Effect of Sampling Rate and Random Position Error on Analysis of Drifter Data

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ABSTRACT

The central question discussed here is how the rate at which drifter positions are determined and the position errors affect the calculation of velocity, acceleration and velocity gradients such as divergence and vorticity. The analysis shows that the mean-square velocity and acceleration errors each are composed of two terms. One arises from the position uncertainty and the discrete sampling rate. The other term is an alias resulting from sampling a continuous velocity or acceleration spectrum discretely. Effects at low and high frequencies and sampling intervals are examined by asymptotic expansions of the spectra. Then optimum smoothing and derivative filters are obtained for the velocity and accelerations, respectively. The efficiency of these filters is determined by comparison with the errors previously established.

The calculation of divergence and vorticity from drifter clusters typically neglects the position error, in which case the errors in the velocity gradients are proportional to the velocity errors. Our analysis shows that this procedure produces estimates of the velocity gradients whose magnitudes are less than the true values. This bias is easily removed. The analysis is concluded with a derivation of formulas for unbiased estimates of the variance and covariance of the velocity gradients.

1. Introduction

A recent area of interest in oceanography has been the description of the horizontal scales of the velocity field of the ocean through observations of drifters. For example, Freeland *et al.* (1975) have presented some intriguing results of deep flow in the western North Atlantic from sofar floats. Earlier work in this area has been performed by Swallow and Worthington (1961) and Rossby and Webb (1971). Dickson and Baxter (1972) and Cresswell (1976) have reported some current observations from satellite-tracked drifters. Kirwan *et al.* (1976) and Richardson (1976) also have employed this technique for studying the Gulf Stream and cyclonic eddies in the North Atlantic. Kirwan and McNally (1975) used Stanford Research Institute Over the Horizon Radar for studying the North Pacific Current. Molinari and Kirwan (1975) determined the differential kinematic properties (DKP) in the vorticity balance in the Yucatan Current by the use of radar-tracked drifters. A similar technique has been used by Stevenson *et al.* (1974) in studying upwelling off the coast of Oregon.

The studies cited above have determined a number of kinematic properties of the ocean. These include trajectories, velocities, Coriolis and inertial accelerations and the DKP such as divergence and vorticity. The sampling rate and position accuracy of the position fixing systems employed in the cited studies have varied tremendously. However, an analysis of how these characteristics affect the accuracy of the calculations has yet to be made.

Kirwan *et al.* (1976) estimated the accuracy of the velocity and acceleration obtained from the Random Access Measurement System (RAMS) by regarding successive position measurements as a random time series in which the position uncertainty σ is the standard deviation of the position measurement. Standard deviations for velocity and acceleration were obtained by familiar formulas for differencing random variables. Such an approach cannot describe completely the errors arising when observations are made in flows where there is significant natural variability whose time scales are comparable to the sampling period. Okubo and Ebbesmeyer (1976) have investigated a number of theoretical problems involved in estimating kinematic quantities from drifter data. However, the effects of position error and sampling rate on the calculations were outside the scope of their analysis.

The purpose here is to extend these last two studies so that the effects of position sampling rate and error in the determination of ocean kine-

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matics from Lagrangian data are assessed realistically. It is hoped that the analysis will result not only in a better feel for the accuracy of previous calculations, but also may serve as a guide for the design of experiments to determine the horizontal scales of variability in the ocean.

It is recognized that other factors may affect the quality of drifter data. For example, with surface drifters winds can produce considerable errors in the trajectories and, perhaps, the velocities. These problems are treated elsewhere (Kirwan *et al.*, 1975, 1978).

2. Errors in velocity and acceleration

The basic kinematic information obtained from drifters is trajectories. Other kinematic properties such as velocity and acceleration are inferred from the trajectory data. It is assumed then that the flow properties can be represented by the Fourier-Stieltjes integrals

$$\begin{bmatrix} \mathbf{X}(t) \\ \mathbf{U}(t) \\ \mathbf{a}(t) \end{bmatrix} = \int e^{i2\pi ft} \begin{bmatrix} d\boldsymbol{\phi}_x(w) \\ d\boldsymbol{\phi}_u(w) \\ d\boldsymbol{\phi}_a(w) \end{bmatrix}, \quad (1)$$

where

$$d\boldsymbol{\phi}_x = d\boldsymbol{\phi}_u/i2\pi f = -d\boldsymbol{\phi}_a/(2\pi f)^2. \quad (2)$$

Typically, trajectory data are obtained at discrete times with each position observation subject to a random error. Thus at time t_i the position vector can be decomposed into

$$\mathbf{X}(t_i) = \mathbf{X}_i = \mathbf{z}_i - \mathbf{r}_i, \quad (3)$$

where $\mathbf{z}_i$ is the observed position and $\mathbf{r}_i$ the position error. For our purposes it is sufficient to take the position error as random with the following properties:

$$\overline{\mathbf{r}_i\mathbf{z}_j} = \overline{\mathbf{r}_i\mathbf{X}_j} = \overline{\mathbf{r}_i} = 0, \quad \overline{\mathbf{r}_i\mathbf{r}_j} = \sigma^2\delta_{ij}, \quad \overline{(\mathbf{r}_i\mathbf{r}_i)}_{x,y} = 0. \quad (4)$$

Eq. (4) asserts that the random position error has zero mean; it is uncorrelated with the position error at other times; and it is also uncorrelated with the observed and true positions; furthermore, the x and y components are also uncorrelated. However, the variance of the components of the position error at any one time is the same as it is at any other time. Also, for convenience it will be assumed that the position sampling interval is constant and is denoted by

$$t_{i+1} - t_i = \Delta$$

for all i. This then gives a Nyquist frequency of $f_N = 1/(2\Delta)$.

Discrete estimates of the velocity can be determined from centered differences of $\mathbf{z}_j$ by

$$\mathbf{V}_j = (\mathbf{z}_{j+1} - \mathbf{z}_{j-1})/2\Delta. \quad (5)$$

Substituting (2), (3) and (5) into (1) yields

$$\begin{aligned}\mathbf{V}_j &= \int (\exp(i2\pi ft_{j+1}) - \exp(i2\pi ft_{j-1}))d\boldsymbol{\phi}_u/i4\pi f \ \Delta \\ &\quad + (\mathbf{r}_{j+1} - \mathbf{r}_{j-1})/2\Delta \\ &= \int \sin 2\pi f\Delta \exp(i2\pi ft_j)d\boldsymbol{\phi}_u/2\pi f \ \Delta \\ &\quad + (\mathbf{r}_{j+1} - \mathbf{r}_{j-1})/2\Delta. \quad (6)\end{aligned}$$

From (1), (4) and (6) the variance of the velocity error is found to be

$$\overline{e_u^2} = \overline{[\mathbf{U}_j - \mathbf{V}_j]^2} = \int [1 - M(2\Delta f)]^2 \times \overline{d\boldsymbol{\phi}_u \cdot d\boldsymbol{\phi}_u^*} + \sigma^2/2\Delta^2, \quad (7)$$

where $M(\alpha) = \sin\alpha/\alpha$. The first term on the right-hand side of (7) gives the contribution to the error variance or covariance arising from motions whose time scales are comparable to the sampling period. This error source is independent of the position error. Also note that frequencies greater than $(2\Delta)^{-1}$ will alias the rest of the spectrum. For low frequencies relative to f_N the expansion of the integrand in (7) yields

$$\overline{e_u^2} = \Delta^4 \int (2\pi f)^4 \{\sum_{p=1}^{\infty} (-1)^{p+1} \times (2\pi f\Delta)^{2p+1}[(2p+1)!]^{-2}\} \overline{d\boldsymbol{\phi}_u \cdot d\boldsymbol{\phi}_u^*} + \sigma^2/2\Delta^2. \quad (8)$$

Thus at low frequencies the leading term increases as Δ^4.

On the other hand, at the high-frequency end the integrand in (7) approaches its maximum value of 1. Here the velocity spectrum is usually assumed to have a power law form

$$\overline{d\boldsymbol{\phi}_u \cdot d\boldsymbol{\phi}_u^*} = S_u(f)df = |f|^{-n}\Gamma df. \quad (9)$$

Then the (7) takes the form

$$\overline{e_u^2} = 2\Gamma(2\pi\Delta)^{n-1} \times \int_{f_0}^{\infty} (1 - \sin X/X)^2 X^{-n} dX + \sigma^2/2\Delta^2. \quad (10)$$

The integral in (10) is finite for $n > 1$. Note that for $n = 5$ Eq. (10) produces the same dependency on Δ as the low-frequency expansion (8). Most spectra from deep moorings, however, suggest that $2 < n < 3$ at the high-frequency end (Pillsbury, *et al.*, 1979). These are Eulerian measurements and strictly speaking are not applicable, but it is not likely that Eulerian–Lagrangian transformation will affect the power law form.

The other contribution to the velocity variances in (7) is due solely to position errors and the sampling period. It is independent of the scale of ocean motions. This is the error assumed by Kirwan *et al.* (1976). This term is proportional to Δ^{-2}, whereas from (8) or (9) the first error term is a positive power of Δ. Thus there is an optimum Δ at which the velocity variance is minimum. The low-frequency expansion given by Eq. (8) shows that the optimum Δ grows as $\sigma^{1/3}$. At high frequencies (10) indicates that the optimum Δ increases as $\sigma^{1/4}$ to $\sigma^{3/10}$ for $n = 2$ and 3. In essence the optimum Δ is not particularly sensitive to σ.

These results are useful in evaluating an optimum smoothing filter for the velocity. The smoothed velocity is obtained by convolving the raw velocity estimates from (6) with a filter $h(\lambda)$, i.e.,

$$\tilde{\mathbf{U}}(t) = \int_{-\infty}^{\infty} h(\lambda)\mathbf{U}(t - \lambda)d\lambda. \quad (11)$$

Here $\tilde{\mathbf{U}}$ is the smoothed velocity and $\mathbf{U}$ the raw velocity estimate obtained at discrete times from (6). Note that (11) specifies h as a two-sided filter in anticipation of analyzing data *a posteriori*. Minimizing the mean-square difference of (6) and (11) leads to the Fourier representation of $h(t)$, namely

$$\int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt = H(f) = S_u(f)M(f\Delta) \times [S_u(f)M^2(2\Delta f) + S_N(f)]^{-1}. \quad (12)$$

Here $S_N(f)$ is the spectrum of the noise whose variance is presumably $\sigma^2/2\Delta^2$. This gives a mean-square error for the smoothing operation of

$$\overline{[\mathbf{U}(t) - \tilde{\mathbf{V}}(t)]^2} = \overline{\epsilon_u^2} = \int_{-\infty}^{\infty} S_u(f)S_N(f)S_N(f) \times [S_u(f)M^2(2\Delta f) + S_N(f)]^{-1}df. \quad (13)$$

At low frequencies it is expected that $S_u(f)M^2(2\Delta f) \gg S_N(f)$. In this region only the noise spectrum will contribute to the error. Since $M^2 \approx 1$, the noise spectra contribution may be enhanced. At higher frequencies M^2 will have a number of zeros and in general will rapidly approach zero. At such frequencies (13) indicates that the main contribution to the error variance is from the velocity spectra. But in no frequency region is the contribution to the error variance greater than the velocity spectrum. This is not true for the unsmoothed record since from (7) it is seen that the velocity spectrum contribution is amplified by $(1 - M)^2$. Since M may have either sign there will be some frequency bands in which $(1 - M)^2 > 1$. Thus the optimum smoother given by (12), which takes into account both the sampling rate and position error, should substantially improve velocity estimates.

The analysis for the acceleration errors proceeds along the lines just given. Taking the centered finite-difference formula for acceleration gives for the acceleration at the center time t_j:

$$\mathbf{a}_j = \int M^2(\Delta f) \exp(i2\pi ft_j)d\boldsymbol{\phi}_a + (\mathbf{r}_{j+1} - 2\mathbf{r}_j + \mathbf{r}_{j-1})/\Delta^2. \quad (14)$$

The variance of the difference between (14) and the true acceleration is

$$\overline{e_a^2} = \overline{(\mathbf{a} - \mathbf{a}_j)^2} = \int_{-\infty}^{\infty} [1 - M^2(\Delta f)]^2 \times \overline{d\boldsymbol{\phi}_a \cdot d\boldsymbol{\phi}_a^*} + 6\sigma^2/\Delta^4. \quad (15)$$

The integrand in (15) differs in two aspects from that in (7). First the argument of M is now Δf rather than $2\Delta f$. This means that the effect on the acceleration error variance from the acceleration spectrum starts at a lower frequency than it does for the velocity error variance. Also the acceleration spectrum is multiplied by a function which is always less than or equal to 1. Thus the acceleration spectrum can never "over contribute" to the acceleration error variance as is the case for the velocity.

The low-frequency expansion of the integrand in (15) yields

$$\overline{e_a^2} = \Delta^4 \int_{-\infty}^{\infty} \{[4(3!)^{-1} + (5!)^{-1}]^2(2\pi f)^4 + O(f^6)\} \times \overline{d\boldsymbol{\phi}_a \cdot d\boldsymbol{\phi}_a^*} + 6\sigma^2/\Delta^4. \quad (16)$$

As with the velocity error the leading term for the low-frequency expansion increases as Δ^4. The second term, however, goes as Δ^{-4} thus requiring the optimum Δ for the acceleration to increase as $\sigma^{1/4}$. At low frequencies acceleration errors are less sensitive than velocity to position errors.

At the high-frequency region the acceleration spectrum should decay as $(2\pi f)^2 S_u(f)$. Through (9) this leads to the limit

$$\overline{e_a^2} = 4\pi^{n-1}\Delta^{n-3}\Gamma \times \int_{-\infty}^{\infty} \{1 - [(\sin X)/X]^2\}^2 X^{2-n}dX + 6\sigma^2/\Delta^4. \quad (17)$$

There is an important difference between (17) and its velocity counterpart (10). For $n \leq 3$, Eq. (17) has no Δ which produces a minimum variance. The smaller the sampling interval the larger the error.

For acceleration, an alternative to an optimum smoother is an optimum derivative filter operating on the velocity. In practice it is generally necessary to smooth the filtered acceleration, but this will not be considered here. For a derivative filter $g(t)$ which minimizes

$$\overline{\epsilon_a^2} = \overline{[\mathbf{a} - \mathbf{a}_j]^2}$$

the filter is given by

$$\int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}dt = G(f) = M^2(\Delta f)S_a(f) \times [M^4(\Delta f)S_a(f) + S_N(f)]^{-1}. \quad (18)$$

Here $S_N(f)$ is now the acceleration noise spectra whose variance is $6\sigma^2/\Delta^4$.

This leads to a minimum error of

$$\overline{\epsilon_a{}^2} = \int_{-\infty}^{\infty} S_a(f)S_N(f) \times [M^4(\Delta f)S_a(f) + S_N(f)]^{-1}df. \quad (19)$$

Because of the presence of M^4 in (19) the noise contribution will be less at lower frequencies than in the velocity case.

3. Errors in differential kinematic properties

In order to analyze the effect of position error and sampling rate on the DKP, it is necessary to indicate both components of position as well as the time and drifter number. This requires a notation change. The coordinates relative to the center of mass of the cluster for the mth drifter are denoted as

$$Z_{im} = X_{im} + r_{im}, \quad i = 1, 2; \quad m = 1, \ldots n. \quad (20)$$

Here the first subscript refers to the coordinate direction and the second to the drifter number. As before, the true position is **X**, and **r** is the random position error which is subject to the restrictions given by (4).

Following Molinari and Kirwan (1975) and Okubo and Ebbesmeyer (1976), the velocity components of the mth drifter of a cluster of n drifters can be expressed as a Taylor's expansion (in space) about the cluster center of mass. Thus

$$\left.\begin{aligned} U_m &= U_0 + \frac{\partial U}{\partial X_i}X_{im} + \ldots . \\ V_m &= V_0 + \frac{\partial V}{\partial X_i}X_{im} + \ldots . \end{aligned}\right\} m = 1 \ldots n; \quad \text{sum on} \quad i = 1, 2. \quad (21)$$

In (21) U_m and V_m can be calculated by the algorithms given above, while the coefficients U_0, V_0, $\partial U/\partial X_i$ and $\partial V/\partial X_i$ can be determined by least squares provided there are three or more drifters (Molinari and Kirwan, 1975; Okubo and Ebbesmeyer, 1976).

The least-squares procedure is followed here except in (21) the true position **X** is replaced by the observed minus the position error. Also from section (2) it appears that position errors and finite-difference velocity errors are uncorrelated.

The following functions are required in the analysis to follow:

$$n\begin{bmatrix} P_{11}P_{12} \\ P_{21}P_{22} \end{bmatrix} = \sum_{m=1}^{n} \begin{bmatrix} (Z_{2m} - \bar{Z}_2)^2 & -(Z_{1m} - \bar{Z}_1)(Z_{2m} - \bar{Z}_2) \\ -(Z_{1m} - \bar{Z}_1)(Z_{22} - \bar{Z}_2) & -(Z_{1m} - \bar{Z}_1)^2 \end{bmatrix}, \quad (22)$$

$$P = \det(P_{ij}), \quad (23)$$

$$M_i = \sum_{m=1}^{n} (U_m - \bar{U})Z_{im}, \quad (24)$$

where $\bar{U}$ is the average of the X component of the center of mass velocity.

From (22)–(24) the least-squares estimates of U_0 and $\partial U/\partial X_i$ are found to be

$$\tilde{U}_0 = n^{-1} \sum_{m=1}^{n} U_m, \quad (25)$$

$$\widetilde{\frac{\partial U}{\partial X_i}} = \sum_{j=1}^{2} M_j P_{ij}/P. \quad (26)$$

Analogous expressions apply for $\tilde{V}_0$ and

$$\widetilde{\frac{\partial V}{\partial X_i}}.$$

We first consider the expansion of P from (23) in terms of the true position and position error

$$P = \sigma_1{}^2\sigma_2{}^2 - \sigma_{12}{}^4 + \tilde{\sigma}^2(\sigma_1{}^2 + \sigma_2{}^2) + \tilde{\sigma}^4, \quad (27)$$

where

$$\sigma_1{}^2 = n^{-1} \sum_{m=1}^{n} (X_{im} - \bar{X}_i)^2,$$

$$\sigma_{12} = n^{-1} \sum_{m=1}^{n} (X_{im} - \bar{X}_i)(X_{2m} - \bar{X}_2),$$

and $\tilde{\sigma}^2$ is the estimate of σ^2, the position error variance. Then from (26) we have

$$\widetilde{\frac{\partial U}{\partial X_1}} = [M_1\sigma_1{}^2 + M_2\sigma_2{}^2 + \tilde{\sigma}^2(M_1 + M_2)]/P, \quad (28)$$

$$\widetilde{\frac{\partial U}{\partial X_2}} = [M_1\sigma_2{}^2 + M_2\sigma_1{}^2 + \tilde{\sigma}^2(M_1 + M_2)]/P. \quad (29)$$

Now the standard least-squares procedure neglects the position error variance σ^2. The resulting formula for the gradients are

$$\widetilde{\frac{\partial U}{\partial X_1}} = [M_1\sigma_1^2 + M_2\sigma_2^2]/\bar{P}, \tag{30}$$

$$\widetilde{\frac{\partial U}{\partial X_2}} = [M_1\sigma_2^2 + M_2\sigma_1^2]/\bar{P}, \tag{31}$$

$$\bar{P} = \sigma_1^2\sigma_2^2 - \sigma_{12}^4 \equiv P - \tilde{\sigma}^2(\sigma_1^2 + \sigma_2^2) - \tilde{\sigma}^4. \tag{32}$$

Comparing (30) and (31) with (28) and (29), it is clear that since $P > \bar{P}$ the first two terms in the latter equations are greater than the corresponding terms in the former. But the last terms in (30) and (31) can be either plus or minus and hence it would be fortuitous if they always compensated the lower bias produced by the first two terms. It is clear then that position errors in general will produce a low bias in the estimate of the velocity gradients.

Note that if an estimate of the position error is known then it can be subtracted from (27), (28) and (29) to produce unbiased estimates of the velocity gradients. In terms of the observed displacements and velocities these are given by

$$\left.\begin{aligned}
\widetilde{\frac{\partial U}{\partial X_1}} &= [M_1P_{11} + M_2P_{12} - \tilde{\sigma}(M_1 + M_2)]/\tilde{P} \\
\widetilde{\frac{\partial U}{\partial X_2}} &= [M_1P_{21} + M_2P_{22} - \tilde{\sigma}(M_1 + M_2)]/\tilde{P} \\
\widetilde{\frac{\partial V}{\partial X_1}} &= [N_1P_{11} + N_2P_{12} - \tilde{\sigma}(N_1 + N_2)]/\tilde{P} \\
\widetilde{\frac{\partial V}{\partial X_2}} &= [N_1P_{21} + N_2P_{22} - \tilde{\sigma}(N_1 + N_2)]/\tilde{P} \\
N_i &= \sum_{m=1}^{n} (V_m - \bar{V})Z_{im} \approx \sum_{m=1}^{n} (V_m - \bar{V})X_{im}
\end{aligned}\right\}. \tag{33}$$

From (33) it is seen that the unbiased estimates of the velocity gradients are linear functions of M_i and N_i which are random. This implies that the unbiased variance of the gradients is given by

$$\overline{\left(\frac{\partial U}{\partial X_i} - \overline{\frac{\partial U}{\partial X_i}}\right)\left(\frac{\partial U}{\partial X_j} - \overline{\frac{\partial U}{\partial X_j}}\right)} = \sigma_u^2[P_{ij} - \tilde{\sigma}^2\delta_{ij}]/n\tilde{P},$$

$$\overline{\left(\frac{\partial V}{\partial X_i} - \overline{\frac{\partial V}{\partial X_i}}\right)\left(\frac{\partial V}{\partial X_j} - \overline{\frac{\partial V}{\partial X_j}}\right)} = \sigma_v^2[P_{ij} - \tilde{\sigma}^2\delta_{ij}]/n\tilde{P}. \tag{34}$$

Here the maximum likelihood estimates of σ_u^2 and σ_v^2 are

$$\tilde{\sigma}_u^2 = \frac{1}{v}\sum_{m=1}^{n}\left\{\left[U_m - U_0 - \widehat{\frac{\partial U}{\partial X_i}}X_{im}\right]^2 - \left(\widetilde{\frac{\partial U}{\partial X_i}}\right)\left(\widetilde{\frac{\partial U}{\partial X_i}}\right)\tilde{\sigma}^2\right\},$$

$$\tilde{\sigma}_v^2 = \frac{1}{n}\sum_{m-1}^{n}\left\{\left[V_m - V_0 - \widehat{\frac{\partial V}{\partial X_i}}X_{im}\right]^2 - \left(\widetilde{\frac{\partial V}{\partial X_i}}\right)\left(\widetilde{\frac{\partial V}{\partial X_i}}\right)\tilde{\sigma}^2\right\}. \tag{35}$$

Eqs. (34) and (35) give the unbiased covariances of

$$\left(\widetilde{\frac{\partial U}{\partial X_i}}\,\widetilde{\frac{\partial U}{\partial X_j}}\right) \quad \text{and} \quad \left(\widetilde{\frac{\partial V}{\partial X_i}}\,\widetilde{\frac{\partial V}{\partial X_j}}\right)$$

in terms of the observed displacements and the center of mass velocity. In essence the equations show that these covariances are all proportional to the mean square velocity variances and the position error variance. For negligible position error these equations reduce to the standard least squares result given by Okubo and Ebbesmeyer (1976) that the covariance is proportional to the variance of the random independent variable U or V.

In many applications the DKP are of more importance than the velocity gradients. As these are linear combinations of $\widetilde{\partial U}/\partial X_i$ and $\widetilde{\partial V}/\partial X_i$ the variances and covariances of the DKP in principle should be readily obtained from (34). Unfortunately the least-squares machinery of Molinari and Kirwan (1975) and Okubo and Ebbesmeyer (1976) does not provide estimates of

$$\overline{\frac{\partial U}{\partial X_i} \pm \frac{\partial V}{\partial X_j}}$$

as the U and V regressions are performed independently. Overcoming this requires a minor generalization of their least-squares procedure to include explicit dependence on the covariance of U and V. This yields for the variances of the DKP

$$\begin{bmatrix}(D-\tilde{D})^2\\(N-\tilde{N})^2\\(\zeta-\tilde{\zeta})^2\\(\dot{S}-\tilde{S})^2\end{bmatrix}=\begin{bmatrix}P_{11}\tilde{\sigma}_u{}^2+P_{22}\tilde{\sigma}_v{}^2+2P_{12}\tilde{\sigma}_{uv}\\P_{11}\tilde{\sigma}_u{}^2+P_{22}\tilde{\sigma}_v{}^2-2P_{12}\tilde{\sigma}_{uv}\\P_{22}\tilde{\sigma}_u{}^2+P_{11}\tilde{\sigma}_v{}^2-2P_{12}\tilde{\sigma}_{uv}\\P_{22}\tilde{\sigma}_u{}^2+P_{11}\tilde{\sigma}_v{}^2+2P_{12}\tilde{\sigma}_{uv}\end{bmatrix}, \quad (36)$$

where

$$D=\frac{\partial U}{\partial X_1}+\frac{\partial V}{\partial X_2}, \quad \zeta=\frac{\partial V}{\partial X_1}-\frac{\partial U}{\partial X_2},$$

$$N=\frac{\partial U}{\partial X_1}-\frac{\partial V}{\partial X_2}, \quad S=\frac{\partial V}{\partial X_1}+\frac{\partial U}{\partial X_2},$$

and the maximum likelihood estimate of σ_{uv} is

$$\tilde{\sigma}_{uv}=n^{-1}\sum_{\substack{m-1\\i=1,2}}^{n}\left(U_m-\tilde{U}_0-\widetilde{\frac{\partial U}{\partial X_i}}\tilde{X}_{im}\right)\times\left(V_m-\tilde{V}_0-\widetilde{\frac{\partial V}{\partial X_i}}\tilde{X}_{im}\right).$$

To summarize it has been shown that position error lowers estimates of the velocity gradients relative to the expected value. However with estimates of the position error it is easy to obtain unbiased estimates of the velocity gradients. Furthermore, unbiased estimates of the covariance matrix of the velocity gradients or the DKP can also be obtained if the position error is known.

It has been our experience that smoothed estimates of velocities should be used in the calculation of velocity gradients.

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