On the “Ranked Probability Score”

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In the preceding note Epstein (1969) has described a new measure, or scoring rule, for evaluating probability forecasts, the “ranked probability score.” The author of this note believes that meteorologists will find this scoring rule of considerable interest and, as a result, the nature and properties of this rule merit particular attention. In this note we prove, as stated by Epstein, that the “ranked probability score” is a proper scoring rule.

Let the (row) vectors \( r=(r_1, \ldots, r_K) \) and \( p=(p_1, \ldots, p_K) \) denote the meteorologist's statement (forecast) and judgment (belief), respectively, for a predictand subjected to a \( K \)-fold partition (\( K \) mutually exclusive and collectively exhaustive categories). The “ranked probability score” when category \( j \) obtains is \( S_j \), where

\[
S_j = \frac{1}{2(K-1)} \sum_{i=1}^{K-1} \left[ \left( \sum_{n=1}^{i} r_n \right)^2 + \left( \sum_{n=i+1}^{K} r_n \right)^2 \right] - \frac{1}{K-1} \sum_{i=1}^{N} |i-j|r_i
\]

(1)

Then, the meteorologist's (subjective) expected score is \( E(S) \), where

\[
E(S) = \sum_{j=1}^{K} p_j S_j
\]

or, from (1),

\[
E(S) = \sum_{j=1}^{K} p_j \left[ \frac{3}{2} - \frac{1}{2(K-1)} \sum_{i=1}^{K-1} \left[ \left( \sum_{n=1}^{i} r_n \right)^2 + \left( \sum_{n=i+1}^{K} r_n \right)^2 \right] - \frac{1}{K-1} \sum_{i=1}^{N} |i-j|r_i \right]
\]

\[
E(S) = -\frac{1}{K-1} \sum_{j=1}^{K} \sum_{i=1}^{N} |j-i| r_j
\]

(2)

Let \( F(r) \) denote \( E(S) \) in (2),

\[
G(r) = \sum_{i=1}^{K} r_i - 1,
\]

and

\[
H(r) = F(r) + \lambda G(r),
\]

(3)

where \( \lambda \) is a Lagrange multiplier. Then, maximizing \( H(r) \) in (3) is equivalent to maximizing the expected score \( E(S) \) in (2). Thus, taking the derivative of \( H(r) \) with respect to \( r_k \) (say) and setting the derivative equal to zero, we obtain

\[
\frac{\partial H(r)}{\partial r_k} = -\frac{1}{K-1} \sum_{j=1}^{K} \left[ (K-1) - |k-j| \right] r_j + \frac{1}{K-1} \sum_{j=1}^{K} |k-j| p_j + \lambda,
\]

(4)

or

\[
\frac{\partial H(r)}{\partial r_k} = -1 + \frac{1}{K-1} \sum_{j=1}^{K} |k-j| r_j + \frac{1}{K-1} \sum_{j=1}^{K} |k-j| p_j + \lambda,
\]

(5)

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\[3\] A paper in which the nature and properties of the “ranked probability score” are considered in some detail is in preparation (Murphy, 1969).
or

\[ \frac{1}{K-1} \sum_{j=1}^{K} |k-j| (r_j - \rho_j) - 1 + \lambda = 0, \quad (k = 1, \ldots, K). \] (5)

Now, (5) implies that

\[ \sum_{j=1}^{K} |k-j| (r_j - \rho_j) = (K-1)(1-\lambda), \quad (k = 1, \ldots, K). \]

Let \( a_j = r_j - \rho_j \) and \( \epsilon = (K-1)(1-\lambda) \). Then,

\[ \sum_{j=1}^{K} |k-j| a_j = \epsilon, \quad (k = 1, \ldots, K). \] (6)

Note that (6) is satisfied if and only if \( a_j = 0 \) for all \( j \). In order to prove this statement we eliminate \( \epsilon \) in (6) between two adjacent equations. Then,

\[ \sum_{j=1}^{K} |k-j| a_j = \sum_{j=1}^{K} |k+1-j| a_j, \quad (k = 1, \ldots, K-1), \]

or

\[ \sum_{j=1}^{k} a_j = \sum_{j=k+1}^{K} a_j, \quad (k = 1, \ldots, K-1). \]

Now, since

\[ \sum_{j=1}^{K} a_j = 0, \quad \left( \sum_{j=1}^{K} r_j = \sum_{j=1}^{K} \rho_j = 1, \right) \]

\[ 2 \sum_{j=1}^{k} a_j = 0, \quad (k = 1, \ldots, K-1). \]

Then, \( a_j = 0 \) for all \( j \), i.e., (6) is satisfied if and only if \( r_j = \rho_j \) for all \( j \). Thus, in order to maximize\( ^{6} \) his expected score \( E(S) \) in (2) the meteorologist should make his statement (r) correspond to his judgment (p), i.e., he should not “hedge.” Therefore, the “ranked probability score” \( S \) in (1) is a proper scoring rule.

REFERENCES


\(^{6}\) Note, from (4), that \( \partial H_r / \partial r^2 = -1 \).