Supplemental Material for 'Scalings for submarine melting at tidewater glaciers from buoyant plume theory'

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S1 Dropping melt and drag terms from plume equations

In the main article we quote a scaling for when the melt and drag terms in Eqs. (1a)-(1d) of the main article become important. These are derived here.

S1.1 Melt terms

We first estimate the height on the calving front Z_m at which the volume added to the plume from melting becomes comparable to the initial volume flux Q_{0m}:

$$\int_0^{Z_m} 2b\dot{m}\, dz \approx Q_{0m} \tag{S1}$$

Since both submarine melt and the initial volume flux have zero salinity, this provides an estimate of when the buoyancy added to the plume by submarine melt becomes comparable to the initial buoyancy flux.

At a first level of approximation, the integral may be evaluated using the solution to the plume equations excluding melt and drag terms (i.e. using Eqs. (6a)-(6c) of the main article), and by taking the point source limit (i.e. neglecting the finite source correction). We also use

$$\dot{m} = \frac{c_w C_d^{1/2} \Gamma_T u (T - T_b)}{c_i(T_b - T_i) + L} \approx \frac{c_w C_d^{1/2} \Gamma_T u (T - T_b)}{L} \approx \frac{c_w C_d^{1/2} \Gamma_T u (T_a - T_0)}{L} \tag{S2}$$

where the first approximation follows as $L \gg c_i(T_b - T_i)$, and the second approximation holds (roughly) for $z \gg z_0$ as in this regime $T \to T_a$. The integral may then be evaluated as

$$\int_0^{Z_m} 2b\dot{m}\,dz = \frac{2c_w C_d^{1/2}\Gamma_T(T_a - T_0)}{L}\left(\frac{9\alpha Q_{0m} g_0'}{5\pi}\right)^{1/3}\int_0^{Z_m} z^{2/3}\,dz \tag{S3}$$

By equating to the initial volume flux as in Eq. (S1) we obtain a relationship between the height on the calving front Z_m and the subglacial discharge Q_{0m} at which total submarine melt flux is similar to the initial volume flux

$$Q_{0m}^{2/3} = \frac{6}{5}\frac{c_w C_d^{1/2}\Gamma_T}{L}\left(\frac{9\alpha g_0'}{5\pi}\right)^{1/3}(T_a - T_0)\,Z_m^{5/3} \tag{S4}$$

S1.2 Drag term

The right hand side of Eq. (1b) of the main article may be written

$$\frac{\pi}{2}b^2 g' - 2C_d b u^2 = \frac{\pi}{2}b^2 g' - 2C_d b^2 g' \frac{u^2}{bg'} \approx \frac{\pi}{2}b^2 g' - 2C_d b^2 g' \frac{5}{8\alpha} = \frac{\pi}{2}b^2 g'\left(1 - \frac{5C_d}{2\pi\alpha}\right) \tag{S5}$$

Where the second equality follows from the definition of Γ in the main article. We see that including drag reduces plume buoyancy by a factor $5C_d/2\pi\alpha \approx 0.08$, and the effect is equivalent to reducing g by the same factor. Therefore an approximate analytical solution to the plume equations with a drag term can be obtained simply by replacing g with $(1 - 5C_d/2\pi\alpha)g$ throughout Eqs. (6a)-(6c) of the main article. Considering particularly plume velocity in the point source limit, this replacement introduces a factor

$$(1 - 5C_d/2\pi\alpha)^{1/3} \approx 1 - 5C_d/6\pi\alpha \tag{S6}$$

Therefore the presence of the drag term reduces plume velocity by $\sim$2.5%. The magnitude of change is such that we can safely neglect the drag term in the plume equations.

S2 Mathematical details from the uniform stratification solution

S2.1 Properties of T_b

In the main article we state a number of results regarding T_b; these are derived here. Consider the melt rate parameterisation Eqs. (2a)-(2c) of the main article. By eliminating $\dot{m}$ from Eq. (2a) using Eq. (2b) and then eliminating T_b in favour of S_b using Eq. (2c) we can obtain a quadratic for S_b

$$a_1 S_b^2 + a_2 S_b + a_3 = 0 \tag{S7}$$

where the coefficients are given by

$$a_1 = \lambda_1(c_i\Gamma_S - c_w\Gamma_T) \tag{S8}$$

$$a_2 = c_w\Gamma_T(T - \lambda_2 - \lambda_3(h - z)) + \Gamma_S(c_i(\lambda_2 + \lambda_3(h - z) - \lambda_1 S - T_i) + L) \tag{S9}$$

$$a_3 = -\Gamma_S S(c_i(\lambda_2 + \lambda_3(h - z) - T_i) + L) \tag{S10}$$

and then S_b is given by the positive branch of the quadratic formula

$$S_b = \frac{1}{2a_1}\left(-a_2 + \sqrt{a_2^2 - 4a_1a_3}\right) \tag{S11}$$

Note that u does not appear in any of the coefficients, it cancels when we substitute Eq. (2b) into Eq. (2a). Therefore $dS_b/du = 0$. Further we use Eq. (2c) to get $dS_b/du = \lambda_1 dT_b/du$, therefore we also have $dT_b/du = 0$.

By direct differentiation of Eq. (S11) we get

$$\frac{dT_b}{dT} = -\frac{\lambda_1 c_w \Gamma_T}{2a_1}\left(1 - \frac{a_2}{\sqrt{a_2^2 - 4a_1a_3}}\right) \tag{S12}$$

For parameters within a reasonable range for glacial fjords (and see also Table S1) it may be shown that $a_1 > 0$, $a_2 > 0$ and $a_3 < 0$, and so $0 < a_2/\sqrt{a_2^2 - 4a_1a_3} < 1$. The prefactor in Eq. (S12) is constant and may be evaluated as $-\lambda_1 c_w\Gamma_T/2a_1 \approx 1/2$ (note $c_i\Gamma_S \ll c_w\Gamma_T$ and $\lambda_1 < 0$), and therefore it follows that $0 < dT_b/dT < 1$, as is stated in the main article.

Lastly, differentiation with respect to S gives (note we use $c_i(T_b - T_i) \ll L$)

$$\frac{dT_b}{dS} = \frac{\lambda_1 \Gamma_S L}{\sqrt{a_2^2 - 4a_1 a_3}} \tag{S13}$$

from which it easily follows (noting $\lambda_1 < 0$) that $dT_b/dS < 0$, which is the final result used in the main article.

S3 Mathematical details from the linear stratification section

S3.1 Non-dimensionalisation of equations

While this procedure has been extensively studied previously (e.g. Morton et al., 1956; Morton, 1959; Turner, 1973) we believe it worthwhile to include a sketch of the non-dimensionalisation of the plume equations and characteristic plume heights as our prefactors differ slightly from others in the literature. We first define plume volume flux $Q = \pi b^2 u/2$, momentum flux $M = \pi b^2 u^2/2$, buoyancy flux $B = \pi b^2 u g'/2$. We define equivalent non-dimensional fluxes $\tilde{Q}$, $\tilde{M}$ and $\tilde{B}$, and non-dimensional depth $\tilde{z}$ by

$$Q = (2\pi)^{1/4} \alpha^{1/2} B_0^{3/4} (N^2)^{-5/8} \tilde{Q} \tag{S14a}$$

$$M = B_0 (N^2)^{-1/2} \tilde{M} \tag{S14b}$$

$$B = B_0 \tilde{B} \tag{S14c}$$

$$z = (2\pi)^{-1/4} \alpha^{-1/2} B_0^{1/4} (N^2)^{-3/8} \tilde{z} \tag{S14d}$$

where B_0 is the initial buoyancy flux of the plume. One can then show with these definitions that the analytical model plume equations reduce to

$$\frac{d\tilde{Q}}{d\tilde{z}} = \tilde{M}^{1/2} \qquad \frac{d\tilde{M}}{d\tilde{z}} = \frac{\tilde{Q}\tilde{B}}{\tilde{M}} \qquad \frac{d\tilde{B}}{d\tilde{z}} = -\tilde{Q} \tag{S15}$$

For a point source of buoyancy only, the relevant initial conditions are $\tilde{Q} = 0$, $\tilde{M} = 0$, $\tilde{B} = 1$ at $\tilde{z} = 0$, starting from which Eqs. (S15) can be solved numerically with solution shown in Fig. S1a. To make contact with the main article define non-dimensional radius by $\tilde{b} = \sqrt{2\tilde{Q}^2/\pi\tilde{M}}$, velocity by $\tilde{u} = \tilde{M}/\tilde{Q}$ and reduced gravity by $\tilde{g}' = \tilde{B}/\tilde{Q}$. The expressions for characteristic plume heights presented in Eqs. (16) of the main article are obtained by combining the solution in Fig. S1a with Eq. S14d and Eq. (6c) of the main article.

S3.2 Total submarine melt rates

In general total submarine melt rates for the plume in a linear stratification are difficult to investigate analytically, but some progress can be made. Following arguments made in the main article, let us take some multiple of plume velocity βu (for constant β) as a proxy for local submarine melt. We can then rewrite total melt in terms of the non-dimensional variables introduced above and in the main article. Consider first the case where the plume does not reach the surface.

$$\dot{M} = 2\beta\,(2\pi)^{-1/4}\,\alpha^{-1/2}\left(Q_0 g_0'\right)^{3/4}(N^2)^{-5/8}\int_{\tilde{z}_0}^{2.57}\tilde{b}\tilde{u}\,d\tilde{z} \tag{S16}$$

If we were considering a point source, then the lower limit of the integral would be 0, such that the integral would be independent of Q_0 and N^2 and then we'd have $\dot{M} \propto Q_0^{3/4}(N^2)^{-5/8}$. Since we in fact have a finite source of subglacial discharge, we need to retain the lower limit $\tilde{z}_0$ which means the integral is not quite independent of Q_0 and N^2. We can split the integral into the point source contribution and the correction for a finite source

$$\int_{\tilde{z}_0}^{2.57}\tilde{b}\tilde{u}\,d\tilde{z} = \int_{0}^{2.57}\tilde{b}\tilde{u}\,d\tilde{z} - \int_{0}^{\tilde{z}_0}\tilde{b}\tilde{u}\,d\tilde{z} \tag{S17}$$

In the main article we noted that for tidewater glacier-relevant parameters we have $z_0 < z_1$ (where z_1 is defined in the main article or using Eq. (S14d) with $\tilde{z} = 1$). In non-dimensional terms this is stated as $\tilde{z}_0 < 1$, and means that the uniform stratification solution $\tilde{b} \propto \tilde{z}$ and $\tilde{u} \propto \tilde{z}^{-1/3}$ can be used to evaluate the second integral

$$\int_{0}^{\tilde{z}_0}\tilde{b}\tilde{u}\,d\tilde{z} \propto \tilde{z}_0^{5/3} \tag{S18}$$

so that the finite source correction term is proportional to $Q_0^{3/4}(N^2)^{-5/8} \cdot Q_0^{1/4}(N^2)^{5/8} = Q_0$. This is an increasing function of Q_0 and therefore its presence slightly reduces the 3/4 exponent which would be obtained taking into account only the first integral. Numerically we find that the second integral is often small compared to the first, therefore a total melt rate-discharge exponent of 3/4 is a good approximation when the plume does not reach the surface.

When the plume does reach the surface we have instead

$$\dot{M} = 2\beta\,(2\pi)^{-1/4}\,\alpha^{-1/2}\,\left(Q_0 g_0'\right)^{3/4}(N^2)^{-5/8}\int_{\tilde{z}_0}^{\tilde{z}_h}\tilde{b}\tilde{u}\,d\tilde{z} \tag{S19}$$

where $\tilde{z}_h = (2\pi)^{1/4}\,\alpha^{1/2}\,(Q_0 g_0')^{-1/4}\,(N^2)^{3/8}(z_0+h)$ is the non-dimensional height corresponding to the fjord surface. This is best investigated numerically as is discussed in the main article, but we can recover the uniform stratification scaling here. Suppose the fjord stratification is sufficiently weak (N^2 close to 0) that we have $\tilde{z}_h < 1$. Then we can evaluate the integral with the uniform stratification solution $\tilde{b} \propto \tilde{z}$ and $\tilde{u} \propto \tilde{z}^{-1/3}$

$$\int_{\tilde{z}_0}^{\tilde{z}_h}\tilde{b}\tilde{u}\,d\tilde{z} \propto \tilde{z}_h^{5/3} - \tilde{z}_0^{5/3} \tag{S20}$$

such that total melt scales as

$$\dot{M} \propto Q_0^{3/4}(N^2)^{-5/8}Q_0^{-5/12}(N^2)^{5/8}\left[(z_0+h)^{5/3} - z_0^{5/3}\right] = Q_0^{1/3}\left[(z_0+h)^{5/3} - z_0^{5/3}\right] \tag{S21}$$

and finally taking the point source limit $h \gg z_0$ we recover the uniform stratification total melt scaling $\dot{M} \propto Q_0^{1/3}h^{5/3}$ as in Eq. (11) of the main article.

S3.3 Derivation of plume temperature contrast for stratification in temperature and salinity

We provide here a derivation of Eq. (18) of the main article. By combining Eqs. (12a) and (12b) of the main article and integrating we obtain

$$\frac{Qg_T' - Q_0 g_{T0}'}{N_T^2} = \frac{Qg_S' - Q_0 g_{S0}'}{N_S^2} \tag{S22}$$

Upon rearranging, noting $g' = g_T' + g_S'$ and using Eq. (S14c) we get

$$Qg_T' = \frac{Q_0(g_{T0}' + g_{S0}')\tilde{B} - Q_0 g_{S0}' + Q_0 g_{T0}' N_S^2/N_T^2}{1 + N_S^2/N_T^2} \tag{S23}$$

At z_{mh} we have $\tilde{B} = -|\lambda|$ where λ is some constant (Fig. S1a). Due to the dominance of salinity in the equation of state we have $g'_{S0}/g'_{T0} \sim 250$. By considering the limits of Eq. (S23) one can show that for $10 < |N_S^2/N_T^2| < 50$ (which covers glacial applications) we have

$$Qg'_T \approx -(1+|\lambda|)\frac{N_T^2}{N_S^2}Q_0 g'_{S0} \tag{S24}$$

giving

$$T_a - T \approx -(1+|\lambda|)\frac{dT_a/dz}{dS_a/dz}\frac{Q_0}{Q}(S_{a,0} - S_0) \tag{S25}$$

which is proportional to $Q_0^{1/4}(N_T^2)(N_S^2)^{-3/8}$ using Eqs. (S14a)-(S14d) and again noting $N^2 \approx N_S^2$.

S4 Line plumes

S4.1 Uniform stratification

In the line plume case the plume is wedge-shaped (Fig. S2) with thickness b. The defining equations which are the equivalent expressions to Eqs. (1a)-(1d) of the main article can be found in Jenkins (2011).

In the line plume case the dimensionless number characterising the balance between buoyancy and momentum of the plume is given by $\Gamma = bg'/\alpha u^2$. As before we fix b_0 and u_0 by setting $Q_0 = b_0 u_0$ and $\Gamma_0 = 1$, giving

$$b_0 = \left(\frac{\alpha Q_0^2}{g'_0}\right)^{1/3} \qquad u_0 = \left(\frac{Q_0 g'_0}{\alpha}\right)^{1/3} \tag{S26}$$

Neglecting melt and drag terms as before and specialising to a uniform stratification, the plume solution is given by (e.g. Linden et al., 1990; Kaye, 2008; Straneo and Cenedese, 2015)

$$b = \alpha\left[z + z_0\right] \qquad u = \left(\frac{Q_0 g'_0}{\alpha}\right)^{1/3} \tag{S27a}$$

$$T = T_0 + (T_a - T_0)\left(1 - \frac{z_0}{z+z_0}\right) \qquad S = S_a\left(1 - \frac{z_0}{z+z_0}\right) \tag{S27b}$$

The finite source correction $z_0 = \left(Q_0^2/\alpha^2 g'_0\right)^{1/3}$ may be obtained by ensuring $Q(z=0) = Q_0$. Regarding submarine melting, Fig. S3 is the equivalent of Fig. 4 of the main article, and shows the result of using Eq. (20) of the main article to predict total submarine melt rates. Points with $h \lesssim 70z_0$ (e.g. $Q_0 = 2.1$ m^3 s^{-1}, $h = 100$ m) have been excluded as the total melt rate-discharge exponent is not well approximated by 1/3 for these points.

S4.2 Linear stratification

Again while this procedure has been discussed extensively in the literature (e.g. Wright and Wallace, 1979; Bush and Woods, 1999) we believe it worth providing an outline here. Define flux quantities for volume $Q = bu$, momentum $M = bu^2$ and buoyancy $B = bug'$. Define non-dimensional equivalents by

$$Q = \alpha^{1/3} B_0^{2/3} (N^2)^{-1/2} \tilde{Q} \tag{S28a}$$

$$M = B_0 (N^2)^{-1/2} \tilde{M} \tag{S28b}$$

$$B = B_0 \tilde{B} \tag{S28c}$$

$$z = \alpha^{-1/3} B_0^{1/3} (N^2)^{-1/2} \tilde{z} \tag{S28d}$$

where $B_0 \approx Q_0 g_0'$ is the initial buoyancy flux per unit width. The plume equations then reduce to

$$\frac{d\tilde{Q}}{d\tilde{z}} = \frac{\tilde{M}}{\tilde{Q}} \qquad \frac{d\tilde{M}}{d\tilde{z}} = \frac{\tilde{Q}\tilde{B}}{\tilde{M}} \qquad \frac{d\tilde{B}}{d\tilde{z}} = -\tilde{Q} \tag{S29}$$

For an infinitessimal line source of buoyancy only, the relevant initial conditions are $\tilde{Q} = 0$, $\tilde{M} = 0$, $\tilde{B} = 1$ at $\tilde{z} = 0$, starting from which Eqs. (S29) can be solved numerically with solution shown in Fig. S1b. To make contact with the main article define non-dimensional width by $\tilde{b} = \tilde{Q}^2/\tilde{M}$, velocity by $\tilde{u} = \tilde{M}/\tilde{Q}$ and reduced gravity by $\tilde{g}' = \tilde{B}/\tilde{Q}$. Expressions for characteristic line plume heights in a linear stratification are (e.g. Wright and Wallace (1979); Bush and Woods (1999), see also Fig. S1b)

$$z_{mh} = 2.09\,(N^2)^{-1/2} \left(\frac{Q_0 g_0'}{\alpha}\right)^{1/3} - \left(\frac{Q_0^2}{\alpha^2 g_0'}\right)^{1/3} \tag{S30a}$$

$$z_{br} = 1.44\,(N^2)^{-1/2} \left(\frac{Q_0 g_0'}{\alpha}\right)^{1/3} - \left(\frac{Q_0^2}{\alpha^2 g_0'}\right)^{1/3} \tag{S30b}$$

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Symbol	Description	Value	Units
α	entrainment coefficient	0.1	-
g	gravitational acceleration	9.81	$\mathrm{m\,s^{-2}}$
C_d	plume-ice drag coefficient	9.7×10^{-3}	-
Γ_T	heat transfer coefficient	1.1×10^{-2}	-
Γ_S	salt transfer coefficient	3.1×10^{-4}	-
c_w	heat capacity of water	3974	$\mathrm{J\,kg^{-1}\,{}^\circ C^{-1}}$
c_i	heat capacity of ice	2009	$\mathrm{J\,kg^{-1}\,{}^\circ C^{-1}}$
L	latent heat of ice melt	334000	$\mathrm{J\,kg^{-1}}$
T_i	ice temperature	-10	$^\circ\mathrm{C}$
λ_1	freezing point salinity slope	-5.73 $\times 10^{-2}$	$^\circ\mathrm{C\,psu^{-1}}$
λ_2	freezing point offset	8.32×10^{-2}	$^\circ\mathrm{C}$
λ_3	freezing point depth slope	-7.53 $\times 10^{-4}$	$^\circ\mathrm{C\,m^{-1}}$
β_S	haline contraction coefficient	7.86×10^{-4}	$\mathrm{psu^{-1}}$
β_T	thermal expansion coefficient	3.87×10^{-5}	$^\circ\mathrm{C^{-1}}$

Table S1: Parameter values used in this study.

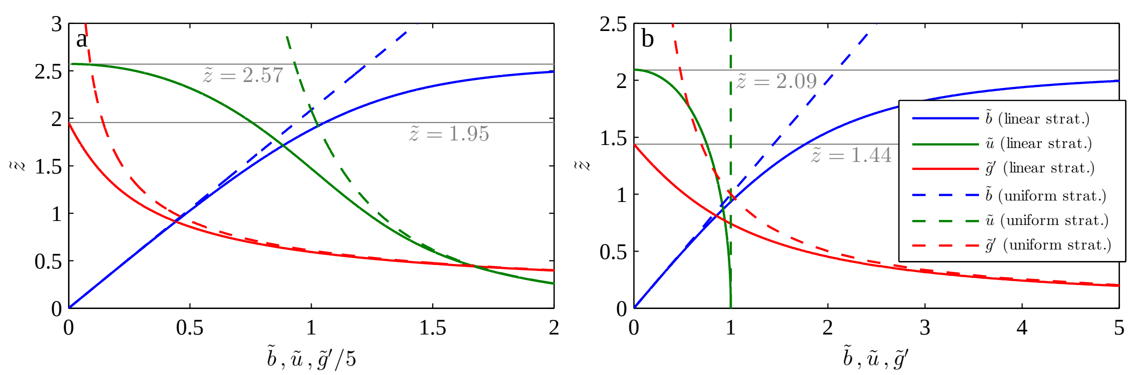

Figure S1: Solution to the non-dimensional plume equations for (a) half-conical plume geometry and (b) line plume geometry. Solid lines are the linear stratification solution, dashed lines are the uniform stratification solution. Note that (a) differs from Fig. 1 of Morton et al. (1956) and Fig. 6.17 of Turner (1973) only in the prefactors of the non-dimensional variables.

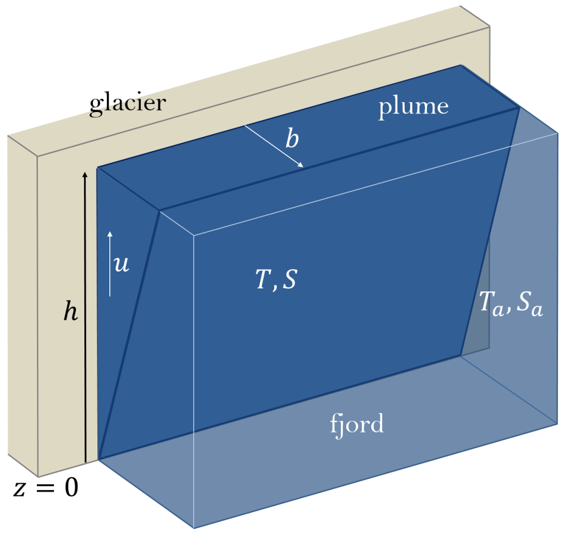

Figure S2: Line plume geometry. In this case b refers to the thickness of the plume.

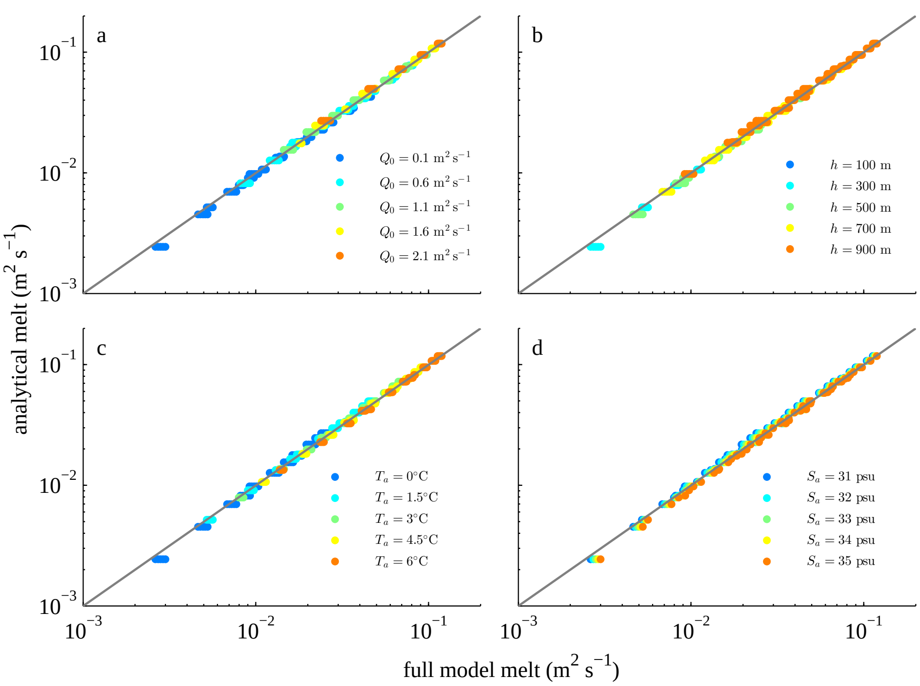

Figure S3: Equivalent of Fig. 4 of the main article for the line plume. Comparison of total submarine melt per unit width as predicted by the full model and by Eq. (20) of the main article. Points with $h \lesssim 70 z_0$ (e.g. $Q_0 = 2.1$ m^3 s^{-1}, $h = 100$ m) have been excluded as the total melt rate-discharge exponent is not well approximated by 1/3 for these points.