A Note on the Utility of Probabilistic Predictions and the Probability Score in the Cost-Loss Ratio Decision Situation\(^1,2\)

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1. The cost-loss ratio decision situation

**Standard framework.** Consider the cost-loss ratio decision situation in which a decision maker must decide whether or not to protect an operation in the face of uncertainty as to whether or not weather adverse to the operation obtains. Let the cost of protective measures, whether or not adverse weather obtains, be \(C\) and the loss incurred if protective measures are not taken and adverse weather obtains (does not obtain) be \(L\) (naught). The situation is summarized in a cost-loss matrix in Table 1. Let \(p(1-p)\) be the probability that adverse weather obtains (does not obtain). Suppose the decision maker selects that course of action, i.e., \(P\) (protect) or \(DNP\) (do not protect), which minimizes the kernel\(^2\) expense \(KE\), where

\[
KE(P) = pC + (1-p)C = C
\]

and

\[
KE(DNP) = pL.
\]

Then the decision maker acts in accordance with the decision rule described in Table 2 in which the ratio \(C/L\), the value of which provides a complete description of the decision situation, is the cost-loss ratio.

Note that a decision situation does not exist unless \(0 < C/L < 1\), for in a situation in which \(C/L \geq 1\) the decision maker selects course of action \(DNP\) (\(P\) if \(C\) is zero or negative and \(DNP\) if \(L\) is negative) regardless of the probability \(p\), i.e., the decision maker’s selection of a course of action does not depend on whether \(W\) (adverse weather) or \(NW\) (no adverse weather)

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<th>Table 1. Cost-loss matrix.</th>
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<td>Protect ((P))</td>
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<td>Do not protect ((DNP))</td>
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<th>Table 2. Decision rule.</th>
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<td>(P), (DNP)</td>
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<td>(DNP)</td>
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<th>Table 3. Equivalent utility matrix.</th>
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<td>(W), (NW)</td>
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<td>(C)</td>
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<tr>
<td>(L)</td>
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obtains. Therefore, the range of the cost-loss ratio \(C/L\), of concern, is the open unit interval \((0, 1)\).

**Utility framework.** Suppose the monetary expenses in the standard framework are negative utilities\(^4\), i.e., suppose the negative utility scale and the positive monetary expense scale are equivalent\(^5\). Now the utility scale is unique only up to a positive linear transformation (e.g., Fishburn, 1964, pp. 6-7), i.e., the origin and unit of measurement of the utility scale are arbitrary. Thus, the cost-loss matrix may be transformed into an equivalent utility matrix which appears in Table 3. The decision maker who selects that course of action which minimizes the kernel expense \(KE\) in the standard framework selects that course of action which maximizes the kernel utility \(KU\), where

\[
KU(P) = p(1 - C/L) + (1 - p)(1 - C/L) = 1 - C/L,
\]

and

\[
KU(DNP) = 1 - p,
\]

in the utility framework. The decision rule which the

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\(^3\) The term kernel is used instead of the standard term expected, when the expense, and subsequently the utility, of a course of action or prediction is considered, in order to forestall the subsequent appearance of the phrase 'the expectation of the average expected utility'.

\(^4\) Utilities express a decision maker’s preferences among consequences (a consequence is that which results when a decision maker selects a course of action and a state of nature obtains) in a particular decision situation. For an introduction to utility theory and a description of the role of utility in decision theory refer to Fishburn (1964) or Luce and Raiffa (1957).

\(^5\) The assumption of equivalence of the positive monetary expense scale and the negative utility scale is equivalent, since the utility scale is unique only up to a positive linear transformation [e.g., Fishburn (1964), pp. 6-7], to an assumption that utility is linear in monetary expense, a necessary and sufficient assumption if we wish to ensure the equivalence of the cost-loss ratio decision situation in the standard and utility frameworks. Such an assumption is, in general, unnecessary since a decision maker's utilities are measured directly [e.g., Fishburn (1964), pp. 77-130].
decision maker follows in this framework is the same as that which the decision maker follows in the standard framework (refer to Table 2).

2. The utility of predictions

The point of view in Section 1 is that of a decision maker whose concern is decision, i.e., the selection of the appropriate course of action. Now the point of view is that of an individual, e.g., a meteorologist, whose concern is evaluation, i.e., the determination of the utility of predictions to a decision maker.

Kernel utility of predictions. The kernel utility of a prediction to a decision maker who acts in accordance with the decision rule is \( U \), where

\[
U = (1 - C/L) d(p, C/L) + \delta d(C/L, p),
\]

(1)

where

\[
d(x, y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases},
\]

and

\[
\delta = \begin{cases} 1 & \text{if } NW \text{ obtains} \\ 0 & \text{if } W \text{ obtains} \end{cases}.
\]

The average kernel utility for \( N \) predictions is \( U_N \), where

\[
U_N = \frac{1}{N} \sum_n [(1 - C/L) d(p_n, C/L) + \delta d(C/L, p_n)],
\]

\(n = 1, \ldots, N\),

(2)

where \( p_n \) is the probability that adverse weather obtains on occasion \( n \) and \( \delta_n \) is zero (one) if adverse weather obtains (does not obtain) on occasion \( n \).

Expectation of kernel utility of predictions. An individual can determine the kernel utility of a prediction \( U \) or the average kernel utility of \( N \) predictions \( U_N \) to a decision maker if the individual's knowledge of the decision maker's cost-loss ratio \( C/L \) is complete [refer to Eqs. (1) and (2)]. However, an individual's knowledge of a particular decision maker's cost-loss ratio is, in general, incomplete. The appropriate treatment of the cost-loss ratio in such a situation is to treat the cost-loss ratio as a random variable\(^6\). A probability distribution for such a random variable which is simple, and, moreover, the use of which leads to results of interest, is the uniform distribution\(^7\). Thus, let \( C/L \) be a random variable with a uniform probability distribution on the open unit interval, i.e., let \( C/L \) be a

random variable with a probability density function

\[
f(C/L) = \begin{cases} 1, & 0 < C/L < 1 \\ 0, & \text{otherwise} \end{cases}.
\]

Then the expectation of the kernel utility of a prediction is \( E(U) \), where

\[
E(U) = \int_{-\infty}^{\infty} U f(C/L) d(C/L),
\]

or

\[
E(U) = \int_{0}^{1} [ (1 - C/L) d(p, C/L) + \delta d(C/L, p) ] d(C/L),
\]

or

\[
E(U) = \int_{0}^{1} \int_{\delta}^{1} d(C/L) d(C/L) + \delta d(C/L, p),
\]

or

\[
E(U) = p - (1/3) p^2 + \delta (1 - p).
\]

The expectation of the average kernel utility for \( N \) predictions is \( E(U_N) \), where

\[
E(U_N) = \frac{1}{N} \sum_n [ p_n - (1/2) p_n^2 + \delta_n (1 - p_n) ],
\]

\(n = 1, \ldots, N\).

(3)

3. The utility of predictions and the probability score

The probability score (Brier, 1950) for a prediction is \( PS \), where

\[
PS = [ p - (1 - \delta)^2 ] + [(1 - p) - \delta]^2,
\]

or

\[
PS = 2 \{ p - (1 - \delta)^2 \}.
\]

The probability score for \( N \) predictions is \( PS_N \), where

\[
PS_N = \frac{1}{N} \sum_n \{ [ p_n - (1 - \delta)^2 ] + [(1 - p_n) - \delta_n]^2 \},
\]

\(n = 1, \ldots, N\),

or

\[
PS_N = \frac{2}{N} \sum_n [ p_n - (1 - \delta_n)^2 ], \quad n = 1, \ldots, N.
\]

(4)

\(^6\) The treatment of the cost-loss ratio as a random variable is appropriate even if an individual's knowledge of the cost-loss ratio for particular decision situations is complete if the individual must simultaneously consider the evaluation of a prediction for a large number of decision situations with different cost-loss ratios.

\(^7\) The uniform distribution is a particularly appropriate distribution for the cost-loss ratio if an individual does not possess any knowledge of the decision maker's utilities in a particular decision situation, i.e., if the probability that the decision maker's cost-loss ratio lies in any sub-interval of the open unit interval is equal to the length of that sub-interval, or if an individual must simultaneously consider a very large number of decision situations with many different cost-loss ratios (knowledge of which on the part of the individual may or may not be complete), such that the cost-loss ratios can be assumed to be uniformly distributed on the open unit interval.
The expectation of the average kernel utility for \( N \) predictions [refer to Eq. (3)] may be expressed as follows:

\[
E(U_N) = -(1/2N) \sum_{n=1}^{N} \left[ \rho_n^2 - 2\rho_n(1-\delta_n) - 2\delta_n \right],
\]

or

\[
E(U_N) = -(1/2N) \sum_{n=1}^{N} \left[ \rho_n - (1-\delta_n) \right]^2 + (1/2)(1+\delta),
\]

where

\[
\delta = (1/N) \sum_{n=1}^{N} \delta_n.
\]

Therefore, from Eq. (4),

\[
E(U_N) = -(1/4)PS_N + (1/2)(1+\delta).
\]

Note that Eq. (5) states that the expectation of the average kernel utility for \( N \) predictions depends only on the probability score \( PS_N \) and the sample-of-\( N \) climatological probability of adverse weather \( 1-\delta \).

The range of the expectation of the average kernel utility \( E(U_N) \) is the closed unit interval \([0,1]\), i.e., \( 0 \leq E(U_N) \leq 1 \). \( E(U_N) \) equals one when \( PS_N \) equals zero and \( 1-\delta \) equals zero, i.e., when each prediction is completely valid* and adverse weather never obtains. \( E(U_N) \) equals zero when \( PS_N \) equals two and \( 1-\delta \) equals one, i.e., when each prediction is completely invalid and adverse weather always obtains.

4. The relative utility of predictions and climatological predictions

Consider a comparison of the expectation of the average kernel utility for \( N \) predictions \( E(U_N) \) and the expectation of the average kernel utility for \( N \) climatological predictions \( E(U_N) \). Let \( \rho(1-\rho) \) be the climatological probability that adverse weather obtains (does not obtain). Then,

\[
E(U_N) = \rho^2 - (1/2)(\rho^2 + \delta(1-\rho)).
\]

Now, the utility of the predictions is greater than the utility of the climatological predictions if

\[
E(U_N) - E(U_N^*) > 0,
\]

i.e., if

\[
PS_N \leq 2(1-\delta - \rho^2) + 2\delta(1-\delta).
\]

For a collection of \( N \) occasions the value assumed by the right hand side of inequality (6) represents an upper bound on the range of values which the probability score can assume for the utility for the \( N \) predictions to be greater than or equal to the utility for \( N \) climatological predictions. The values of the upper bound for selected values of \( \rho \) and \( 1-\delta \) are presented in Table 4. Note that the first term on the right hand side of (6) is twice the square of the difference between the sample-of-\( N \) climatological probability of adverse weather \( 1-\delta \) and the climatological probability of adverse weather \( \rho^2 \). The difference between \( 1-\delta \) and \( \rho^2 \) is, in general, small, particularly if \( N \) is large. Thus the second term on the right hand side of (6), i.e., \( 2\delta(1-\delta) \), represents a least upper bound (for a particular collection of \( N \) occasions). The values of the least upper bound are, of course, located on the principal diagonal in Table 4 and are italicized.

5. Summary

An expression has been developed which relates the utility and the validity of probabilistic predictions for the cost-loss ratio decision situation in which knowledge of the decision maker's utilities is incomplete (the knowledge of the decision maker's utilities is assumed to take a specific, simple form). The expression indicates that the utility of \( N \) predictions in such a situa-

<table>
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<tr>
<th>Sample-of-( N ) climatological probability of adverse weather ( 1-\delta )</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
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<td>0.0</td>
<td>0.00</td>
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<td>0.40</td>
<td>0.60</td>
<td>0.80</td>
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<td>1.60</td>
<td>1.80</td>
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<td>0.1</td>
<td>0.02</td>
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<td>0.82</td>
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* The probability score is a measure of the attribute validity (Epstein and Murphy, 1965).
tion depends only on the probability score and the sample-of-N climatological probability of adverse weather. In addition, an expression has been obtained for this situation which indicates the range of values of the probability score for which the predictions are of greater utility than climatological predictions.

The relationship between the natural measure of the "value" of probabilistic predictions, i.e., utility, and artificial measures of the "value" of probabilistic predictions, e.g., the probability score, for cost-loss ratio decision situations in which knowledge of the decision maker's utilities assumes different forms and for more complex decision situations is considered elsewhere (Murphy, 1966).

REFERENCES