Is Weather Chaotic?
Coexistence of Chaos and Order within a Generalized Lorenz Model
Bo-Wen Shen, Roger A. Pielke Sr., Xubin Zeng, Jong-Jin Baik,
Sara Faghih-Naini, Jialin Cui, and Robert Atlas

https://doi.org/10.1175/BAMS-D-19-0165.2
Corresponding author: Bo-Wen Shen, bshen@sdsu.edu
This document is a supplement to https://doi.org/10.1175/BAMS-D-19-0165.1
©2021 American Meteorological Society
For information regarding reuse of this content and general copyright information, consult the AMS Copyright Policy.

AFFILIATIONS: Shen—Department of Mathematics and Statistics, San Diego State University, San Diego, California; Pielke—Cooperative Institute for Research in Environmental Sciences, University of Colorado Boulder, Boulder, Colorado; Zeng—Department of Hydrology and Atmospheric Science, The University of Arizona, Tucson, Arizona; Baik—School of Earth and Environmental Sciences, Seoul National University, Seoul, South Korea; Faghih-Naini—Department of Mathematics and Statistics, San Diego State University, San Diego, California, and University of Bayreuth, Bayreuth, and Friedrich-Alexander University Erlangen—Nuremberg, Erlangen, Germany; Cui—Department of Mathematics and Statistics, and Department of Computer Sciences, San Diego State University, San Diego, California; Atlas—National Oceanic and Atmospheric Administration/AOML, Miami, Florida
The generalized Lorenz model and the Lorenz 1963 model

Here, we provide a brief summary on the generalized Lorenz model (GLM) (Shen 2019a,b; Shen et al. 2019). The original Lorenz 1963 model was derived from a system of two partial differential equations (PDEs) that describe the evolution of vorticity and temperature. Such a system is used for studying Rayleigh–Bénard convection (RBC) (e.g., Saltzman 1962; Lorenz 1963). Each of the two PDEs is an equation that involves temporal and spatial partial derivatives. The spectral method, that uses spatial Fourier modes as basis functions for representing solutions, can help convert time-dependent PDEs into a system of time-dependent ODEs in order to understand fundamental physical processes. The related procedures were well documented in Lorenz (1963) and have been reported in many other studies (e.g., Shen 2014, and references therein). Over the past several years, a series of papers regarding high-dimensional Lorenz models that have applied a different number of Fourier modes (Shen 2014, 2015, 2016, 2017, 2018; Faghih-Naini and Shen 2018) have yielded the following GLM (Shen 2019a,b; Shen et al. 2019):

\[
\frac{dX}{d\tau} = \sigma Y - \sigma X, \quad (\text{ES1})
\]

\[
\frac{dY}{d\tau} = -XZ + rX - Y, \quad (\text{ES2})
\]

\[
\frac{dZ}{d\tau} = XY - XY_1 - bZ, \quad (\text{ES3})
\]

\[
\frac{dY_j}{d\tau} = jXZ_{j-1} - (j + 1)XZ_j - d_{j-1}Y, \quad j \in \mathbb{Z} : j \in [1, N], \quad (\text{ES4})
\]

\[
\frac{dZ_j}{d\tau} = (j + 1)XY_j - (j + 1)XY_{j+1} - \beta_j Z, \quad j \in \mathbb{Z} : j \in [1, N], \quad (\text{ES5})
\]

\[
N = \frac{M - 3}{2}; \quad d_{j-1} = \frac{(2j + 1)^2 + a^2}{1 + a^2}; \quad \beta_j = (j + 1)^2 b; \quad b = \frac{4}{(1 + a^2)}. \quad (\text{ES6})
\]

Here, \( \tau \) is dimensionless time. The three integers \( j, M, \) and \( N \) are related to the number of additional Fourier modes within higher dimensional LMs. While \( M \) represents the total number of modes (or equations), \( N \) indicates the total number of pairs \( (Y_j, Z_j) \) for higher-wavenumber modes that do not appear within the original 3DLM (Lorenz 1963). Time-independent parameters include \( \sigma, r, a, b, d_{j-1}, \) and \( \beta_j \). The first two represent the Prandtl number and the normalized Rayleigh number (or the heating parameter), respectively (e.g., Shen 2014). Parameter \( a \) is defined as the ratio of the vertical scale of the convection cell to its horizontal scale, which is equal to \( 1/\sqrt{2} \). The last three parameters, \( b, d_{j-1}, \) and \( \beta_j \), are coefficients for the dissipative terms. Detailed discussions for each of the above terms can be found in Shen (2019a). Variable \( X \) denotes the amplitude of the Fourier mode for the streamfunction. Variables \( (Y, Z), (Y_1, Z_1), (Y_2, Z_2), \) and \( (Y_3, Z_3) \) are referred to as the primary, secondary, tertiary, and quaternary modes, respectively, and represent the amplitudes of the Fourier modes at different wavenumbers for temperature. The GLM with \( M = 5, 7, \) or \( 9 \) is referred to as the 5DLM, 7DLM, or 9DLM, respectively, and the classical 3DLM can be obtained using Eqs. (ES1)–(ES3) without the nonlinear term \(-XY_1\), written as follows:

\[
\frac{dX}{d\tau} = \sigma Y - \sigma X, \quad (\text{ES7})
\]
Here, it should be noted that although the Lorenz 1963 model was originally derived from the Saltzman model with seven ODEs (e.g., Saltzman, 1962; Lorenz, 1993), the Lorenz 1963 model, with its simplicity, has been used as a powerful tool for revealing chaotic solutions. Additionally, it can be shown that the analytical solution of homoclinic orbits within the nondissipative 3DLM is mathematically identical to the solitary solution of the Korteweg–de Vries (KdV) equation in traveling-wave coordinates (Shen 2018, 2020).

\[
\frac{dY}{d\tau} = -XZ + rX - Y, \tag{ES8}
\]

\[
\frac{dZ}{d\tau} = XY - bZ. \tag{ES9}
\]

References


