A New Analysis to Diagnose Ageostrophic Winds from Wind and Temperature Measurements Made by an Observational Network

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ABSTRACT

The present paper is devoted to a new analysis of winds and temperature measurements made by an observational network in order to provide the 3D ageostrophic circulations. This analysis may be used with data from instrumented networks providing wind measurements such as Doppler radars or ST (stratospheric-tropospheric) radars networks, as long as thermodynamic data are available. This analysis referred to as AVAG (analyse du vent ageostrophique) is mathematically described. An application of the method to simulated and real data is also presented. The method consists in expanding the three ageostrophic wind components in terms of orthogonal functions. Physical constraints on the geostrophic part and on the ageostrophic part of the wind are also included in the analysis as variational constraints in order to improve the results. The data of any number of stations may be included in the analysis without reformulating the problem (allowing to deal with inhomogeneous datasets where, for example, the number of dynamic and thermodynamic measurements are different). This analysis may be also used in order to obtain the detailed structure of the temperature field from wind measurements provided that a few temperature measurements are available. In this case the analysis is used as an interpolation method (based on physical constraints) between sparse temperature measurements using the available dynamical information (case of a network that involves, for example, ST radars and radiosounding stations providing continuous wind measurements and 12-h temperature measurements).

I. Introduction

Numerous studies have been devoted to the kinematic structure of the troposphere in the presence of intense upper-level waves and jet-streak phenomena or in the presence of low-level frontogenesis. Some of them (i.e., Namas and Clapp 1949; Uccellini and Johnson 1979) attempt to investigate the transverse vertical circulations associated with these upper-tropospheric jet streaks, their role in the coupling of upper-level and low-level frontogenesis and their effects related to the cyclogenesis (Brill et al. 1985; Uccellini et al. 1987). Others scrutinize the ageostrophic circulation associated with low-level frontogenesis (i.e., Hoskins 1982; Emanuel 1985; Thorpe and Emanuel 1985; Xu 1989; Huang and Emanuel 1991) and the role of mesoscale and small-scale processes on these circulations. Some attempts (i.e., Lagouvardos et al. 1992) have been made to retrieve these mesoscale circulations from real data using the Sawyer–Eliassen equation under various forms based on either the assumption of alongfront geostrophy, known as the geostrophic momentum approximation (Hoskins and Bertherton 1972), or on the primitive equation formulation, where an ageostrophic alongfront motion is admitted (Keyser and Pecnick 1985). However, the experimental study of the mesoscale dynamical processes that lead to these ageostrophic winds has been generally difficult because of the limited number of instruments and tools allowing a detailed and accurate description of the associated 3D mesoscale circulation. Thus, it appears important to develop new tools in order to investigate the 3D ageostrophic circulations associated with jet streaks in the upper atmosphere or associated with low-level coldfrontal systems.

Given the temporal resolution and accuracy of the systematic wind measurements (provided by wind profilers, Doppler radars, etc.) and temperature measurements [provided by radio acoustic sounding system (RASS) facilities], which will be certainly implemented in future networks for short-time weather prediction, we chose to develop an analysis based on the equation of motion. This analysis differs from the previous ones (i.e., Zamora et al. 1987) by a variational formulation with physical constraints included in the process, and by a functional expansion of the ageostrophic wind components allowing an analytical representation of them and thus a filtering effect. This analysis is based on the same technique as that in the MANDOP (multiple analytical Doppler) analysis (Scialom and Lemaître 1990) for the retrieval of three-dimensional mesoscale wind fields from ground-based multiple-Doppler radar (Lemaître and Scialom 1992, 1994) or

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from airborne radar (Dou 1993). The present paper describes this analysis referred to as AVAG (analyse du vent agéostrophique) in the following.

Section 2 gives the mathematical formulation of the AVAG analysis with special emphasis on the analytical and variational aspects. The adequacy of this analytical variational approach to retrieve the ageostrophic circulations is evidenced through applications to simulated (section 3) and real (section 4) data. Section 5 gives possible extensions or particular applications allowed by the matricial form of this analysis.

2. The AVAG analysis

a. Principle

As pointed out in the introduction, this analysis consists of expressing each of the searched ageostrophic wind components as a product of three expansions in terms of orthonormal functions series, each of these expansions in turn depending upon each spatial coordinate. The coefficients that define the analytical forms of these three wind components are retrieved by means of a variational adjustment of their analytical forms to the observed ones (or to the components calculated from observations through the equation of motion). The anelastic continuity equation (for the geostrophic and the ageostrophic wind components) and the thermal wind equation, which are simultaneously satisfied by the analytical form, are included in the process after rewriting these equations. The AVAG analysis thus allows the three components of the ageostrophic wind to be analytically obtained under physical constraints. We will consider two cases depending on the instruments involved in the network: the case with (hereafter referred to as WAVAG) and the case without (hereafter referred to as WOAVAG) measured vertical velocities. In the second case a lower kinematic boundary condition (which must be satisfied by the analytical form of the vertical velocity) is included in the process in order to be able to retrieve the vertical wind field.

The basic equations for the WAVAG case and the analytical formulation of the wind field will be, respectively, described in subsections 2b and 2c, and the role of the different terms of this variational analysis will be examined in subsections 2d and 2e, namely, the wind adjustment (2d), and the additional constraints (2e) associated with the continuity equations and the thermal wind equations. The variational formulation of the analysis in the WAVAG case is described in subsection 2f. The changes in the variational formulation of this analysis for the WOAVAG case will be given in the subsection 2g. The introduction of any other additional constraints and remarks concerning the application of the method will be considered in subsections 2h and 2i, respectively.

b. Basic equations for the WAVAG case

Rewriting the equation of motion, the continuity equation, and the thermal wind relations in order to separate the ageostrophic part of the wind and the total wind measured by the network yields the following basic set of equations:

\[ f u_a = - \frac{Dv}{Dt} + \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right), \]  
\[ f v_a = \frac{Dv}{Dt} - \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right), \]  
\[ w_a = w, \]  
\[ \rho \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} \right) = -w \frac{\partial \rho}{\partial z}, \]  
\[ \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \]  
\[ \frac{\partial u_a}{\partial z} = \frac{\partial u}{\partial z} + \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial y}, \]  
\[ \frac{\partial v_a}{\partial z} = \frac{\partial v}{\partial z} - \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial x}, \]

with \( K, \rho, f, g, \theta_0, \) the eddy viscosity, the air density, the Coriolis parameter, the gravitational acceleration, and a reference temperature, respectively.

The first three equations are the main equations that allow us to estimate the ageostrophic winds \((u_a, v_a, w_a)\). The other equations are the physical constraints that these ageostrophic winds must verify. These constraints are applied to the horizontal [Eqs. (4) and (5)] and the vertical [Eqs. (6) and (7)] derivatives of the ageostrophic winds.

The right-hand terms of these equations are experimental and deduced from the wind and temperature measurements. They necessitate the estimation of the total wind, its temporal \((D/Dt)\), and spatial variations \((\partial/\partial x, \partial/\partial y, \partial/\partial z)\) and the determination of the horizontal gradient of potential temperature \(\theta\). This evidence the requirement of a network of radiosounding stations (or other instruments such as RASS systems) to obtain these temperature gradients. The wind gradients should be estimated thanks to a rawinsonde network, an ST (stratospheric–tropospheric) radar network, or thanks to Doppler radars performing conical scannings.

c. Analytical formulation of the wind

As in MANDOP analysis, the analytical form of each of the three ageostrophic wind components is expressed as

\[ V_i = \prod_{l=1}^{3} f_{ij}(\chi_l), \]
with
\[ f_{i}(x_{i}) = \sum_{k=1}^{n_{i}} a_{ik} F_{ik}(x_{i}), \]  
where \( V_{1} = u_{a}, V_{2} = v_{a}, \) and \( V_{3} = w_{a} \); \( a_{ik} \) are the
coefficients of expansion. The base of the orthonormal functions \( F_{ik} \) and
the order of expansion \( n_{i} \) of the component \( i \) on the corresponding
\( X_{i} (X_{1} = x, X_{2} = y, X_{3} = z) \) axis are chosen in order to best
represent the observations.

Here, \( x_{1}(x_{2}) \) will be equal to either \( x (y) \) or to \( c_{s} \Delta t
\) \( (c_{s} \Delta t) [ \Delta t \) being the local time relative to a reference
time and \( c_{s} (c_{y}) \) the translation speed in the \( x (y) \) direction],
depending on the available dataset. In the first
case, data are obtained by a great number of stations
covering the \( x \) and \( y \) directions. In the second case, the
2D vertical structure of the front is obtained by a few
stations (one Doppler radar or three rawinsondes or
three wind profilers, and three radiosoundings in order
to obtain the spatial variations) using the advection of
the system in order to obtain a 2D vertical cross-front
section through a space-time conversion. For various
practical reasons explained in section 3, this latter
configuration will be considered in the application of
the analysis to simulated and real data. However, this
analysis is developed in the most general form allowing
its application to all possible situations (dataset with
2D or 3D coverage, number and type of instruments,
etc.).

Combining (8) and (9) yields
\[ V_{i} = \prod_{i=1}^{3} [ \sum_{k=1}^{n_{i}} a_{ik} F_{ik}(x_{i}) ]; \]  
Thus, the analytical form (8) of the three ageostrophic
wind components \( V_{i} \) may in turn be rearranged as
\[ V_{i} = \sum_{k=1}^{N_{i}} b_{ki} g_{ki}(x_{1}, x_{2}, x_{3}), \]  
with
\[ N_{i} = n_{1} n_{2} n_{i3}, \]  
and \( b_{ki} \) the product of three coefficients \( a_{i1k'}, a_{i2k''}, \)
and \( a_{i3k'''} \). Coefficients \( k', k'', \) and \( k''' \) range from 1 to
\( n_{1}, n_{2}, \) and \( n_{i3}, \) respectively, and \( K_{i} \) is given by the
equation
\[ K_{i} - 1 = (k' - 1)n_{2}n_{i3} + (k'' - 1)n_{i3} + k''' - 1. \]  
Here, \( g_{ki}(x_{1}, x_{2}, x_{3}) \) is the product of the three
corresponding orthonormal functions:
\[ g_{ki}(x_{1}, x_{2}, x_{3}) = F_{i1k'}(x_{1}) F_{i2k''}(x_{2}) F_{i3k'''}(x_{3}). \]  
\[ d. \text{ Adjustment} \]

The \( N_{i} \) coefficients \( b_{ki} \) are obtained by minimizing
in the least-squares sense, for all the experimental
points (denoted exp) in the 3D (2D) retrieval domain,
the expressions \( P_{i} \) corresponding to the first three
equations (1), (2), and (3) and defined as
\[ P_{i} = \sum_{\text{exp}} (V_{i} - V_{i}^{\text{obs}})^{2}, \]
with respect to the \( N_{i} \) coefficients \( b_{ki} \), where \( V_{i} \) is the
searched analytical form of the ageostrophic component,
and \( V_{i}^{\text{obs}} \) is its “experimental” value calculated
from data using the right-hand term of the equations.
See appendix A for more details.

We thus solve the following linear system of \( N_{i} \)
equations with \( N_{i} \) unknowns \( b_{ki} \):
\[ \frac{\partial P_{i}}{\partial b_{ki}} = 0 \text{ for } K_{i} = 1, 2, \cdots N_{i}. \]  
Rearranging system (16) yields, for \( K_{i} = 1, 2, \cdots, N_{i} \):
\[ \frac{\partial P_{i}}{\partial b_{ki}} = \sum_{\text{exp}} \left[ b_{ki} g_{ki}(x_{1}, x_{2}, x_{3}) g_{ki}(x_{1}, x_{2}, x_{3}) \right] \]
\[ - \sum_{\text{exp}} V_{i}^{\text{obs}} g_{ki}(x_{1}, x_{2}, x_{3}) = 0. \]  
System (17) may be written as
\[ \sum_{K_{i}} C_{K_{i}K_{i}} b_{K_{i}} = A_{K_{i}} \]  
for \( K_{i} = 1, 2, \cdots N_{i} \) (\( K_{i} \) and \( K'_{i} \) are line and column
indices, respectively).

This is equivalent to the matrix equation:
\[ C_{K_{i}K_{i}} = A_{K_{i}}, \]  
in which \( B_{i} \) is the \( N_{i} \)-dimensional vector of the
unknowns \( b_{ki} \); \( C_{i} \) is an \( N_{i} \times N_{i} \) symmetric matrix,
the elements of which, \( C_{K_{i}K'_{i}} \), consist of analytical information
[orthonormal functions through their products
\( g_{ki}(x_{1}, x_{2}, x_{3}) \)]
\[ C_{K_{i}K'_{i}} = \sum_{\text{exp}} g_{ki}(x_{1}, x_{2}, x_{3}) g_{K_{i}}(x_{1}, x_{2}, x_{3}); \]  
and \( A_{i} \) is a \( N_{i} \)-dimensional vector, the elements of
which, \( A_{K_{i}} \), contain experimental information \( V_{i}^{\text{obs}} \)
\[ A_{K_{i}} = \sum_{\text{exp}} V_{i}^{\text{obs}} g_{ki}(x_{1}, x_{2}, x_{3}). \]  
Then, the unknowns \( b_{ki} \) may be determined using the
following relation:
\[ B_{i} = C_{i}^{-1} A_{i}, \]  
where \( C_{i}^{-1} \) is the inverse matrix.

Thus matrix equation (22) allows a direct determination
of the ageostrophic wind component \( V_{i} \) (\( i = 1, 2 \) or 3) using (11).

The relation (22) may be rewritten for the \( x \) component
\( u_{a} \) (\( i = 1 \)
\[ \mathbf{C}_u \mathbf{B}_u = \mathbf{A}_u, \]  

where \( \mathbf{B}_u \) is the \( N \)-dimensional vector of the unknowns \( b_K \), with \( \mathbf{B}_u = \mathbf{B}_1 \) for \( K = K_1 \) (with \( K_1 = 1, N_1 \)), or \( \mathbf{B}_u = 0 \) for \( K > N_1 \) and \( N = N_1 + N_2 + N_3 \); \( \mathbf{C}_u \) is an \( N \times N \) symmetric matrix with \( \mathbf{C}_u = \mathbf{C}_1 \) for \( K = K_1 \) and \( K' = K_1' \) (with \( K_1 \) and \( K_1' = 1, N_1 \)) and null for \( K \) and \( K' > N_1 \); \( \mathbf{A}_u \) is the \( N \)-dimensional “experimental” vector with \( \mathbf{A}_u = \mathbf{A}_1 \) for \( K' = K_1' \) (with \( K_1' = 1, N_1 \)) or \( \mathbf{A}_u = 0 \) for \( K' > N_1 \).

In the same way, for the \( y \) and \( z \) components \((i = 2 \text{ and } i = 3)\) we can write

\[ \mathbf{C}_v \mathbf{B}_v = \mathbf{A}_v \]  

and

\[ \mathbf{C}_w \mathbf{B}_w = \mathbf{A}_w \]  

with \( \mathbf{B}_v = 0, \mathbf{C}_v = 0, \mathbf{A}_v = 0 \) except for \( K = N_1 + N_2 \) and \( K' = N_1 + K_2' \) (with \( K_2 \) and \( K_2' = 1, N_2 \)) and \( \mathbf{B}_w = 0, \mathbf{C}_w = 0, \mathbf{A}_w = 0 \) except for \( K = N_1 + N_2 + N_3 \) and \( K' = N_1 + N_2 + K_3' \) (with \( K_3 \) and \( K_3' = 1, N_3 \)).

\[ e. \text{ Additional constraints} \]

1) **Continuity equations**

Equation (4) may be written at the first order as

\[ \text{div}(\rho_0 \mathbf{V}) = 0, \]  

where \( \mathbf{V} \) is the ageostrophic wind vector \((V_1, V_2, V_3)\), \( \rho_0(z) \) is the air density of the basic state under hydrostatic conditions. The air density may then be written as

\[ \rho_0(z) = \rho_0(z_r) \exp \left( -\frac{z - z_r}{H} \right), \]  

where \( z_r \) is a reference altitude and \( H \) is the scale height for density variations.

Substituting (26) in (27) yields

\[ \rho_0(z_r) \exp \left( -\frac{z - z_r}{H} \right) \left[ \text{div} \mathbf{V} - \left( \frac{V_3}{H} \right) \right] = 0. \]  

Condition (28) has to be satisfied by the analytical form for all the experimental points of the domain. It can be also expressed as a constraint statistically verified in the least-squares sense:

\[ D = \sum_{\text{exp}} \left[ \text{div} \mathbf{V} - \left( \frac{V_3}{H} \right) \right]^2 \exp \left( -\frac{2(z - z_r)}{H} \right) \text{ minimum}. \]  

This condition is equivalent to the matrix equation

\[ \mathbf{C}_d \mathbf{B} = \mathbf{A}_d, \]  

in which \( \mathbf{C}_d \) is an \( N \times N \) matrix that contains “analytical” information, \( \mathbf{B} \) an \( N \)-dimensional vector defined by \( \mathbf{B} = \mathbf{B}_u \) if \( K = 1, N_1 \); \( \mathbf{B} = \mathbf{B}_v \) if \( K = N_1 + 1, N_1 + N_2 \); \( \mathbf{B} = \mathbf{B}_w \) if \( K = N_1 + N_2 + 1, N_1 + N_2 + N_3 \); and \( \mathbf{A}_d \) the experimental vector that depends on the measured vertical velocity and the density (see appendix B).

In the same way (5) leads to the matrix equation

\[ \mathbf{C}_e \mathbf{B} = \mathbf{A}_e, \]  

where \( \mathbf{A}_e \) contains the horizontal divergence information and is null for \( K > N_1 + N_2 \), and \( \mathbf{C}_e \) is null for \( K \) and \( K' > N_1 + N_2 \) (see appendix C).

2) **Thermal wind relations**

Equations (6) and (7), which must be satisfied by the analytical form for all experimental points of the domain, may be also expressed as a constraint statistically verified in the least-squares sense:

\[ F = \sum_{\text{exp}} \left[ \frac{\partial u}{\partial z} - \left( \frac{\partial u}{\partial z} + \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial y} \right) \right]^2 \text{ minimum}, \]  

\[ G = \sum_{\text{exp}} \left[ \frac{\partial v}{\partial z} - \left( \frac{\partial v}{\partial z} - \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial x} \right) \right]^2 \text{ minimum}. \]  

In the same way we show (see appendix D) that these conditions are equivalent to the matrix equations

\[ \mathbf{C}_f \mathbf{B} = \mathbf{A}_f \]  

\[ \mathbf{C}_g \mathbf{B} = \mathbf{A}_g. \]  

f. **Mathematical formulation of the variational problem**

The variational problem to be solved consists in finding the coefficients that give the best fit, in the least-squares sense, to the ageostrophic winds in the whole retrieval domain, under the subsidiary conditions that the continuity equations and the thermal wind relations (expressed in terms of the total wind and of the ageostrophic wind) are verified by the analytical form of the ageostrophic wind field.

This may be expressed as

\[ \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3 + \lambda_4 \mathbf{P}_d + \lambda_5 \mathbf{P}_e + \lambda_6 \mathbf{P}_f + \lambda_7 \mathbf{P}_g \]  

minimum (36)

with respect to the \( N \) coefficients to be determined, so that the matrix equation (19) has now to be replaced by

\[ M \mathbf{B} = [\lambda_1 \mathbf{C}_u + \lambda_2 \mathbf{C}_v + \lambda_3 \mathbf{C}_w + \lambda_4 \mathbf{C}_d \]  

\[ + \lambda_5 \mathbf{C}_e + \lambda_6 \mathbf{C}_f + \lambda_7 \mathbf{C}_g] \mathbf{B} \]  

\[ = \lambda_1 \mathbf{A}_u + \lambda_2 \mathbf{A}_v + \lambda_3 \mathbf{A}_w + \lambda_4 \mathbf{A}_d \]  

\[ + \lambda_5 \mathbf{A}_e + \lambda_6 \mathbf{A}_f + \lambda_7 \mathbf{A}_g, \]  

(37)
where $\lambda_1, \lambda_2, \ldots, \lambda_g$ are weighting factors whose evaluation is discussed in the following. The main effect of including the physical constraints in the retrieval process is to improve wind components accuracy. Since these constraints are applied to the derivatives of the analytical form, they avoid classical problems of methods using polynomial expansions. Note that the present variational formulation corresponds to a formalism with weak constraints as claimed by Sasaki (1970), since these constraints are not strictly verified at each point of the retrieval domain but are statistically verified in the least-squares sense.

An estimate of the weighting factors may be done following the calculus of variations (Courant and Hilbert 1953). In the present simple case, a standard deviation approach is used. It consists in attributing to each constraint a confidence inversely proportional to the square of the error on the estimation of the right-hand terms of the corresponding equation. This error may be easily obtained from error estimation using the standard deviation of each basic data $u, v, w, \text{and } \theta$. As an example for (5) the variance $\sigma^2(5)$ is $4\sigma^2(u)/\Delta x^2$. After calculations, the following estimate for the weighting factors used in the applications performed in the paper are $\sigma^{-2}(1) = \sigma^{-2}(2) = 3, \sigma^{-2}(3) = 10^{-2}, \sigma^{-2}(4) = 10^2, \sigma^{-2}(5) = 4 \times 10^2, \text{and } \sigma^{-2}(6) = \sigma^{-2}(7) = 10^{-2}$. Various tests (not shown) show that the method is not too sensitive to the value of these weighting factors.

The WOAVAG case

1) REFORMULATION OF Eqs. (1), (2), AND (4)

Without estimations of the vertical velocities, the analysis previously described may be also applied. This is simply performed by substituting $w$ with its analytical form in (1), (2), and (4) and by canceling (3). These equations are thus rewritten as

$$u_a + w_a \frac{1}{f} \frac{\partial v}{\partial z} = \frac{1}{f} \left[ - \frac{D_p v}{D_l} + \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right) \right], \quad (38)$$

$$v_a - w_a \frac{1}{f} \frac{\partial u}{\partial z} = \frac{1}{f} \left[ \frac{D_p u}{D_l} - \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right) \right], \quad (39)$$

and

$$\text{div}(\rho V) = 0, \quad (40)$$

where $D_3$ denotes a temporal derivation including only the horizontal part of the advection term.

Then we show (see appendices A and B) that the minimization of these equations in the least-squares sense leads to the matrix equations

$$C_u B_u = A_u \quad (41)$$

$$C_v B_v = A_v \quad (42)$$

$$C_p B = 0. \quad (43)$$

Note that in this WOAVAG case the analysis acts as a retrieval method for the vertical velocity contrary to the WAVAG case where it acts as a filtering one.

2) BOUNDARY CONDITION AT GROUND LEVEL

To improve the estimation of the vertical velocities through this retrieval method, it is necessary (as in other classical retrieval methods) to introduce a boundary condition at ground. If orography is neglected, that is, if one assumes flat ground of altitude $z_0$ over the whole domain, the ground-level boundary condition implies a zero vertical wind at ground level, so that

$$P_h = \sum \limits_g v \frac{1}{z} \min. \quad (44)$$

Condition (44) is equivalent to the matrix equation

$$C_g \mathbf{B} = 0, \quad (45)$$

in which $C_h$ is an $N \times N$ matrix. The nonzero terms of matrix $C_h$ occupy the $N_3$ last lines and columns of it. Its elements $C_{kk'}$ contain analytical information concerning the third component.

Equation (44) shows that the variational form of the analysis generates a boundary condition best fulfilled in the least-squares sense. This means that the condition $w = 0$ at ground level is statistically verified, that is, the vertical velocity at ground level fluctuates about 0 with a variance depending in this case on the weighting factor attributed to this constraint.

If orography is taken into account, the ground-level condition is established as in the MANDOP analysis (see Scialom and Lemaître 1990). It may be expressed as the constraint statistically verified in the least-squares sense

$$P_h = \sum \limits_g \left[ V_3 + (u \cos \xi + v \sin \xi) \tan \alpha \right] \min, \quad (46)$$

where $\xi$ and $\alpha$ are orographic angles giving the orientation of the local ground-normal unit vector relative to the $x$ and $z$ axes, respectively.

h. Other conditions

Additional information, of experimental nature, or mathematical constraints on the wind field characteristics could also be taken into account under a variational form and expressed in the same matricial manner.

i. Characteristic scale of phenomena resolved by the analysis

The cutoff wavelength $\lambda_c$ of the analysis (as for the MANDOP analysis) results, on the one hand, from the horizontal (vertical) resolution of the data (grid resolution) and on the other hand, from the order of ex-
pansion of the analytical winds in the given domain. As explained in Scialom and Lemaître (1990), if the spatial sample resolution is $\Delta x$, it means that no structure with a wavelength smaller than $\lambda_2 = 2\Delta x$ can be resolved, according to Shannon’s theorem. Concerning the expansion, for the Legendre polynomial base used in the following applications, for an order of expansion $n$ along all coordinates, a wave along each axis reaches zero at most $(n - 1)$ times, this implying a minimal wavelength $\lambda_{c2}$ of about $2/(n - 1)$ times the total domain $D$. Therefore, parameters $n$, $D$, $\Delta x$ will be chosen such that $\lambda_1 < \lambda_{c2}$. Let us recall that if the cutoff wavelength is $\lambda$, it means that the minimum observable scale is $l_c = 0.26\lambda$.

**j. Remarks for the application of the method**

In conclusion, the AVAG analysis is expressed in a matrix notation and with a variational formulation. It allows for the use of information either of the vertical component of the air motion, or of the statistical boundary condition adjustment.

Practically, it leads to a matricial equation $\mathbf{CB} = \mathbf{A}$ in which $\mathbf{C}$ depends on the observations locations $(x, y, z)$, and on the base orthonormal functions. Thus, $\mathbf{C}$ is obtained by giving as input data of the $\mathbf{C}$-building software, the various locations of the observations and the expansion base. Additional conditions (continuity equations, thermal wind equations, boundary conditions, etc.) are expressed in the same matricial form and introduced in the retrieval process by simply adding them to $\mathbf{C}$. Thus, any additional constraint is taken into account by a simple matricial addition. Note that any change of the base of orthogonal functions requires no change in the retrieval software, but simply consists of a change of the definition of the functions at the program input, since this analysis is not built on special characteristics of a particular base. Using a base of orthogonial functions in the analysis allows us to add higher-order terms in the expansion without changing the expansion coefficients of lower order. Inversely, as in the MANDOAP analysis, if the expansion is performed at a higher order than the phenomenon itself, the coefficients of these higher-order terms are found to be null.

An additional advantage of the analysis is that any dataset (eventually inhomogeneous and without data on the retrieval domain boundary) can be processed without reformulation. Among these data, airborne radar data can also be included in the retrieval process. The analysis permits to also bypass the interpolation, and to use data points where they occur (without defining a regular grid mesh) as densely or sparsely as they occur, and to perform a sequential pass through all the data in any order, simply accumulating terms in the matrices. Finally, this analysis may be also applied in the case where temperature data are not available by simply canceling the matrices ($\mathbf{C}_s$, $\mathbf{C}_f$) that involve these measurements in the analysis. This formulation of the method is the most general one, allowing its application to all possible situations (2D or 3D dataset, number, and types of instruments, bases of orthogonal functions, etc.).

**3. Application to simulated data**

These applications are an illustration of the capabilities of the AVAG analysis to study the dynamical processes involved in the mesoscale upper and lower frontal dynamics. Numerical tests are carried out in order to put in evidence the role of each constraint in the present analytical variational analysis and to define the characteristics of future experimental networks that would allow us to respond to the actual scientific purposes relevant to the mesoscale frontal dynamics. Simulated winds were chosen in order to represent a classical front. The interest of the analytical and variational approach used in the analysis will be particularly evidenced by comparison with classical simpler methods. This application is carried out in the particular case of a Legendre polynomial base. This base is simply selected to illustrate the application. Let us recall that the analysis is not linked to a particular base and any other base of orthogonal functions may be used without reformulation of the analysis.

The application to real data (section 4) will be performed using dynamical and thermodynamical observations made by the mesoscale network of three radiosounding stations involved in the MFDP/Fronts 87 experiment. This mesoscale network allowed us to resolve the thermodynamic and dynamic details of frontal zones thanks to accurate thermodynamic measurements at a spacing about 20–30 km (see Lemaître and Scialom 1994; Lagouvardos et al. 1992). This experiment, held from October 1987 to January 1988, is the first field observation campaign of the Mesoscale Frontal Dynamics Project (MFDP), a long-term program of research into the dynamics of active cold fronts with participation of groups in France, the United Kingdom, and Germany (Thorpe et al. 1987; Lunnon and Lemaître 1987).

The used observations were carried out on 9 January 1988 in a 27-h intensive observation period (called IOP 7) around the time of the frontal passage at Brest, center of the intensive ground observations (Lemaître and Scialom 1994). Let us recall that the present method can be used for any type of dynamic and thermodynamic information without reformulation in particular for continuous wind profiler (ST) observations. For simulated data, it is assumed that the three considered stations are located as in the FRONTS 87 experiment, that is, 80 km spaced apart. Their locations are shown in Fig. 1.

**a. Simulated data**

As indicated previously, this application of the AVAG method to simulated data is performed in the
The diagnostic equation for the ageostrophic motion in the \((x-z)\) plane may be written as (Sawyer 1956; Eliassen 1962)
\[
N^2 \psi_{xx} - S^2 \psi_{xz} + F^2 \psi_{zz} = -Q_g. \tag{48}
\]
This equation is the so-called Sawyer–Eliassen equation on its inviscid, adiabatic form; its coefficients are the classical basic frequencies of the flow and may be written as
\[
S^2 = \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial x}, \tag{49}
\]
\[
F^2 = \frac{f}{\theta_0} \left( f + \frac{\partial u}{\partial x} \right), \tag{50}
\]
\[
N^2 = \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial z}. \tag{51}
\]

The geostrophic forcing \(Q_g\), appearing in the right-hand side of Eq. (48) is written as
\[
Q_g = - \frac{2 g}{\theta_0} \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial \theta}{\partial y} \right) - 2 \frac{g}{\theta_0} \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial \theta}{\partial x} \right), \tag{52}
\]
where the first term, \((\partial u / \partial x)(\partial \theta / \partial y)\), is the so-called shear term and the second term, \((\partial u / \partial x)(\partial \theta / \partial x)\), shows the effect of confluence on the horizontal temperature gradient. The three coefficients \(N^2, F^2, S^2\), as well as the forcing \(Q_g\) are calculated for every grid point \((x, z)\) of the computational domain using centered differences. Equation (48) is elliptical provided that the term \(PV = F^2 N^2 - S^4\) is always positive. This condition corresponds physically to a positive potential vorticity. In the present case the aforementioned condition is satisfied in all grid points of the computational domain. The SE equation is numerically solved using a successive overrelaxation scheme. The lateral and vertical boundary conditions for the ageostrophic streamfunction are \(\psi = 0\), which implies \(u_a = 0\) along the lateral boundaries and \(w = 0\) along the lower and upper boundaries.

The numerical solution is performed in a domain of 1100 km horizontally and 10 km vertically; however, in order to avoid boundary effects on the lateral boundaries, the fields analyzed by the AVAG method have been limited to a rectangular box 550 km \(\times\) 5 km with a grid spacing 20 km \(\times\) 0.25 km.

2) DESCRIPTION OF THE SIMULATED “OBSERVED DATA”

The basic fields deduced from the Orlanskii and Ross formulation and obtained solving the SE equation are given in Figs. 2–8. The cross section of the alongfront geostrophic wind component shown in Fig. 2 is char-

The case of a 2D dataset. This choice results first from the fact that available real data to test the method are 2D. These data come from the previously described MFDP/FRONTS 87 experiment devoted to the study of quasi-two-dimensional active cold fronts. Thus, in order to determine the accuracy and the limitations of the analysis in this case, we perform these numerical tests on the same configuration as that of the real dataset. Moreover, it is the simplest experimental method, largely used actually, to describe the frontal structure. This choice has been also conducted because, in this 2D case, a numerical tool exists to construct a geostrophic flow and the associated ageostrophic flow. This tool is based on the numerical solution of the Sawyer–Eliassen equation and allowed us to build the set of simulated “observed data.”

1) CONSTRUCTION OF THE SIMULATED DATA

Thus we consider for these tests a conceptual straight front, oriented in the \(y\) direction, with the \(x\) coordinate increasing toward warmer air and with the alongfront wind component \(v\) approximated by its geostrophic value \(v_g\) (Lagouvardos and Lemaitre 1994). This approximation (quasigeostrophic momentum) describes a flow in which the alongfront motion is exactly balanced, while the cross-front motion is not, that is, an ageostrophic component \(u_a\) is admitted. This conceptual front has been obtained using the analytical formulation of the midtropospheric case given by Orlanski and Ross (1977).

The diagnosis of the secondary circulation \((u_a, w_a)\) has been performed by numerically solving the two-dimensional Sawyer–Eliassen (SE) equation. Introducing a cross-front ageostrophic streamfunction \(\psi\) such that

\[
u_a = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x}, \tag{47}
\]
acterized by a low-level jet located just ahead of the frontal surface. This jet extends horizontally about to 250 km and its maximum value exceeds 23 m s\(^{-1}\). Behind the front the wind weakens rapidly, reaching values about 6–10 m s\(^{-1}\). The associated potential temperature field is displayed in Fig. 3. Strong cross-front gradients of \(\theta\) are present within the first 2 km of the atmosphere, with the strongest values of 4–5 K (100 km\(^{-1}\) observed at \(z = 800\) m. The comparison of these fields with those deduced from the FRONTS 87 experiment and analyzed by the method in section 4 shows that these data simulate very well the real ones (see Figs. 16a,d). Figure 4 gives the cross-front geostrophic wind imposed to these fields in order to simulate the forcing due to the confluence between the cold and warm air masses. Figures 5 and 6 provide the

**Fig. 2.** Vertical cross section of the alongfront geostrophic wind component (at 2 m s\(^{-1}\) intervals) in the limited domain processed by the AVAG analysis.

**Fig. 3.** Potential temperature (at 1-K intervals) field in geostrophic balance with the alongfront geostrophic wind field given in Fig. 2.

**Fig. 4.** Vertical cross section of the cross-front geostrophic wind component (at 2 m s\(^{-1}\) intervals) imposed in the geostrophic forcing.

**Fig. 5.** Ageostrophic cross-front wind field deduced from the numerical solution of the SE equation.
corresponding \( u_\theta \) and \( w_\theta \) fields obtained from the numerical solution of the diagnostic equation [Eq. (48)]. The main feature of the resulting circulation is a thermally direct cell (warm air rising, cold air sinking), centered within the frontal zone at \( z = 2000 \) m. A minor indirect cell is partly evident in the right side of the domain centered at \( z = 4.2 \) km. It leads to subsiding motions in the warm side of the front.

The input data of the analysis are chosen to be \( u, v, w, \frac{Du}{Dt}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \) etc. This choice allows its application to various types of data provided either by Doppler radars or by networks of ST [or radiosounding (RS)] stations. In the present application to the simulated case previously described, the partial derivatives with respect to \( x \) and \( z \) are obtained using centered differences, and other derivatives (with respect to \( y \) and \( t \)) are deduced from the relations (1)–(5).

b. Tests of the AVAG analysis

The basic data of these tests are thus the along-front wind field (Fig. 2), the potential temperature field (Fig. 3), and the total (\( u_v + u_\theta \)) cross-front wind field (Fig. 7) in the same rectangular grid box with a grid resolution of 20 km \( \times \) 0.250 km close to the one used in the real case application. Thus the minimum observable wavelength (see section 2) due to the horizontal grid spacing is \( \lambda_{v1} = 40 \) km (i.e., a minimum scale of 11 km) and \( \lambda_{\theta1} = 500 \) m vertically. We consider in the following the case with (WAVAG case) and the case without (WOAVAG case) information on the vertical velocity. The first AVAG analyses (WAVAG case and WOAVAG case) are done without noise added to the basic data in order to put in evidence (i) the cutoff effect of the analysis due to the order of expansion of the analytical form and its capability to represent the discontinuity associated with the frontal zone (representation limited by the sampling), and (ii) the effect of the discretization to obtain vertical and horizontal gradients of winds (and \( \theta \)) that are compared with the exact derivative of the analytical forms of these quantities. The second AVAG analyses (WAVAG case and WOAVAG case) are done with added noises with 0.5 m s\(^{-1}\), 0.5 m s\(^{-1}\), 1 cm s\(^{-1}\), and 0.5-K standard deviations, simulating the statistical error on the measured \( u, v, w \) wind components and potential temperature, respectively. These tests allowed us to evaluate the capability of the method to filter out the noise. To put it in evidence, comparisons will be made with the direct determination of \( u_\theta, v_\theta, \) and \( w_\theta \) using the basic equations (1), (2), and (3) without constraints.

1) Without noise

The analysis in the WAVAG case retrieves easily at the order 17 (in \( x \) and \( z \)) the ageostrophic circulation (see Figs. 8a,b). This expansion up to 17 on this domain (\( D = 550 \) km) leads to a minimum observable wavelength of 70 km. Thus scale motions less than 20 km are filtered out by the analysis. The minimum observable vertical wavelength is 0.6 km. The typical standard deviations are \( 8 \times 10^{-2} \) m s\(^{-1}\), \( 6 \times 10^{-2} \) cm s\(^{-1}\), and \( 10^{-2} \) m s\(^{-1}\) on \( u_\theta, v_\theta, \) and \( w_\theta \), respectively. The only limit concerns the order of expansion that must represent the strong gradients in particular those present in the \( w \) field. An illustration of this limitation

![Diagram](https://via.placeholder.com/150)

**Fig. 7.** Vertical cross section of the total cross-front wind component (at 2 m s\(^{-1}\) intervals) processed by the AVAG analysis.
is given in Figs. 9a,b, which show the results if the analysis is performed at the order 7. While the $u_a$ component is relatively well retrieved, a strong discrepancy appears for the $w_a$ component in the strong gradient area. At the order 17, discrepancies, which persist at the bottom of the domain, result essentially, as suggested previously, from the discretization of the input data.

In the WOAVAG case (order 17) the ageostrophic $u_a$ component (not shown) appears identical to the one obtained in the WAVAG case. The $w_a$ component appears also relatively well retrieved (see Fig. 10). The typical standard deviations are now $10^{-1} \text{ m s}^{-1}$, $1.5 \times 10^{-1} \text{ cm s}^{-1}$, and $0.1 \text{ m s}^{-1}$ on $u_a$, $w_a$, and $v_a$, respectively. The increase of errors in particular on $w_a$ results essentially from the fact that, as explained previously, in this WOAVAG case the analysis does not play the role of a filtering method for $w_a$ but of a retrieval method. Here, $w_a$ is now retrieved through (38), (39), and (40). In the present application, $w_a$ is in fact retrieved essentially thanks to (40) and (39). Indeed its contribution to (38) is relatively weak, the typical value of $w_a f^{-1} \partial u/\partial z$ being about 10% of $u_a$.

2) WITH NOISE

The addition of a noise with 0.5 m s$^{-1}$, 0.5 m s$^{-1}$, 1 cm s$^{-1}$, and 0.5-K standard deviation to the measured $u$, $v$, $w$ wind components and potential temperature, respectively, leads to the input fields given in Figs. 1a–d (grid resolution of 20 km $\times$ 0.250 km). The retrieved $w_a$ wind field in the WAVAG case, given Fig. 12b (filtering case for $w$) is found to be very close to the original wind. It shows that a satisfying filtering of the noise is performed by the method. On the contrary a discrepancy appears for the $u_a$ component (Fig. 12a). This results from the absence of data along the $y$ axis allowing a filtering effect of the method on the $y$ derivatives, contrary to the $x$ axis. This is clearly evidenced by Fig. 12c, which shows the retrieved $u_a$ fields when noise is canceled on the $y$ derivative. This result shows the need for alongfront measurements in future experiments devoted to the study of ageostrophic circulations. Nevertheless the comparison between the retrieved $u_a$ field by the analysis and the $u_a$ field obtained directly using the basic equations (1)–(3) (see Fig. 12d) shows that a satisfying filtering of the noise is performed by the method. On the other hand, the relative errors appear equally distributed in the domain. Corresponding errors are about 0.6 and 1 m s$^{-1}$ on the horizontal $u_a$ and $v_a$ wind components (see Fig. 12e), and 0.3 cm s$^{-1}$ on $w_a$, respectively.

The application of the analysis to real data will be made in section 4 using as input data those obtained during the MFDP/Fronts 87 experiment. These data will be distributed in a rectangular grid (with a grid resolution 1 h $\times$ 0.250 km, that is, 25 km $\times$ 0.250 km using a space–time conversion) using a cubic-spline scheme. The interest of this scheme is to reduce the noise on data. Since this process is performed for each RS station of the network before evaluating the derivatives [using for the derivatives a method (see Lagouvardos et al. 1992) identical to that proposed by Zamora et al. (1987)], the noise on the derivatives (in particular on the $y$ derivative) is also reduced. Thus
again that a satisfying filtering of the residual noise is performed by the analysis. The results of the analysis for the WOAVAG case (retrieval case for $w_0$) are given in Figs. 14a,b.

The boundary equation (ground-level) constraint is now necessary and is thus introduced in the analysis (adding a matrix). These results show that the inclusion of the continuity equation constraint improves the vertical velocity retrieval. Indeed, the vertical component appears close to the original one at the top level, implying no numerical errors related to the classical upward integration of the continuity equation used in other methods. On the other hand, the relative error is equally distributed between the three wind components. The corresponding maximum error on $w_0$ is about 0.6 cm s$^{-1}$.

The residual error in the retrieval of the $w$ field results essentially from the fact that the imposed boundary condition ($w = 0$) does not exactly fit the simulated one.

4. Application to real data

As indicated previously, the data used in the present section concern the IOP 7 of the FRONTS 87 experiment. They were extracted from the FRONTS 87 experiment data bank. Several aspects of this frontal case have been reviewed by Lemaître and Scialom (1994), Lagouvardos (1992), and Lagouvardos et al. (1992). This cold front passed over Brest Airport at 1900 UTC 9 January 1988. Its speed was evaluated to be approximately 7 m s$^{-1}$, coming from the 315$^\circ$ sector. The...
Fig. 11. Vertical cross section of the basic noisy fields processed by the AVAG analysis. (a) Alongfront wind component (at 1 m s$^{-1}$ intervals). (b) Potential temperature (at 1-K intervals). (c) Cross-front wind component (at 2 m s$^{-1}$ intervals). (d) Vertical wind component (at 0.5 cm s$^{-1}$ intervals).

Following observations have been selected to perform the present experimental tests: vertical profiles of wind components deduced from soundings performed by the network of three rawin sondes, and vertical profiles of potential temperature deduced from the associated upper-air soundings (RS).

A cubic-spline scheme has been used to construct fields in a rectangular grid. The sounding data consist of 27 h of observations (10 h before and 17 h after the frontal passage), corresponding to 680 km in length assuming a mean frontal velocity of 7 m s$^{-1}$. Eleven radiosondes were released from each of the three sites within this time interval. In the vertical direction there are 20 levels that define a domain between the surface and $z = 5$ km (grid spacing $\Delta z = 250$ m). Note that this is a particular application of the AVAG analysis,
Fig. 12. Ageostrophic wind field retrieved by the AVAG analysis in the WAVAG case with noise at the order 17. (a) Cross-front wind component (at 1 m s\(^{-1}\) intervals). (b) Vertical wind component (at 0.5 cm s\(^{-1}\) intervals). (c) Ageostrophic cross-front wind component (at 1 m s\(^{-1}\) intervals) retrieved by the AVAG analysis canceling the noise on the y derivatives. (d) Ageostrophic cross-front wind component (at 1 m s\(^{-1}\) intervals) obtained directly using Eqs. (1)-(3). (e) Ageostrophic alongfront wind component (at 1 m s\(^{-1}\) intervals) retrieved by the AVAG analysis.
since the normal way is to process data without defining a grid mesh by simply accumulating terms in the matrices. However, as pointed out previously, the case of a 2D dataset needs an interpolation on a regular grid mesh that allows us to reduce the noise, in particular on the \( y \) derivatives. In the case of a 3D dataset, this procedure is not necessary.

All terms in the right-hand side of (1)–(7) involving horizontal gradients \( (\partial/\partial x, \partial/\partial y) \) are evaluated through the network of three French sounding stations, 80 km spaced apart, using a method identical to that proposed by Zamora et al. (1987) assuming the linearity of thermodynamic and dynamic fields within the area defined by the three stations. This method has been applied to the fields deduced from the cubic-spline interpolation scheme performed on the temporal measurements made at each station.

a. Case of 9 January 1988 dataset

Figures 15a,b show the surface and 500-mb analysis chart of the present synoptic situation, at 1200 UTC.
The cold front is oriented southwest–northeast at 45° clockwise from north. In the vicinity of the experimental area the front appears relatively two-dimensional. The cross section of the alongfront wind component shows a classical pattern of a frontal system (Fig. 16a). It permits to identify a low-level jet formed just ahead of the frontal surface. This jet extends horizontally up to 150 km and its maximum value exceeds 24 m s$^{-1}$. Behind the front the wind weakens rapidly, reaching values of 3–5 m s$^{-1}$. The field of the relative wind component (Fig. 16b) perpendicular to the front (in the reference frame of the front) shows positive
values located within 150 km (or 6 h) at the rear of the discontinuity at ground. Also seen is the ageostrophic convergence in the low layer below the low-level jet resulting from the friction at ground. This field differs from the one used in the simulated case by a confluence area confined essentially in the low troposphere. The potential temperature field is also displayed in Fig. 16d. Strong cross-front gradients of $\theta$ are present within the first two kilometers of the atmosphere, with the strongest values of 3.5–4 K (100 km)$^{-1}$ observed at $z = 1000$ m.

Since direct measurements of the vertical wind speed $w$ were not available, in order to perform tests in the WAVAG case, an estimation of $w$ was made through the continuity equation. The horizontal total wind divergence $(\partial u/\partial x) + (\partial v/\partial y)$ was calculated using the network data (under the assumption of linearity of the wind field within the area defined by the three stations) and then, so as to obtain an estimation of $w$, it was numerically integrated with a linear correction with height of the divergence in each vertical column, assuming $w = 0$ at the ground surface and at altitude $z = 10$ km. This correction was performed in order to avoid the classical numerical errors related to the upward integration of the continuity equation. It can be shown that this upper boundary condition did not in-
fluence significantly the $w$ field within the first 5 km of the atmosphere where the analysis was applied. Isotachs of the resulting vertical velocity $w$ are shown in Fig. 16c. Positive values (indicating ascending motions) are evident before the frontal passage, with two marked cells at $z = 1$ km (letter A) and at $z = 2$ km (letter B), respectively, just ahead of the frontal surface. Negative values (subsidence) are evident in the cold sector, underneath the frontal zone.

b. WOAVAG case

Figures 17a–c, provide the $u_a$, $v_a$, and $w_a$ fields obtained by the analysis. The main feature of the obtained wind fields ($u_a, w_a$) is a thermically direct cell centered within the frontal zone at $z = 2000$ m, with strong ascending motions ahead of the frontal discontinuity at ground and subsidence behind. The core of strong ascending motions in the lowest layers just ahead of the frontal discontinuity at ground is related to the narrow cold-frontal rainband (NCFR) observed during this IOP. A minor indirect cell is also evident in the warm side of the frontal zone, centered around the low-level jet at about 0.9 km with a well-defined return branch (positive values of $u_a$) above and subsiding motions on the eastern edge of the jet. This application of the AVAG analysis allows us to test the assumptions.

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**Fig. 17.** Ageostrophic wind field retrieved by the AVAG analysis in the WOAVAG case on 9 January 1988 data at the order 17. (a) Cross-front wind component (at 2 m s$^{-1}$ intervals). (b) Alongfront wind component (at 4 m s$^{-1}$ intervals). (c) Vertical wind (at 1 cm s$^{-1}$ intervals) component.
underlying the SE diagnostic tool with the geostrophic momentum approximation) in particular the alongfront geostrophy. Indeed, it appears that the alongfront wind component is close to a thermal wind balance with the temperature field. The alongfront ageostrophic component is observed globally constant in the warm side (around 6 m s$^{-1}$ ± 2 m s$^{-1}$) and in the cold side (around 2 m s$^{-1}$ ± 2 m s$^{-1}$) of the frontal surface. This indicates that the mean vertical gradient of the $v_c$ component in each side of the frontal zone (respectively, warm and cold sectors) is close to zero and thus the total wind $v$ is in thermal wind balance (since $\partial v/\partial z = \partial v_t/\partial z + \partial v_a/\partial z$) in each of these sectors. This is consistent with the study of Lagouvardos et al. (1992) performed on this case, which shows that the geostrophic momentum approximation is able to reproduce the gross features of the secondary circulation developed around the observed transition zone. However, in the frontal surface area, strong local variations are observed. This is the reason why a more elaborate approach is necessary to reproduce the main characteristics of the observed ageostrophic circulation in the frontal surface area (Lagouvardos et al. 1992). This result allows us to confirm one of the specific hypotheses of the FRONTS 87 experiment (Thorpe et al. 1987), that is, the frontal motion is balanced on horizontal scales down to about 50 km. As shown by the $u_c$ field, it also confirms the other hypothesis that most of the ageostrophic convergence in the frontal region is in the boundary layer. The general pattern of the $w_c$ field appears close to the one deduced from the classical integration of the continuity equation using boundary condition at ground and at $z = 10$ km (Fig. 16c). Note that no upper-limit condition is imposed in the WOAVAG analysis. This shows that including the continuity equation in the variational problem, which allows the retrieval of the third component homogeneously throughout the vertical domain at the same time as the horizontal component, minimizes the numerical errors related to the classical upward integration of the continuity equation. The small scales differences result from the fact that the order of expansion used in the present application is limited (expansion limited in order to filter out the noise). Small-scale motions less than 25 km are filtered by the analysis (13 km in the direct determination of $w$). These differences may result also from the stationarity hypothesis used in the present AVAG application.

c. WAVAG case

This application to the WAVAG case allows us to qualify the WOAVAG results. Indeed the $u_c$ and $v_c$ fields (see Figs. 18a,b) appear close to those obtained in the WOAVAG case. The main differences are evidenced on the $u_c$ field in the lowest layers of the warm sector. This results essentially from the fact that the boundary condition $w = 0$ (applied only in the WOAVAG case) is imposed, in the present case, at the first measurement level, instead of at ground.

5. Possible extensions

As indicated in the abstract, this analysis may be used to retrieve the detailed structure of the temperature field between sparse temperature measurements. In this case the dense or continuous wind measurements provided by, for example, ST radars allow us to reconstruct the corresponding temperature field.

FIG. 18. Ageostrophic wind field retrieved by the AVAG analysis in the WOAVAG case on 9 January 1988 data at the order 17. (a) Cross-front wind component (at 2 m s$^{-1}$ intervals). (b) Alongfront wind component (at 4 m s$^{-1}$ intervals).
through the thermal wind equations included in the retrieval process, provided that a few temperature measurements are available. In this case the analysis is used as an interpolating method (based on physical constraints) between sparse temperature measurements using the dense dynamical information. Practically, the AVAG analysis provides the ageostrophic components of the wind in each point of the domain (where a dynamic measurement is available), which allows us to compute, using the total winds filtered out from noise, the left-hand side of (6) and (7). This procedure provides an estimate of horizontal gradients of potential temperature and thus of the potential temperature fields between thermodynamic measurements by integrating these gradients and using these measurements as boundary conditions. An illustration of this procedure is given in Fig. 19, which gives the obtained $\theta$ field using only three vertical temperature profiles (12-h or 300-km spaced, 2 on the boundary of the domain and one in the middle) for the WAVAG case on the noise-simulated data (to be compared with the original field displayed, Fig. 3). The $\theta$ field is retrieved, in this particular case, by simply integrating the horizontal gradients using as a boundary condition each of the three basic vertical temperature profiles, and performing a distance-weighted average (relative to the considered RS) of the three obtained estimates of $\theta$. The rms error is about 0.1 K.

6. Conclusions

The study previously developed describes a new analysis, AVAG, for the retrieval of the 3D ageostrophic circulations from observations by a network providing thermodynamic and dynamic data. It gives the mathematical principle of the wind retrieval. This analysis, based on a variational concept, lies upon the expressions of the ageostrophic wind components as products of expansions in series of orthonormal functions on each axis, thus allowing the wind to be analytically expressed. Then, the analytical form of the ageostrophic winds is variationally adjusted to the observed wind by including in the minimization process the additional physical conditions satisfied by the ageostrophic wind such as mass conservation, boundary condition at ground level and thermal wind equations. The variational process allows the ageostrophic wind to be retrieved with good accuracy.

The present paper also deals with the application of the method to real frontal cases and specifies how to operate in these cases. It illustrates how the analytical form of the wind implies data filtering and interpolating. The application to real data obtained during the FRONTS 87 Experiment in a 2D frontal case has been made. The analysis can be also used as an interpolation method (based on physical constraints) between sparse temperature measurements using the dynamical information (case of a network which involves, e.g., ST radars and radiosounding stations providing continuous wind measurements and 12-h temperature measurements).

Thus, in conclusion, the AVAG analysis constitutes a new tool now available to detect the 3D ageostrophic circulations acting at the mesoscale. It will be intensively used on FRONTS 87 data, in particular, the wind profiler data in order to investigate the transverse vertical circulations associated with the upper-tropospheric jet streaks, their role in the coupling of upper-level and low-level frontogenesis, and their effects related to the detected frontal waves.

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APPENDIX A

Analytical Retrieval of the Wind

We seek to determine the analytical form $V_i$ ($i = 1$–$3$) of the ageostrophic wind components $u_\alpha$, $v_\alpha$, and $w_\alpha$ [see Eq. (10)]:

![Fig. 19. Potential temperature (at 1-K intervals) field retrieved by the AVAG analysis in the WAVAG case on the simulated case using only three vertical profiles of the potential temperature.](image-url)
\[ V_i = \prod_{j=1}^{3} \left[ \sum_{k=1}^{n_d} a_{ik} F_{ik}(x_j) \right] \]  

(A1)

by writing each component \( V_i \) under the form of the product of three functions, each of these functions being in turn an expansion on the corresponding axis \( x_i \) in terms of orthonormal functions \( F_{ik}(x_j) \). This formulation may be rewritten as [see Eq. (11)]

\[ V_i = \sum_{k=1}^{n_i} b_{ki} g_k(x, y, z). \]  

(A2)

\[ a. \text{ Case with available information on the vertical velocity} \]

If there is available information on the vertical velocity, we use Eqs. (1)–(3):

\[ f u_a = -\frac{Dv}{Dt} + \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right) = H(x, y, z), \]  

(A3)

\[ f u_a = \frac{Dv}{Dt} - \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right) = G(x, y, z), \]  

(A4)

\[ w_a = w. \]  

(A5)

The analytical form of the component \( u_a \) is then written as

\[ V_1 = \sum_{K_i=1}^{N_i} b_{K_i} g_{K_i}(x, y, z), \]  

(A6)

The coefficients \( b_{K_i} \) are determined minimizing in the least-squares sense the difference between \( V_1 \) and \( u_a \) extended to the experimental points:

\[ \sum_{\text{exp}} \left[ f u_a - \left\{ -\frac{Dv}{Dt} + \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right) \right\} \right]^2 \]

\[ = \sum_{\text{exp}} \left[ f u_a - H(x, y, z) \right]^2 \]  

(A7)

(with \( \sum_{\text{exp}} \) being the sum extended to the experimental points). Deriving (A7) with respect to the unknowns \( b_{K_i} \) yields

\[ \frac{\partial}{\partial b_{K_i}} \sum_{\text{exp}} \left[ f u_a - H(x, y, z) \right]^2 = 0; \]  

(A8)

\[ \sum_{\text{exp}} \left[ 2 f g_{K_i}(x, y, z) \left\{ f \sum_{K_i} b_{K_i} g_{K_i}(x, y, z) \right\} - H(x, y, z) \right] = 0, \]  

(A9)

which can be written under a matricial form:

\[ C_1 B_1 = A_1, \]  

(A10)

in which \( B_1 \) is the \( N_i \)-dimensional vector of the unknowns \( b_{K_i} \); and \( C_1 \) is an \( N_i \times N_i \) symmetric matrix, the elements of which, \( C_{K_i K_j} \), consist of analytical in-

formation [orthonormal functions through their products \( g_{K_i}(x, y, z) \)]. Here,

\[ C_{K_i K_j} = \sum_{\text{exp}} f g_{K_i}(x, y, z) g_{K_j}(x, y, z), \]  

(A11)

and \( A_1 \) is an \( N_i \)-dimensional vector, the elements of which \( A_{K_i} \) contain experimental information data \( V_1^{\text{obs}} \):

\[ A_{K_i} = \sum_{\text{exp}} V_1^{\text{obs}} g_{K_i}(x, y, z) \]

\[ = \sum_{\text{exp}} g_{K_i}(x, y, z) H(x, y, z). \]  

(A12)

In the same way, the coefficients \( b_{K_i}(b_{K_j}) \) of the component \( v_a(w_a) \) are determined minimizing the difference between \( V_2 \) and \( v_a(V_3 \) and \( w_a) \) extended to the experimental points, which similarly provides the matricial equations:

\[ C_2 B_2 = A_2 \] (corresponding to \( N_2 \) equations), \[(A13)\]

\[ C_3 B_3 = A_3 \] (corresponding to \( N_3 \) equations), \[(A14)\]

with

\[ C_{K_i K_j} = \sum_{\text{exp}} f g_{K_i}(x, y, z) g_{K_j}(x, y, z), \]  

(A15)

\[ A_{K_i} = \sum_{\text{exp}} V_2^{\text{obs}} g_{K_i}(x, y, z) \]

\[ = \sum_{\text{exp}} g_{K_i}(x, y, z) G(x, y, z), \]  

(A16)

\[ C_{K_i K_j} = \sum_{\text{exp}} g_{K_i}(x, y, z) g_{K_j}(x, y, z), \]  

(A17)

\[ A_{K_i} = \sum_{\text{exp}} V_3^{\text{obs}} g_{K_i}(x, y, z) \]

\[ = \sum_{\text{exp}} g_{K_i}(x, y, z) w_a. \]  

(A18)

To retrieve simultaneously the three ageostrophic wind components [i.e., to process simultaneously (A3)–(A5)], it is necessary to rewrite the matrix equations (A10), (A13), and (A14) under the form

\[ C_u B_u = A_u, \]  

(A19)

\[ C_v B_v = A_v, \]  

(A20)

\[ C_w B_w = A_w, \]  

(A21)

where the dimensions of the systems are \( N = N_1 + N_2 + N_3 \), instead of \( N_1 \), \( N_2 \), and \( N_3 \), respectively, the additional terms being null. This procedure can thus be extended to all the additional conditions (4)–(7), as it is shown in the subsequent appendices.

\[ b. \text{ Unknown vertical velocity} \]

If there is no information on the vertical velocity, (A5) is injected into (A3) and (A4) which become
\[ fu_a + w_a \frac{\partial v}{\partial z} = \left[ -\frac{Dv}{Dt} + \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right) \right] = H_1(x, y, z), \quad \text{(A22)} \]

\[ fv_a - w_a \frac{\partial u}{\partial z} = \left[ \frac{Du}{Dt} - \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right) \right] = G_1(x, y, z). \quad \text{(A23)} \]

Thus, the unknown coefficients are found by minimizing, respectively, the expressions:

\[ \sum_{\exp} \left[ fu_a + w_a \frac{\partial v}{\partial z} - H_1(x, y, z) \right]^2, \quad \text{(A24)} \]

\[ \sum_{\exp} \left[ fv_a - w_a \frac{\partial u}{\partial z} - G_1(x, y, z) \right]^2, \quad \text{(A25)} \]

with respect to these coefficients.

This leads to the expressions:

\[ \frac{\partial}{\partial b_{K_1}} \sum_{\exp} \left[ fu_a + w_a \frac{\partial v}{\partial z} - H_1(x, y, z) \right]^2 = 0, \]

\[ \frac{\partial}{\partial b_{K_3}} \sum_{\exp} \left[ fu_a + w_a \frac{\partial v}{\partial z} - H_1(x, y, z) \right]^2 = 0, \]

for (A24), and

\[ \frac{\partial}{\partial b_{K_2}} \sum_{\exp} \left[ fv_a - w_a \frac{\partial u}{\partial z} - G_1(x, y, z) \right]^2 = 0, \]

\[ \frac{\partial}{\partial b_{K_3}} \sum_{\exp} \left[ fv_a - w_a \frac{\partial u}{\partial z} - G_1(x, y, z) \right]^2 = 0, \]

for (A25). Thus, we may write

\[ \sum_{\exp} \left\{ f g_{K_1}(x, y, z) \left[ f \sum_{K_1} b_{K_1} g_{K_1}(x, y, z) \right. \right. \]

\[ + \frac{\partial v}{\partial z} \sum_{K_1} b_{K_1} g_{K_1}(x, y, z) - H_1(x, y, z) \left. \right] = 0, \]

\[ \left. \sum_{\exp} \left\{ f g_{K_2}(x, y, z) \left[ f \sum_{K_1} b_{K_2} g_{K_2}(x, y, z) \right. \right. \]

\[ + \frac{\partial v}{\partial z} \sum_{K_1} b_{K_2} g_{K_2}(x, y, z) - H_1(x, y, z) \left. \right] = 0, \]

\[ \sum_{\exp} \left\{ 2 f g_{K_3}(x, y, z) \left[ f \sum_{K_1} b_{K_2} g_{K_2}(x, y, z) \right. \right. \]

\[ - \frac{\partial u}{\partial z} \sum_{K_1} b_{K_2} g_{K_2}(x, y, z) - G_1(x, y, z) \left. \right] = 0, \]

\[ \sum_{\exp} \left\{ -\frac{\partial u}{\partial z} g_{K_1}(x, y, z) \left[ f \sum_{K_1} b_{K_1} g_{K_1}(x, y, z) \right. \right. \]

\[ - \frac{\partial u}{\partial z} \sum_{K_1} b_{K_1} g_{K_1}(x, y, z) - G_1(x, y, z) \left. \right] = 0. \]

Coefficients $K_1$ and $K_1'$ vary between 1 and $N_1$, coefficients $K_2$ and $K_2'$ vary between 1 and $N_2$, and coefficients $K_3$ and $K_3'$ vary between 1 and $N_3$. The corresponding matricial equations

\[ C_a B_a = A_a, \quad \text{(A26)} \]

\[ C_b B_b = A_b, \quad \text{(A27)} \]

may be written thanks to the following change of index: $K_1 - K_2$ and $K_1' - K_2'$ are replaced by $K$ and $K'$, respectively, both varying between 1 and $N (= N_1 + N_2 + N_3)$. When $K$ and $K'$, respectively, replace $K_1$ and $K_1'$, they vary between 1 and $N_1$; when they replace $K_2$ and $K_2'$, they vary between $N_1 + 1$ and $N_1 + N_2$; when they replace $K_3$ and $K_3'$, they vary between $N_1 + N_2 + 1$ and $N$.

Thus, $C_a$ terms are given by

\[ C_{kk'} = \sum_{\exp} f^2 g_k(x, y, z) g_{k'}(x, y, z) \]

for $K$ and $K'$ varying from 1 to $N_1$;

\[ C_{kk'} = \sum_{\exp} f \frac{\partial v}{\partial z} g_k(x, y, z) g_{k'}(x, y, z) \]

for $K$ (or $K'$) varying from $(N_1 + N_2 + 1)$ to $N$, and $K'$ (or $K$) varying from 1 to $N_1$;

\[ C_{kk'} = \sum_{\exp} \left( \frac{\partial v}{\partial z} \right)^2 g_k(x, y, z) g_{k'}(x, y, z) \]

for $K$ and $K'$ varying from $N_1 + N_2 + 1$ to $N$;

\[ C_{kk'} = 0 \]

for $K$ or $K'$ varying from $N_1 + 1$ to $N_1 + N_2$.

Here, $A_a$ terms are given by

\[ A_K = -\sum_{\exp} f g_k(x, y, z) H_1(x, y, z) \]

for $K$ varying from 1 to $N_1$;

\[ A_K = -\sum_{\exp} \frac{\partial v}{\partial z} g_k(x, y, z) \]

for $K$ varying from $N_1 + N_2 + 1$ to $N$; and

\[ A_K = 0 \text{ for } K \text{ varying from } N_1 + 1 \text{ to } N_1 + N_2. \]

In the same way, $C_a$ terms are given by
\( C_{kk'} = \sum_{\exp} f^2 g_k(x, y, z) g_{k'}(x, y, z) \)

for \( K \) and \( K' \) varying from \( N_1 + 1 \) to \( N_1 + N_2 \);

\( C_{kk'} = -\sum_{\exp} f \frac{\partial u}{\partial x} g_k(x, y, z) g_{k'}(x, y, z) \)

for \( K \) (or \( K' \)) varying from \( N_1 + 1 \) to \( N_1 + N_2 \);

and for \( K' \) (or \( K \)) varying from \( N_1 + N_2 + 1 \) to \( N \);

\( C_{kk'} = \sum_{\exp} \left( \frac{\partial u}{\partial z} \right)^2 g_k(x, y, z) g_{k'}(x, y, z) \)

for \( K \) and \( K' \) varying from \( N_1 + N_2 + 1 \) to \( N \); and

\( C_{kk'} = 0 \) for \( K \) or \( K' \) varying from 1 to \( N_1 \).

Here, \( \Delta u \) terms are given by

\( A_K = -\sum_{\exp} f g_k(x, y, z) G_1(x, y, z), \)

for \( K \) varying from \( N_1 + 1 \) to \( N_1 + N_2 \);

\( A_K = -\sum_{\exp} \frac{\partial u}{\partial z} g_k(x, y, z), \)

for \( K \) varying from \( N_1 + N_2 + 1 \) to \( N \); and

\( A_K = 0 \), for \( K \) varying from 1 to \( N_1 \).

### APPENDIX B

**Matricial Formulation of the Continuity Equation for the Ageostrophic Wind**

#### a. Available measurements on \( w \)

Equation (4) expresses the continuity equation for the ageostrophic wind if there is available information on the vertical velocity:

\[ \rho \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} \right) = -w \frac{\partial p}{\partial z}. \]  

(B1)

Then the expression to be minimized is

\[ \sum_{\exp} \left[ \rho \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} \right) + w_a \frac{\partial p}{\partial z} \right]^2. \]  

(B2)

Considering (A8) yields

\[ \frac{\partial u_a}{\partial x} = \frac{\partial V_1}{\partial x} = \sum_{k=1}^{N_1} b_k g_k(x, y, z) \]

and similar equations for \( \partial v_a/\partial y \) and \( \partial w_a/\partial z \).

Here, \( g_k(x, y, z) \) is the derivative of the function \( g_k(x, y, z) \) with respect to the corresponding coordinate \( x \).

As in appendix A, the unknown coefficients \( b_k \) of the ageostrophic wind are derived by minimizing (B2) with respect to these coefficients leading to the matricial equation:

\[ C_B B = A_d. \]  

(B3)

Matrix \( C_d \) elements are given by

\[ C_{kk'} = 4 \sum_{\exp} \rho \frac{d_x}{d_k d_{k'}} g_{k'}(x, y, z) g_k(x, y, z) \]

(B4)

with \( d_k \) as the domain size on the corresponding axis \( x_1(d_k = d_x) \), if \( K \) is between 1 and \( N_1 + 1 \); \( d_k = d_x \) if \( K \) is between \( N_1 + 1 \) and \( N_1 + N_2 + 1 \); and \( d_k = d_x \) if \( K \) is between \( N_1 + N_2 + 1 \) and \( N \).

Here, \( A_d \) elements are given by

\[ A_K = -2 \sum_{\exp} \frac{\rho}{d_k} w \frac{\partial p}{\partial z} g_k(x, y, z). \]  

(B5)

#### b. No available measurements on \( w \)

If there is no available information on the vertical velocity, condition (B1) becomes

\[ \rho \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} \right) + w_a \frac{\partial p}{\partial z} = 0. \]  

(B6)

The formulation of this condition in terms of a matricial equation is formally identical to that given in the MANDOP analysis (Scialom and Lemaître 1990), except that in the MANDOP analysis, the considered divergence was calculated on the total wind. The condition (B6) must be verified in a statistical sense and minimized with respect to the unknowns:

\[ \sum_{\exp} \left[ \rho \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} \right) + w_a \frac{\partial p}{\partial z} \right] \]  

minimum

\[ \frac{\partial}{\partial b_K} \sum_{\exp} \left[ \rho \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} \right) + w_a \frac{\partial p}{\partial z} \right] = 0. \]  

(B7)

This leads to the matricial equation:

\[ C_d B = 0. \]  

(B8)

Here, \( C_d \) matrix terms are given by

\[ C_{kk'} = C_1 + C_2 + C_3 + C_4, \]

with

\[ C_1 = 4 \sum_{\exp} \frac{\rho^2}{d_k d_{k'}} g_k(x, y, z) g_{k'}(x, y, z), \]  

(B9)

for any \( K \) and \( K' \) between 1 and \( N_1 \);

\[ C_2 = -2 \sum_{\exp} \frac{\rho^2}{d_k H} g_k(x, y, z) g_{k'}(x, y, z) \]  

(B10)

for any \( K \) and \( K' \) greater than \( N_1 + N_2 + 1 \);

\[ C_3 = -2 \sum_{\exp} \frac{\rho^2}{d_k H} g_k(x, y, z) g_{k'}(x, y, z) \]  

(B11)
for any $K'$ and for $K$ greater than $N_1 + N_2$;

$$C_4 = \sum_{\exp} \frac{p^2}{H^2} g_k(x, y, z) g_{K'}(x, y, z)$$  \hspace{1cm} \text{(B12)}$$

for $K$ and $K'$ both greater than $N_1 + N_2$. In the expressions (B9)-(B12), $H$ is the density scale height.

APPENDIX C

Matricial Formulation of the Continuity Equation for the Geostrophic Wind

Equation (5) expresses the continuity equation for the geostrophic wind:

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \hspace{1cm} \text{(C1)}$$

Then the expression to be minimized is

$$\sum_{\exp} \left[ \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2.$$  \hspace{1cm} \text{(C2)}

This leads to the matrix equation:

$$C_a B = A_a. \hspace{1cm} \text{(C3)}$$

The $C_a$ terms are as follows:

$$C_{KK'} = \sum_{\exp} g_k(x, y, z) g_{K'}(x, y, z)$$

for $K$ and $K' < N_1 + N_2$, and

for $K$ or $K' > N_1 + N_2$, $C_{KK'} = 0. \hspace{1cm} \text{(C4)}$

The $A_a$ terms are as follows:

$$A_k = \sum_{\exp} g_k(x, y, z) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

for $K < N_1 + N_2$, and

for $K > N_1 + N_2$, $A_k = 0. \hspace{1cm} \text{(C5)}$

APPENDIX D

Matricial Formulation of the Thermal Wind Relations

Equations (6) and (7) express the thermal wind relations

$$\frac{\partial u_a}{\partial z} = \frac{\partial u}{\partial z} + \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial y}, \hspace{1cm} \text{and (D1)}$$

$$\frac{\partial v_a}{\partial z} = \frac{\partial v}{\partial z} - \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial x}. \hspace{1cm} \text{(D2)}$$

Considering first Eq. (D1), the expression to be minimized with respect to the unknowns is

$$\sum_{\exp} \left[ \frac{\partial u_a}{\partial z} - \left( \frac{\partial u}{\partial z} + \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial y} \right) \right]^2 \hspace{1cm} \text{minimum. (D3)}$$

Since the analytical form of $u_a$ is [see Eq. (A2)]

$$u_a = V_1 = \sum_{K_i=1}^{N_1} b_{K_i} g_{K_i}(x, y, z), \hspace{1cm} \text{(D4)}$$

then

$$\frac{\partial u_a}{\partial z} = \frac{\partial V_1}{\partial z} = \sum_{K_i=1}^{N_1} b_{K_i} \frac{\partial g_{K_i}}{\partial z}(x, y, z)$$

$$= \sum_{K=1}^{N_1} b_k \frac{\partial g_k}{\partial z}(x, y, z), \hspace{1cm} \text{(D5)}$$

Thus, including (D5) in (D3) and minimizing with respect to the unknowns yields the matricial equation:

$$C_f B = A_f, \hspace{1cm} \text{(D6)}$$

The $C_f$ and $A_f$ coefficients are, respectively, given by

$$C_{KK'} = \sum_{\exp} \frac{\partial g_k}{\partial z}(x, y, z) \frac{\partial g_{K'}}{\partial z}(x, y, z), \hspace{1cm} \text{(D7)}$$

$$A_k = \sum_{\exp} \frac{\partial g_k}{\partial z}(x, y, z) \left[ \frac{\partial u}{\partial z} + \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial y} \right], \hspace{1cm} \text{(D8)}$$

with $K$ and $K'$ ranging from 1 to $N_1$. If $K$ (or $K'$) is greater than $N_1$, $C_{KK'}$ and $A_k$ are null. Considering now Eq. (D2) similarly yields the matricial equation:

$$C_g B = A_g, \hspace{1cm} \text{(D9)}$$

The $C_g$ and $A_g$ coefficients are, respectively, given by

$$C_{KK'} = \sum_{\exp} \frac{\partial g_k}{\partial z}(x, y, z) \frac{\partial g_{K'}}{\partial z}(x, y, z), \hspace{1cm} \text{(D10)}$$

$$A_k = \sum_{\exp} \frac{\partial g_k}{\partial z}(x, y, z) \left[ \frac{\partial v}{\partial z} - \frac{1}{f} \left( \frac{g}{\theta_0} \right) \frac{\partial \theta}{\partial x} \right], \hspace{1cm} \text{(D11)}$$

with $K$ and $K'$ ranging from $N_1 + 1$ to $N_2$. For other values of $K$ (or $K'$), $C_{KK'}$ and $A_k$ are null.

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