha\\ 0, & i\neq\alpha\end{cases}\quad\text{is the Kronecker delta.}$$

Approximation (A3) is constructed by analogy with viscous stress approximation in the theory of Newtonian fluid. The Boussinesq-type approximation (A4) is used to close the correlation fluctuations of velocity and of scalar variables. The approximation (A5) of the rate of turbulent decay can be given a physically clear interpretation as a superposition of two limiting forms of turbulent eddy resistance: at a low rate the eddy resistance is proportional to radius of the eddy, while at a high rate the eddy resistance is proportional to the cross-sectional area and to the square of velocity.

Having applied approximations (A3)–(A5) to some terms in (A2) we obtain the turbulent energy equation

$$\frac{\partial b}{\partial t}+\bar{u}_\alpha\frac{\partial b}{\partial x_\alpha}=K\left(\frac{\partial \bar{u}_i}{\partial x_\alpha}+\frac{\partial \bar{u}_\alpha}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_\alpha}+\frac{1}{\rho_0}\frac{\partial}{\partial x_\alpha}\left(K_P\frac{\partial \bar{P}_1}{\partial x_\alpha}\right)+\frac{\partial}{\partial x_\alpha}\left((\nu+K_b)\frac{\partial b}{\partial x_\alpha}\right)-2\nu(\Delta/l^2)b+g_i\alpha K_T\frac{\partial \bar{T}_1}{\partial x_i}. \tag{A6}$$

To close the set of turbulent transport equations we deduce an equation for the decay function

$$D\equiv\frac{1}{2}\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}.$$

Differentiating D with respect to time yields

$$\frac{\partial D}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}=\overline{\frac{\partial}{\partial x_k}\left(\frac{\partial u_i'}{\partial t}\frac{\partial u_i'}{\partial x_k}\right)}. \tag{A7}$$

By substituting the equation of fluctuating velocity in (A8) for $\partial u_i'/\partial t$, we obtain

$$\frac{\partial D}{\partial t}=-\overline{\frac{\partial}{\partial x_k}\left(u_\alpha\frac{\partial u_i}{\partial x_\alpha}+\frac{1}{\rho_0}\frac{\partial \mathrm{P}_1}{\partial x_i}-\nu\frac{\partial^2 u_i}{\partial x_\alpha^2}-g_i\alpha T_1\right)\frac{\partial u_i'}{\partial x_k}}. \tag{A8}$$

Performing some simple manipulations in (A8) we obtain a correlation equation for the function D:

$$\frac{\partial D}{\partial t}+\bar{u}_\alpha\frac{\partial D}{\partial x_\alpha}=-\underbrace{\overline{\frac{\partial u_\alpha'}{\partial x_k}\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_k}}}_{\mathrm{I}}-\underbrace{\frac{\partial \bar{u}_\alpha}{\partial x_k}\overline{\left(\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_k}\right)}}_{\mathrm{II}}-\underbrace{\frac{\partial \bar{u}_i}{\partial x_\alpha}\overline{\left(\frac{\partial u_\alpha'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{\mathrm{III}}-\underbrace{\frac{\partial}{\partial x_k}\overline{\left(u_k'\frac{1}{2}\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_\alpha}\right)}}_{\mathrm{IV}}-\underbrace{\frac{\partial^2 \bar{u}_i}{\partial x_\alpha\partial x_k}\overline{\left(u_\alpha'\frac{\partial u_i'}{\partial x_k}\right)}}_{\mathrm{V}}-\underbrace{\frac{1}{\rho_0}\frac{\partial}{\partial x_i}\overline{\left(\frac{\partial P_1'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{\mathrm{VI}}-\underbrace{g_i\alpha\overline{\frac{\partial T_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}}_{\mathrm{VII}}+\underbrace{\nu\frac{\partial^2 D}{\partial x_\alpha^2}}_{\mathrm{VIII}}-\underbrace{\nu\overline{\left(\frac{\partial}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_k}\right)^2}}_{\mathrm{IX}}. \tag{A9}$$

The term V can be neglected as a quantity of a higher order with respect to the other terms. We combine the term I, denoting the rate of eddy fluctuation generation due to self-extension of turbulence eddies, with the term IX, denoting viscous dissipation of the function D, to obtain

$$-\overline{\frac{\partial u_\alpha'}{\partial x_k}\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_k}}-\nu\overline{\left(\frac{\partial}{\partial x_\alpha}\frac{\partial u_i'}{\partial x_k}\right)^2}=-2\nu\frac{\Delta'}{l^2}D, \qquad \text{(A10)}$$

where $\Delta'=c_3+c_4K/\nu$. Having closed the right-hand terms in (A9) by approximations (A3)–(A5), we arrive at an equation for D:

$$\frac{\partial D}{\partial t}+\bar{u}_\alpha\frac{\partial D}{\partial x_\alpha}=a_0\frac{\Delta}{l^2}K\left(\frac{\partial \bar{u}_i}{\partial x_\alpha}+\frac{\partial \bar{u}_\alpha}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_\alpha}+\frac{\partial}{\partial x_\alpha}\left[(\nu+K_D)\frac{\partial D}{\partial x_\alpha}+\frac{\Delta}{l^2}K_P\frac{1}{\rho_0}\frac{\partial \bar{P}_1}{\partial x_\alpha}\right]-2\nu\frac{\Delta'}{l^2}D+g_i\frac{\Delta'}{l^2}\alpha K_T\frac{\partial \bar{T}_1}{\partial x_i}. \qquad \text{(A11)}$$

Finally we rewrite (A5) in the more convenient form

$$D=(\Delta/l^2)b. \qquad \text{(A12)}$$

The dynamic turbulent equations (A6) and (A11) and the algebraic relations (A1) and (A12), together with the equations of free convection, form a closed system. It is assumed that all turbulent transport coefficients can be expressed in terms of a turbulent eddy coefficient $K_D=\alpha_D K$, where α_D is an empirical constant or a function of dimensionless parameters.

Eqs. (A1), (A6), (A11) and (A12) were tested on stationary turbulent flows in a tube. Numerical calculations have shown that there is no satisfactory agreement with experimental data simultaneously in the viscous sublayer and in the region of free turbulence. This is why we have developed a turbulence model for only the region of free turbulence and excluded the viscosity sublayer from our consideration.

The Reynolds numbers $\mathrm{Re}_T=K/\nu$ for the turbulent processes in which we are interested in are greater than or equal to 10. This means that 1) according to contemporary experimental concepts in the approximation of dissipation terms, one is led to the well-known Kolmogoroff law, i.e., having taken into account only the final terms of dimensionless functions Δ, Δ',

$$\Delta=\frac{c_2}{c_0}\frac{K}{\nu},\quad \Delta'=c_4\frac{K}{\nu}, \qquad \text{(A13)}$$

we have

$$2\nu D=2\nu\frac{\Delta}{l^2}b=2\nu\frac{c_2}{c_0}\frac{K}{\nu}\frac{b}{l^2}\approx\frac{b^{3/2}}{l};$$

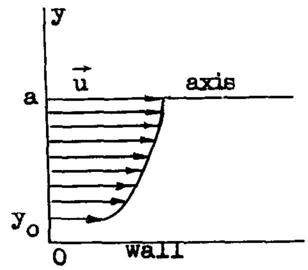

Fig. 7.

and 2) in all equations one can neglect molecular diffusion relative to turbulent diffusion.

Having assumed $\mathrm{Re}_T\geqslant 10$, we now consider stationary turbulent flow in a plane duct without the viscosity sublayer (see Fig. 7), to define empirical constants. The turbulence equations reduce to

$$K\left(\frac{\partial u}{\partial y}\right)^2+\alpha_b\frac{\partial}{\partial y}\left(K\frac{\partial b}{\partial y}\right)-2\nu D=0, \qquad \text{(A14)}$$

$$a_0\frac{D}{b}K\left(\frac{\partial u}{\partial y}\right)^2+\alpha_D\frac{\partial}{\partial y}\left(K\frac{\partial D}{\partial y}\right)-2\nu\beta\frac{D}{b}D=0, \qquad \text{(A15)}$$

$$K=c\frac{b^2}{2\nu D}, \qquad \text{(A16)}$$

$$y=a:\quad K\frac{\partial b}{\partial y}=0,\quad K\frac{\partial D}{\partial y}=0,$$

$$y=y_0:\quad \frac{\partial b}{\partial y}=0,\quad 2\nu D=u_\tau^2\frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial y}=\frac{u_\tau}{0.42y_0},\quad u_\tau^2=u_{\tau_w}^2\left(1-\frac{y_0}{a}\right).$$

In deriving Eq. (A14) we exclude pressure by using the equation of motion

$$\frac{\partial}{\partial y}\left(\frac{P}{\rho_0}+\frac{2}{3}b\right)=0,$$

and in Eq. (A15) this term was neglected as insignificant. The terms Δ/l^2 and Δ'/l^2 in (A15) were excluded using relations (A12) and (A13) to obtain

$$\frac{\Delta}{l^2}=\frac{D}{b},\quad \frac{\Delta'}{l^2}=\frac{\Delta'}{\Delta}\frac{\Delta}{l^2}=\left(\frac{c_4c_0}{c_2}\right)\frac{D}{b}=\beta\frac{D}{b}.$$

In (A14)–(A16) there are five undefined constants—α_b, α_D, a_0, β, c. The constant β is found from the law involving the degeneration of isotropic turbulence and is

$$1.4 \leqslant \beta \leqslant 2. \quad \text{(A17)}$$

The upper limit corresponds to high Re_T, the lower limit to low Re_T. The constants a_0 and c are determined from semi-empirical laws satisfied in the constant stress layer near the wall ($50 < yu_\tau/\nu < 200$):

$$K\frac{\partial u}{\partial y} = u_\tau^2 = \text{constant}, \quad \frac{\partial u}{\partial y} = \frac{u_\tau}{0.42 \cdot y}, \quad 2\nu D = u_\tau^2 \frac{\partial \bar{u}}{\partial y}, \quad b = 3.5 u_\tau^2. \quad \text{(A18)}$$

Having substituted (A18) in (A15) we obtain a relation for a_0, i.e., $a_0 = 2 - 0.62\alpha_D$. The turbulent eddy coefficient by definition is equal to $K = u_\tau^2 (\partial \bar{u}/\partial y)^{-1}$. Setting the right-hand part of this expression to that of Eq. (A16) and introducing relations (A18) we obtain $c = 0.08$. The constants α_b, α_D are found by optimization of the numerical solution of (A14)–(A16) with the boundary conditions and the equation of motion over the limits $y_0 \leqslant y \leqslant a$ using experimental data. The values $\alpha_b = \alpha_D = 1$ and $\beta = 1.9$ are the optimum constants.

For a stratified medium, we add terms $g\alpha K_T (\partial \bar{T}_1/\partial y)$ and $g\beta(D/b)\alpha K_T(\partial \bar{T}_1/\partial y)$ in Eqs. (A14) and (A15), respectively, to give

$$K\left(\frac{\partial \bar{u}}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(K\frac{\partial b}{\partial y}\right) - 2\nu D + g\alpha K_T \frac{\partial \bar{T}_1}{\partial y} = 0, \quad \text{(A19)}$$

$$1.38\frac{D}{b}K\left(\frac{\partial \bar{u}}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(K\frac{\partial \bar{D}}{\partial y}\right) - 2\nu\beta\frac{D}{b}D + g\beta\frac{D}{b}\alpha K_T\frac{\partial \bar{T}_1}{\partial y} = 0. \quad \text{(A20)}$$

The constant β of the last term in (A20) is found from fulfilment of the condition of balance between dissipation and stratification terms in (A19) and (A20) for the case of turbulent convection at high Rayleigh numbers. If the variable D is changed in (A16), (A19) and (A20) to $\epsilon = 2\nu D$, the molecular viscosity ν will be eliminated from the equations. The constructed system of turbulence equations is invariant relative to molecular viscosity of the medium. In investigation of oceanic microscale turbulence the constant β is taken equal to 1.4 which corresponds in (A17) to the limit of very weak turbulence.

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