Interior Shelf Waves on an Equatorial β-Plane

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ABSTRACT

In this note the properties of long-period waves on an interior exponential shelf on an equatorial β-plane are determined. When both the shelf and deep-sea region are in the same hemisphere the wave properties are similar to those of an exponential continental shelf. However, when the shelf and deep-sea regions are in opposite hemispheres a finite number of westward propagating modes may exist for a fixed wavenumber. All westward modes, except the fundamental mode, have a zero group velocity and a long-wave cutoff for trapping. The fundamental approaches the cutoff as the wavenumber decreases. When the shelf straddles the equator westward, eastward propagating modes may exist. In the case of the southern edge of the Melanesian Basin, the relevant resonant frequency at the zero group velocity in the fundamental mode is found to be between 25 and 33 days.

1. Introduction

A general theory of homogeneous, nondivergent zonally propagating shelf waves on an equatorial beta plane has been given by Mysak (1978a) for the case of a continental shelf parallel to the equator. Assuming an exponential shelf profile independent

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of longitude, Mysak (1978b) showed that the presence of long-period oscillations along the Gulf of Guinea, West Africa, may be attributed to shelf waves.

As was noted by Buchwald and Adams (1968), however, shelf waves may propagate on "interior shelves" in the deep ocean. It may be conjectured that such waves will occur in regions of the Pacific Ocean, such as the Melanesian Basin. As is now well known, there are possibilities of resonance when zero group velocities occur. In this note the dispersion curves for equatorial interior shelf waves are computed, in order to determine their properties, and, in particular, to locate the frequencies for which there is a zero group velocity.

Let $\Psi(x,y,t)$ be the mass transport streamfunction, and assume that Ψ has a traveling wave form

$$\Psi(x,y,t) = \psi(y)e^{i(kx-\omega t)}, \tag{1}$$

where x, y are eastward, northward coordinates, respectively. The horizontal components of the velocity are

$$Hu = \frac{\partial \Psi}{\partial y}, \quad Hv = -\frac{\partial \Psi}{\partial x}, \tag{2}$$

and it can be shown (Mysak, 1978a) that the linearized equations for unforced motion are satisfied when

$$\left(\frac{\psi'}{H}\right)' - \left[\frac{k^2}{H} + \left(\frac{k\beta}{\omega}\right)\left(\frac{y}{H}\right)'\right]\psi = 0, \tag{3}$$

where the prime represents d/dy and $\beta = 2\Omega_E/R_E$, where Ω_E and R_E are the earth's angular velocity and radius, respectively.

We assume a continuous depth profile of the form

$$H(y) = \begin{cases} H_s, & y < y_s \\ H_e(y), & y_s \leq y \leq y_n \\ H_n, & y > y_n. \end{cases} \tag{4}$$

Continuity of pressure and volume transport implies that ψ and $d\psi/dy$ are continuous at $y = y_s$ and y_n. For trapping to occur, it is necessary that $\psi \to 0$ as $|y| \to \infty$. Consequently in the regions $y < y_s$ and $y > y_n$ the solution to (3) is

$$\psi = Ae^{-K|y|}, \quad K = \left(k^2 + \frac{k\beta}{\omega}\right)^{1/2} < 0. \tag{5}$$

In the shelf region Eq. (3) is to be solved in $y_s \leq y \leq y_n$ subject to the boundary conditions

$$\psi' - K\psi = 0, \quad y = y_s, \tag{6}$$

$$\psi' + K\psi = 0, \quad y = y_n. \tag{7}$$

These conditions are obtained by substituting (5) into the continuity conditions at y_s, y_n.

2. The exponential profile

Let $H_e(y) = H_0 e^{2by}$, whereupon (3) reduces to

$$\psi'' - 2b\psi' - \left(K^2 - \frac{2bk\beta y}{\omega}\right)\psi = 0,$$
$$y_s \leq y \leq y_n. \tag{8}$$

It is sufficient to consider the case $b > 0$ (deep water to the north) and introduce the normalized variable

$$\eta = \frac{y - y_s}{y_n - y_s} = \frac{y - y_s}{l}, \tag{9}$$

where l is the shelf width, so that (8) is transformed to

$$\frac{d^2\psi}{d\eta^2} - 2B\frac{d\psi}{d\eta} - \left[\chi^2 - (\eta + \gamma)\frac{2B\kappa}{\Omega}\right]\psi = 0,$$
$$0 \leq \eta \leq 1, \tag{10}$$

with $B = bl > 0$, $\gamma = y_s/l$, $\kappa = kl$, $\Omega = \omega/\beta l$ and $\chi^2 = \kappa^2 + \kappa/\Omega > 0$. κ and Ω are the nondimensional wavenumber and frequency, respectively, B characterizes the curvature of the shelf, and γ denotes the relative position of the shelf with respect to the equator. When $\gamma > 0$ the shelf is entirely north of the equator, when $\gamma < -1$ the shelf is entirely south of the equator, and when $-1 < \gamma < 0$ the shelf straddles the equator. The boundary conditions (6) and (7) transform into

$$\frac{d\psi}{d\eta} - \chi\psi = 0 \quad \text{at} \quad \eta = 0, \tag{11}$$

$$\frac{d\psi}{d\eta} + \chi\psi = 0 \quad \text{at} \quad \eta = 1. \tag{12}$$

The case $b < 0$ (deep water to the south) is dealt with by making the substitution $\eta' = (y_n - y)/l$, which produces an identical set of equations with the parameters B and γ replaced by $-B$ and $-(\gamma + 1)$, respectively.

We now distinguish between the cases of eastward and westward propagating waves.

a. *Eastward propagating waves; $\Omega > 0$*

We let

$$\lambda^2 = 2B\kappa/\Omega > 0. \tag{13}$$

The substitution

$$\psi = e^{B\eta}\phi, \tag{14}$$

where

$$\mu = \eta + \gamma - (\chi^2 + B^2)/\lambda^2, \tag{15}$$

yields the Airy equation (Abramowitz and Stegun, 1965, p. 446)

$$\frac{d^2\phi}{d\mu^2} + \lambda^2\mu\phi = 0, \quad \mu_0 \leq \mu \leq \mu_1 = \mu_0 + 1, \tag{16}$$

with boundary conditions

$$\frac{d\phi}{d\mu} + (B - \chi)\phi = 0 \quad \text{at} \quad \mu = \mu_0, \tag{17}$$

$$\frac{d\phi}{d\mu} + (B + \chi)\phi = 0 \quad \text{at} \quad \mu = \mu_1, \tag{18}$$

where $\mu_0 = \gamma - (\chi^2 + B^2)/\lambda^2$. The general solution of (16), in terms of Airy functions Ai, Bi, is

$$\phi = c_1\text{Ai}[\tau\mu] + c_2\text{Bi}[\tau\mu], \tag{19}$$

where $\tau = -\lambda^{2/3} \leq 0$ and c_1, c_2 are arbitrary constants. For nontrivial solutions of (16) to (18) it is necessary that

$$[\tau\text{Ai}'(\tau\mu_0) + (B - \chi)\text{Ai}(\tau\mu_0)][\tau\text{Bi}'(\tau\mu_1) + (B + \chi)\text{Bi}(\tau\mu_1)] - [\tau\text{Ai}'(\tau\mu_1) + (B + \chi)\text{Ai}(\tau\mu_1)][\tau\text{Bi}'(\tau\mu_0) + (B - \chi)\text{Bi}(\tau\mu_0)] = 0. \tag{20}$$

This relation is an implicit form of the dispersion relation. The solutions $\Omega = \Omega_n(\kappa)$ to this relation, where $n = 0, 1, 2, \ldots$, are the eigenfrequencies. For fixed κ these can be ordered such that $\Omega_0 > \Omega_1 > \ldots \Omega_n > 0$. The eigenfunction corresponding to an eigenfrequency Ω_n can be described in terms of the mass transport streamfunction $\psi_n(\eta)$ such that

$$\psi_n(\eta) = e^{B\eta}\left\{\text{Bi}(\tau\mu) - \left[\frac{\tau\text{Bi}'(\tau\mu_0) + (B - \chi)\text{Bi}(\tau\mu_0)}{\tau\text{Ai}'(\tau\mu_0) + (B - \chi)\text{Ai}(\tau\mu_0)}\right]\text{Ai}(\tau\mu)\right\}, \quad 0 \leq \eta \leq 1, \tag{21}$$

where $\chi^2 = \kappa^2 + \kappa/\Omega_n$ and $\lambda^2 = 2B\kappa/\Omega_n$, and μ is given in (15).

b. Westward propagating waves: $\Omega < 0$

We let

$$\lambda^2 = -2B\kappa/\Omega. \tag{22}$$

Then the substitution of (14), but with

$$\mu' = \eta + \gamma - (\chi^2 + B^2)/\lambda^2, \tag{23}$$

into (10)–(12) yields

$$\frac{d^2\phi}{d\mu^2} - \lambda^2\mu'\phi = 0,$$

$$\mu_0' \leq \mu' \leq \mu_1' = \mu_0' + 1, \tag{24}$$

with the boundary conditions

$$\frac{d\phi}{d\mu} + (B - \chi)\phi = 0,$$

at

$$\mu' = \mu_0' = \gamma + (\chi^2 + B^2)/\lambda^2, \tag{25}$$

$$\frac{d\phi}{d\mu} + (B + \chi)\phi = 0, \quad \text{at} \quad \mu' = \mu_1'. \tag{26}$$

The solution to this variant of the Airy equation is

$$\phi = d_1\text{Ai}[\tau\mu'] + d_2\text{Bi}[\tau\mu'], \tag{27}$$

where $\tau = \lambda^{2/3} \geq 0$ and d_1, d_2 are arbitrary constants. The implicit dispersion relation is identical to (20), with μ' substituted for μ. When μ' is negative the Airy functions are oscillatory; thus, as in the case $\Omega > 0$, there will be an infinity of eigenfrequencies $\Omega = \Omega_n(\kappa), n = 0, 1, 2, \ldots$. These can be ordered such that $\Omega_0 < \Omega_1 < \ldots < 0$. Since $\chi^2 = \kappa^2 + \kappa/\Omega > 0$ for trapping to occur, $\Omega = -\kappa^{-1}$ is an upper bound for the eigenfrequencies $\Omega = \Omega_n(\kappa)$. The eigenfunction corresponding to an eigenfrequency $\Omega = \Omega_n(\kappa)$ may now be obtained in a form similar to (21).

3. Bounds on γ for eastward propagating waves

In this section it is assumed that the deep water is to the north. If Eq. (16) is multiplied by ϕ and then integrated with respect to η over (0,1), using the boundary conditions (17) and (18), it can be shown that

$$\frac{\Omega}{\kappa} = \frac{2B\int_0^1 \eta\phi^2 d\eta + (2B\gamma - 1)\int_0^1 \phi^2 d\eta}{\int_0^1 (G^2\phi^2 + \phi'^2)d\eta + (B + \chi)\phi^2(1) + (\chi - B)\phi^2(0)}, \tag{28}$$

where $G^2 = B^2 + \kappa^2$. This relation can be rearranged to give

$$\frac{\Omega}{\kappa} = \frac{2B\int_0^1 \eta\phi^2 d\eta + (2B\gamma - 1)\int_0^1 \phi^2 d\eta}{G^2\int_0^1 (\phi + \phi' G^{-1})^2 d\eta + \phi^2(1)(B + \chi - G) + \phi^2(0)(G + \chi - B)}. \tag{29}$$

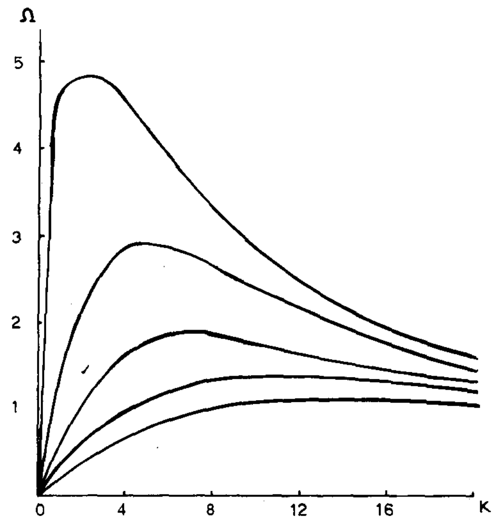

FIG. 1. The dispersion curves of the first five modes (n = 0, 1, 2, 3, 4) for $B = 3$ and $\gamma = 5.0$. These curves also apply to the case $B = -3$ and $\gamma = -6.0$ (shelf in the Southern Hemisphere with deep water to the south).

By inspection, when $\kappa > 0$, $G + \chi - B > 0$ for all Ω, and also $B + \chi - G > 0$ if $\Omega > 0$. Therefore, when $\Omega > 0$ the denominator of (29) is greater than zero, and a lower bound of $\dot{\gamma}$ can be found such that there will exist an eastward propagating wave. This bound can be found by equating the numerator of (29) to zero. Thus, eastward propagating waves will exist only if $\gamma > \gamma^*$, where

$$\gamma^* = \frac{1}{2B} - \delta, \quad 0 < \delta = \frac{\int_0^1 \eta\phi^2 d\eta}{\int_0^1 \delta^2 d\eta} < 1. \quad (30)$$

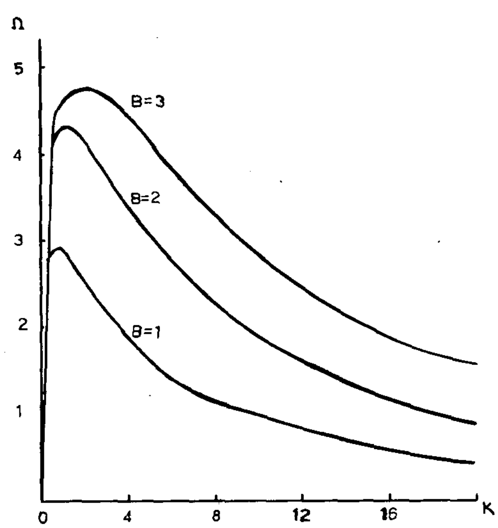

FIG. 2. The fundamental mode for $\gamma = 5.0$ and $B = 1, 2, 3$.

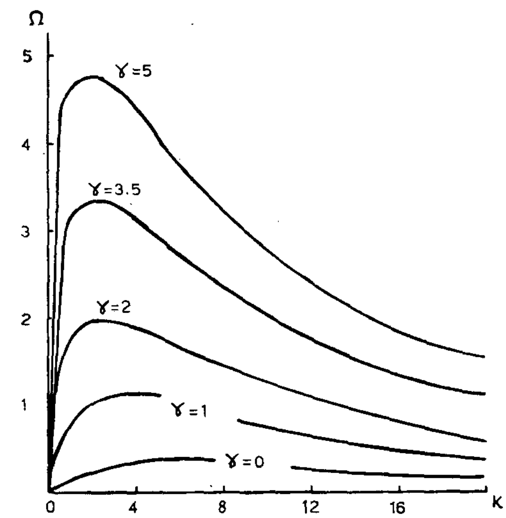

FIG. 3. The fundamental mode for $B = 3$ and $\gamma = 0.0$, 1.0, 2.0, 3.5 and 5.0.

Hence, when the shelf is entirely in the Southern Hemisphere (when the deepwater extends infinitely to the north) no eastward propagating waves may exist.

4. Numerical results

As in Section 3 it is assumed that the deep water is to the north. The dispersion curves, Ω vs κ, were computed for the three ranges of the parameter γ. These ranges are 1) $\gamma > 0$, where the shelf is in the Northern Hemisphere and where eastward propagating waves exist; 2) $-1 < \gamma < 0$, where the shelf straddles the equator and where, in general,

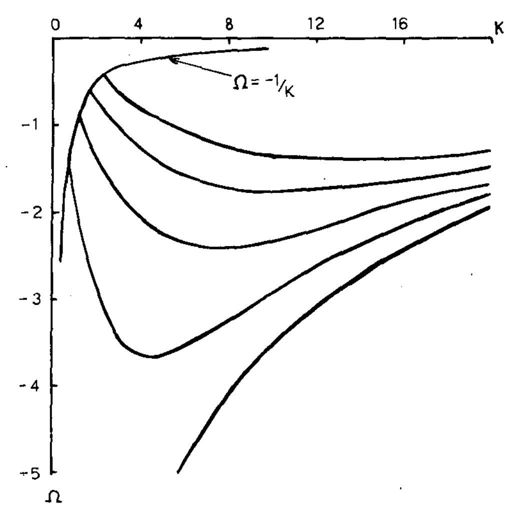

FIG. 4. The dispersion curves of the first five modes for $B = 3$ and $\gamma = -7.0$. The curve $\Omega = -1/\kappa$ represents the long-wave cutoff for trapped modes when $\Omega < 0$. These curves also apply to the case $B = -3$ and $\gamma = -7.0$ (shelf in Northern Hemisphere with deep water to the south).

both eastward and westward propagating waves exist; and 3) $\gamma < -1$, where the shelf is entirely in the Southern Hemisphere and where it has been proved that only westward propagating waves may exist.

Fig. 1 shows the dispersion curves for the first five modes for $B = 3$, $\gamma = 5$. By applying the transformation $B \rightarrow -B$ and $\gamma \rightarrow -(\gamma + 1)$, Fig. 1 also applies to the case $B = -3$, $\gamma = -6$. These curves represent the case where the shelf and deep water are in the same hemisphere, and these curves are qualitatively similar to the corresponding case for an exponential continental shelf (Mysak, 1978b, Fig. 2c). All modes are found to propagate eastward. Each mode has a zero-group velocity at some intermediate wavenumber, $\kappa = \kappa_n^*$, where κ_n^* increases as n (the mode number) increases.

Figs. 2 and 3 show the effects on the frequency and phase speed of changes in the shelf curvature parameter B and the distance γ of the shelf from the equator. As B or γ increases, the phase speed and frequency increase. As B increases, κ_0^* increases, but when γ increases, κ_0^* decreases.

Fig. 4 shows the dispersion curves for the first five modes for $B = 3$, $\gamma = -7$ (and $B = -3$, $\gamma = 6$). These curves represent the case where the shelf and deep water are on opposite sides of the equator. In this case all modes are found to propagate westward. For the modes $n \geqslant 1$, a zero group velocity exists for an intermediate wavenumber κ_n^*. For the fundamental mode, $n = 0$, there is no zero group velocity. Comparison with Fig. 4c of Mysak (1978b) shows that this result is the major difference between the continental shelf case and the interior shelf case. Buchwald and Adams (1968) also found that the fundamental mode of the midlatitude interior shelf also has no zero group velocity. At small κ the fundamental mode approaches the long-wave cutoff, $\Omega = -1/\kappa$, which represents the dispersion relation $\omega = -\beta/\kappa$. This is the usual dispersion relation for a one-dimensional, westward propagating Rossby wave.

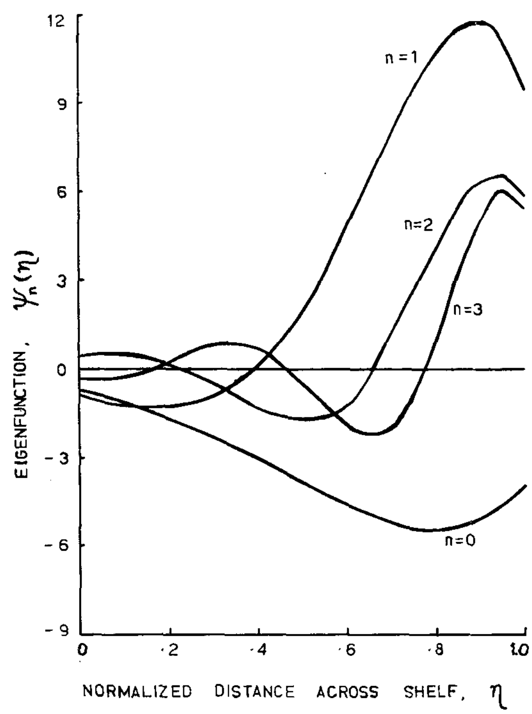

FIG. 5. The eigenfunctions ψ_n for the case $B = 3$, $\gamma = -7.0$ and $\kappa = 5.0$.

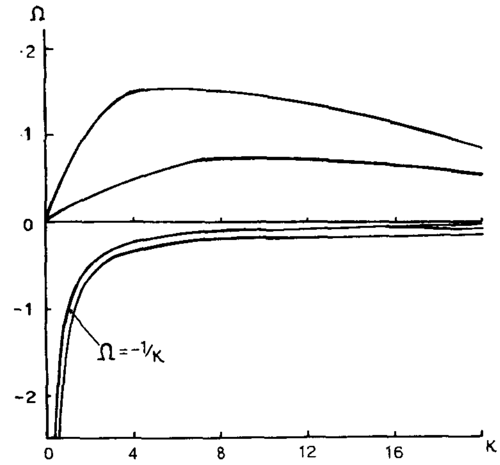

FIG. 6. The dispersion curves of the first two westward and eastward modes for $B = 3$ and $\gamma = -1/3$. These curves also apply to the case $B = -3$ and $\gamma = -2/3$ (shelf straddling equator with deep water to the south).

In Fig. 5 the eigenfunctions for the first three modes for the case $B = 3$, $\gamma = -7$ are represented. These functions are oscillatory with the nth mode having n zero-crossings. The eigenfunctions for the other cases are qualitatively similar.

Finally, Fig. 6 shows the dispersion relation for the case where the shelf straddles the equator, $B = 3$, $\gamma = -1/3$. In this case as in the continental shelf case (Mysak, 1978), both eastward and westward modes exist. The fundamental westward propagating mode is very close to the long-wave cutoff, $\Omega = -1/\kappa$, for longer waves. The eastward propagating modes and $n \geqslant 1$ westward modes have a zero group velocity.

5. Discussion

It has been shown that trapped long-period waves may exist on an interior exponential shelf in an equatorial β-plane. The qualitative properties of these waves are not very different from those of similar waves on a continental shelf. In most modes, zero group velocities occur, and, as has been suggested by Barton and Buchwald (1977) and Buchwald (1977), there are linear and nonlinear ways in which there may be a resonant response at frequency corresponding to a zero group velocity.

For instance, the southern edge of the Melanesian Basin (167–173°E, 6°S) in the Pacific Ocean could be regarded as an interior shelf with B in the range -0.27 to -0.35. In this case Fig. 1 applies and the $n = 0$ mode has a resonance between 25 and 33 days. It would be interesting to see if any anomalous 28-day tides are found in this region.

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