Variational Approach to Gravity Waves in Terms of Streamfunction[1]

R. Benzi
I.F.A.-C.N.R., Roma, Italy

E. Salusti
INFN-Istituto Fisica, Università di Roma, Roma, Italy

A. Sutera[2]
Istituto Fisica, Università di Roma, Roma, Italy

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ABSTRACT

A variational principle is presented for periodic, finite-amplitude gravity waves in terms of a streamfunction.

1. Introduction

Usually gravity waves are described in terms of the velocity potential under the assumption of no vorticity in the fluid. This hypothesis is rather difficult to justify and not useful if the induced boundary layer must be studied. In the case of long crested waves, i.e., for a two-dimensional problem, a streamfunction representation can be also introduced. Dean (1965) first studied, by means of a streamfunction, the case of zero vorticity for a periodic wave system travelling with constant speed C. He showed that this method can be employed to represent wave conditions, with a prescribed uniform steady current and pressure distribution on the free surface. Dalrymple (1973) has enlarged Dean's study for the case of constant vorticity and more recently Dalrymple and Cox (1976) and Dalrymple (1977) have studied, in the case of a vorticity field proportional to the streamfunction, some interesting current profiles related to nonlinear waves.

The aim of this note is to show how to formulate the streamfunction method by means of a variational principle in analogy with that of Luke (1967) for the velocity potential.

2. The problem

We consider a two-dimensional, incompressible and inviscid fluid in a Cartesian reference frame x', z'. The z' axis is directed upward and $\eta(x',t')$ will be the free surface. We discuss only the case in which $\eta(x',t')$ has the form

$$\eta(x',t') = \eta(x' + ct') = \eta(x' + ct' + KL),$$

where c is the constant wave speed, L the wavelength and $K = 0, \pm1, \pm2, \ldots.$

Performing a Galilean transformation

$$\left.\begin{array}{l} z \leftarrow z' \\ x \leftarrow x' + ct' \\ t \leftarrow t' \end{array}\right\},$$

we obtain the steady case. In this new moving frame the free surface $\eta(x)$ is periodic, i.e.,

$$\eta(x) = \eta(x + L).$$

For the streamfunction $\Psi(x,z)$, the steady two-dimensional Euler equation is

$$J(\Psi,\nabla^2\Psi) = 0$$

or

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[2] On leave of absence from Istituto di Ricerche "G. Donegani", Via G. Fauser 4, Novara, Italy.

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$$\nabla^2\Psi = F(\Psi).$$

The pressure p is function of Ψ, $p = p(\Psi)$; the function $F(\Psi)$ is not determined by the theory.

The boundary conditions at the flat bottom are

$$\Psi = \text{constant} = \Psi_D, \quad z = -D,$$

and at the free surface

$$\left.\begin{array}{ll} \Psi = \text{constant} = \Psi_\eta, & z = \eta(x) \\ \tfrac{1}{2}(\nabla\Psi)^2 + gz = Q = \text{constant}, & z = \eta(x) \end{array}\right\},$$

i.e., the Bernoulli equation.

3. The variational formulation

We now derive a variational formulation of our problem. Following Whitham (1974), the natural candidate is

$$I(\eta,\Psi) = \int_0^L \int_{-D}^{\eta} \{\tfrac{1}{2}|\nabla\Psi|^2 + gz + G(\Psi)\}dxdz,$$

where $F(\Psi) = dG/d\Psi$ is related to the pressure. Our statement is that the preceding equations are equivalent to the equations

$$\frac{\delta I}{\delta\Psi} = 0, \quad \frac{\delta I}{\delta\eta} = 0,$$

with the boundary conditions

$$\eta(x) = \eta(x + L),$$

$$\Psi(x) = \Psi(x + L),$$

$$\Psi = \Psi_\eta = \text{constant}, \quad z = \eta(x),$$

$$\Psi = \Psi_D = \text{constant}, \quad z = -D.$$

To prove our statement we compute explicitly $\delta_\eta I$ and $\delta_\Psi I$.

For $\delta_\eta I$ we easily obtain

$$\delta_\eta I = [\tfrac{1}{2}|\nabla\Psi|^2 + gz + G(\Psi)]\delta\eta = 0 \quad \text{at} \quad z = \eta.$$

This relation is equivalent to the Bernoulli equation, using the definition

$$Q = -G(\Psi_\eta).$$

We compute now $\delta_\Psi I$:

$$\delta_\Psi I = \int_0^L dx \int_{-D}^{\eta} \frac{\partial\Psi}{\partial x}\,\delta\,\frac{\partial\Psi}{\partial x}dz + \int_0^L dx \int_{-D}^{\eta} \frac{\partial\Psi}{\partial z}\,\delta\,\frac{\partial\Psi}{\partial z}dz + \int_0^L dx \int_{-D}^{\eta} \frac{dG}{d\Psi}\delta\Psi dz,$$

which transforms to

$$\delta_\Psi I = \int_0^L dx \int_{-D}^{\eta} \left[\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial x}\delta\Psi\right) + \frac{\partial}{\partial z}\left(\frac{\partial\Psi}{\partial z}\delta\Psi\right)\right]dz - \int_0^L dx \int_{-D}^{\eta} \left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \frac{dG}{d\Psi}\right)\delta\Psi dz$$

or

$$\delta_\Psi I = \int_{-D}^{\eta} \left[\left(\frac{\partial\Psi}{\partial x}\delta\Psi\right)_{x=L} - \left(\frac{\partial\Psi}{\partial x}\delta\Psi\right)_{x=0}\right]dz + \int_0^L \left[-\frac{\partial\eta}{\partial x}\left(\frac{\partial\Psi}{\partial x}\delta\Psi\right)_{z=\eta} + \left(\frac{\partial\Psi}{\partial z}\delta\Psi\right)_{z=\eta} - \left(\frac{\partial\Psi}{\partial z}\delta\Psi\right)_{z=-D}\right]dx - \int_0^L \int_{-D}^{\eta} \left(\nabla^2\Psi - \frac{dG}{d\Psi}\right)\delta\Psi dz.$$

The first integral on the right vanishes because of the periodic condition on Ψ; the second integral vanishes because $\delta\Psi = 0$ at $z = \eta$ and $z = -D$. Hence for $\delta_\Psi I = 0$ we obtain

$$\nabla^2\Psi = dG/d\Psi$$

which completes the proof.

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