Use of the Vorticity Equation in Dual-Doppler Analysis of the Vertical Velocity Field

JOHN J. MEWES
Coastal Meteorology Research Program, University of Oklahoma, Norman, Oklahoma

ALAN SHAPIRO
School of Meteorology, University of Oklahoma, Norman, Oklahoma

(Manuscript received 12 February 2001, in final form 24 August 2001)

ABSTRACT

The anelastic vertical vorticity equation is imposed as a flow constraint in dual-Doppler wind analysis techniques, with emphasis on improving retrievals of the vertical velocity field. Several new techniques in which the vorticity equation is imposed as a weak constraint are presented. In one method, the vorticity equation is used to obtain the vertical velocity boundary condition field required when mass continuity is used as a strong constraint. In a second method, the vorticity equation is used to solve for the set of two boundary condition fields required when the vertical velocity is expressed in a “weak constraint form.” A final correction step to the horizontal wind field ensures that mass continuity is satisfied exactly in the final analysis. This latter method can be easily adapted to retrieve a single boundary condition when an opposing boundary condition can be accurately assessed using another method.

These new analysis techniques are tested and compared with each other and to common traditional approaches in which the vorticity equation is not used. These tests use simulated data sampled from Advanced Regional Prediction System (ARPS) simulations of a supercell thunderstorm and of a thunderstorm evolving in a typical dry microburst environment. The results suggest that the vorticity equation holds significant potential for improving dual-Doppler analyses of the vertical velocity field, especially under less than optimal data coverage.

1. Introduction

Accurate retrieval of the vertical velocity field in dual-Doppler analyses is an ongoing problem in radar meteorology. Reliable synthesis of the vertical velocity field is desirable in many of the situations for which dual-Doppler analyses are commonly applied, such as in studies of the microphysical (Ziegler et al. 1983), kinematic (Brandes 1977), dynamic (Parsons and Kropfli 1990), and thermodynamic (Gal-Chen and Kropfli 1984) structures of thunderstorms, in studies of tornadogenesis (Dowell and Bluestein 1997), and in initialization for numerical weather prediction models (Lin et al. 1993).

The vertical velocity field is typically the most difficult component of the wind field to accurately synthesize in dual-Doppler analyses (Clark et al. 1980; Ray et al. 1980; Testud and Chong 1983). Anelastic mass continuity is often incorporated into dual-Doppler analyses as a means of bringing closure to a system that otherwise contains two equations in three unknowns, assuming the terminal velocity of the scatterers is a known quantity. Scan geometries typically dictate that the vertical velocity is the least well-observed wind component and is therefore most often constructed using mass continuity. Armijo (1969) pioneered modern dual-Doppler analysis techniques with his derivation of the (unique) solution for the horizontal and vertical velocity fields that exactly satisfy the anelastic mass continuity equation as well as radial wind observations from two Doppler radars.

In practice, errors in specifying the vertical velocity boundary condition on the upper or lower data surfaces and cumulative errors arising from integration of the mass continuity equation in the presence of horizontal divergence errors make reliable synthesis of the vertical velocity field difficult. Spatial filtering (Ray et al. 1975) and techniques that allow for vertical velocity specification at both the upper and lower data boundaries (Lafeef 1967; O’Brien 1970; Ray et al. 1980) are popular means of reducing the latter source of errors. The latter approach often involves application of mass continuity as a weak constraint while determining \( w \), followed by a correction step to the horizontal wind field that ensures mass continuity is satisfied. Protat and Zawadzki (1999) proposed an alternative technique in which the vertical velocity field at a given level is a linear combination of
interim vertical velocity fields obtained from an upward and a downward integration of the continuity equation, with greater weight being placed on the integration that is expected to have accumulated fewer errors at that height. This technique also requires that vertical velocity boundary conditions be specified at both the top and bottom of the analysis domain.

Unfortunately, errors arising from inadequacies in specification of the vertical velocity boundary condition(s) have been more difficult to overcome. The earth’s curvature, beam blockage, and ground clutter contamination frequently prevent adequate radar data from being available near the earth’s surface, where the impermeability condition could safely be applied. The impermeability condition could also be applied at storm top, provided that data are available and the storm top lies in a stably stratified layer of the atmosphere. However, typical scan geometries often fail to reach storm top, and the vertical velocities at storm top may be far from zero during storm development. Alternative means of specifying the required vertical velocity boundary conditions are therefore desirable.

Ad hoc approaches, such as assuming that the horizontal divergence below the lowest data level is equal to (or some fraction of) the horizontal divergence at the lowest data level (Brandes 1977), are common. Chong and Testud (1983, hereafter CT83) used variational techniques to adjust the boundary condition field at ground level so as to maximize the “mathematical regularity” of the vertical velocity field. With this technique a subsidiary condition ensures that the vertical velocity preserves a statistical mean of zero at ground level while being allowed a prespecified variance. The technique was found to exhibit statistically similar errors to downward integration of the anelastic mass continuity equation from storm top, with the advantage that it preserves the physically based mean of zero vertical velocity at the ground (provided data are available down to ground level).

The CT83 technique actually takes advantage of the same properties of density stratification that appear to make downward integration of the mass continuity equation preferable to upward integration. By maximizing the regularity of the \( w \) field, noise associated with the accumulation of divergence errors is preferentially distributed to lower levels where the vertical mass flux \( (\rho w) \) errors are realized as smaller \( w \) errors. Similar results could be obtained by an upward-integration technique that identifies upper-level noise, then makes high wavenumber adjustments in the original lower boundary condition to negate the noise in the upper levels.

Downward integration usually appears preferable to upward integration for two reasons. First, density stratification dictates that \( w \) errors at a level will be amplified (diminished) as one integrates upward (downward) from that level. This is true only for \( w \) errors, however, as the vertical mass flux \( (\rho w) \) errors do not systematically change with height. Second, the geometrical feedback of \( w \) into the horizontal divergence estimates is typically greatest at upper levels with ground-based radars, so it is desirable that divergence-related noise be distributed toward lower levels in order to diminish the potential for negative feedback. However, in both cases it is important to note that it is not the direction of integration, but rather where the noise associated with this integration is distributed, that ultimately determines the accuracy of the final vertical velocity field.

If this noise can be distributed to lower levels regardless of the direction of integration (as with the CT83 technique), the “best” direction for integration is determined only by the location at which a \( \rho w \) boundary condition (not including noise adjustments) can be specified most accurately. For example, an auxiliary condition of \( \rho w \), maintaining a statistical mean of zero at storm top (rather than the ground) in the CT83 technique may yield superior results if that boundary condition were to be more accurate or applicable in a given case. This subtle distinction about the preference toward downward integration is important for this work, since the techniques to be presented here are much less sensitive to the direction of integration than are traditional methods (we will elaborate on this in section 3). Since traditional approaches to boundary condition specification have not attempted to counter divergence-related noise, it is not surprising that upward integration has long appeared inferior to downward-integration techniques.

Sun and Crook (1996) retrieved the thermodynamic and velocity fields from an observed collapsing cold pool by fitting the dry, Boussinesq equations of motion and the continuity equation to radial wind data from two Doppler radars using an adjoint technique. The vertical velocity field was thus constrained to satisfy the continuity equation and the equations of motion. The adjoint technique was found to yield lower rms errors in the vertical velocity field when compared to a traditional upward integration of the continuity equation [section 4.b(5) of Sun and Crook (1996)], largely because it does not suffer from divergence error accumulation. However, the adjoint technique introduces pressure and temperature as extra control variables. Moreover, if the adjoint technique was applied in a moist-convective case, moisture and hydrometeor species would also need to be added as control variables. The increased number of control variables generally increases the possibility of nonuniqueness in solutions. Extra constraints such as smoothness or background constraints can diminish the uniqueness concerns but may sacrifice resolution in the solution or introduce errors into the solution when the background field contains significant errors.

Matejka and Bartels (1998, hereafter MB98) have recently conducted a thorough examination of the errors in the vertical velocity fields from various multiple-Doppler analysis techniques. Using a Monte Carlo meth-
od they examined the responses of eight different analysis techniques to different magnitudes and behaviors of errors in the input data, including random, bias, and trend errors. Errors in vertical velocity boundary condition specification were identified as a major remaining obstacle. A potential solution to this problem when using methods similar to some of MB98’s four “best” analysis methods will be presented in this work.

In the present investigation the anelastic vertical vorticity equation is introduced as an additional constraint in dual-Doppler analyses as a means of improving retrievals of the vertical velocity. The approach taken here bears little resemblance to that of Protat and Zawadzki (2000), where the vertical vorticity equation was applied to improve retrievals of the perturbation pressure and temperature fields in multiple-Doppler analyses. Although introduction of the vorticity equation as an analysis constraint in Protat and Zawadzki left the three-dimensional wind solution largely unchanged, the derivatives of the three wind components were changed slightly in a way that allowed for substantial improvement in the retrieval of the perturbation pressure and temperature fields. In this work we apply the vorticity equation in an entirely different manner, with the hope of achieving significant improvement to retrievals of the vertical velocity field. These new techniques are formulated using concepts from the calculus of variations (Sasaki 1970; Courant and Hilbert 1953). In each new method we introduce the anelastic vertical vorticity equation as a weak constraint with the goal of obtaining the boundary condition(s) required when applying mass continuity in dual-Doppler analyses. However, before proceeding with application of the vorticity equation as a weak constraint in dual-Doppler analyses it is useful to first explore its nature when applied as a strong (exact) constraint.

2. The anelastic vertical vorticity equation: Exact solution and error analysis

a. The exact solution

In order to improve retrievals of the vertical velocity in dual-Doppler analyses, it is desirable to incorporate new analysis constraints that contain information about the vertical velocity field. The vertical vorticity equation would appear to be a good candidate for this purpose since it contains information about the vertical velocity field, and, under the anelastic approximation, it takes a form that is particularly well-suited for use in dual-Doppler analyses. In the absence of mixing terms, the anelastic vertical vorticity equation (henceforth referred to only as the vorticity equation) is given by

\[
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + (\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left( \frac{\partial w}{\partial x} \frac{\partial \zeta}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial \zeta}{\partial z} \right) = 0,
\]  

(1)

where \( \zeta \) is the vertical vorticity. Notice that under the anelastic approximation the baroclinic term present in the general vertical vorticity equation does not appear in (1). Scale analysis indicates that the baroclinic generation of vertical vorticity will generally be at least an order of magnitude smaller than the other terms in (1) for a variety of deep convective phenomena (Ray 1976). Neglect of this vertical baroclinic term does not imply that baroclinic processes are irrelevant. However, baroclinic processes are generally of secondary importance in the direct generation of vertical vorticity. This is convenient, since knowledge of the pressure and density fields is not required when using (1) as an analysis constraint. We have neglected the mixing terms in (1) because mixing processes are expected to be of secondary importance to our results. However, it would be a simple matter to include parameterized mixing terms, such as \( \nu \nabla^2 \zeta \), in (1) as well as in the weak constraint analyses proposed in section 3.

With \( u \) and \( v \) (and hence \( \zeta \)) regarded as known, (1) can be thought of as a linear first-order partial differential equation (PDE) for the vertical velocity \( w \). It can be solved using the method of characteristics, that is, by reducing (1) to an ordinary differential equation (ODE) to be solved along special “characteristic” curves (Sneddon 1957). The characteristic curves of (1) are specified by

\[
\frac{dx}{ds} = \frac{\partial u}{\partial z}, \quad \frac{dy}{ds} = \frac{\partial v}{\partial z}, \quad \frac{dz}{ds} = 0,
\]  

(2)

where \( s \) denotes distance along a characteristic curve, and \( S = (\partial u/\partial z) \mathbf{i} + (\partial v/\partial z) \mathbf{j} \) is the local shear vector. Note that the characteristic curves run perpendicular and to the right of the local shear vector. Along these characteristic curves (1) reduces to a linear first-order ODE for \( w \):

\[
\frac{dw}{ds} + a(s)w + b(s) = 0,
\]  

(3)

where

\[
a(s) = \left[ \frac{\partial \zeta}{\partial z} \right], \quad b(s) = \left[ \frac{\partial \zeta + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left( \frac{\partial w}{\partial x} \frac{\partial \zeta}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial \zeta}{\partial z} \right)}{S} \right].
\]  

(4)

Use of the method of characteristics can be thought of as a transformation to a new curvilinear coordinate system in which the horizontal vorticity available for tilting is always confined to one coordinate direction, reducing the PDE to an ODE.

Equation (3) can be solved analytically as
\[
\begin{align*}
w(s) &= -\int_0^s b(s') \exp \left[-\int_0^{s'} a(s'') \, ds'' \right] \, ds' \\
&\quad + w_0 \exp \left[-\int_0^s a(s'') \, ds'' \right], \quad (5)
\end{align*}
\]

where \( w_0 \) is a horizontal boundary value for \( w \). It can be shown that errors in \( u \) and \( v \) can lead to significant errors in the calculated locations of the characteristic curves as well as in the solution for \( w \) along these curves. The exact solution (5) is therefore not a practical means of retrieving the vertical velocity in the presence of observational errors. However, it is useful as a means of investigating the underlying sensitivities to errors in the forcing terms (i.e., errors in \( u \) and \( v \)).

**b. Error analysis**

One significant feature of the exact solution is its sensitivity to the magnitude of the vertical wind shear. It is immediately evident that the solution for \( w \) is undefined in regions with no vertical wind shear. Further, any characteristic curves entering such an area end abruptly, leaving open the possibility of “orphaned” regions unreachable by characteristic curves extending from the data boundaries. It is also apparent that the solution will be most sensitive to errors in \( u \) and \( v \) when the magnitude of the vertical wind shear is small. Physically, this is a reflection of the fact that more tilting is necessary to provide a given change in the vertical vorticity (to satisfy the vorticity equation) when the magnitude of the vertical wind shear is small.

Another potentially significant source of errors is poor representation of the temporal derivative of vorticity. Typical radar scan period times range from 90 s for some research radars to 6 min for the Weather Surveillance Radar-1988 Doppler (WSR-88D) (Klazura and Imy 1993). Using a second-order centered difference approximation, the temporal derivative is then estimated over time spans ranging from 3 to 12 min. If \( \xi \) changes in a nearly linear manner with time (i.e., \( \delta \xi/\delta t \) remains nearly constant over the discretization period), the discretized estimate of the temporal derivative should be reasonably accurate. However, as changes in \( \xi \) become more nonlinear over the discretization period, the \( \delta \xi/\delta t \) estimate will become increasingly poor. Since advection of small-scale features is a potentially significant source for this nonlinear behavior for changes in \( \xi \), it is desirable to perform retrievals in a moving reference frame in which the effects of advection are minimized (Gal-Chen 1982).

A final issue to be discussed is the need for spatial filtering of the horizontal wind fields prior to retrieval of the vertical velocity field. Since the vorticity equation is nonlinear in \( u \) and \( v \), scale interaction can create scales in the forcing for the vorticity equation that are unresolvable on the analysis grid. For instance, interaction of a \( 2 \Delta x \) and a \( 4 \Delta x \) wave on a Cartesian grid can create a \( 1.33 \Delta x \) wave in the forcing for the vorticity equation. Thus, in order to prevent unresolvable scales from appearing in the retrieved vertical velocity fields, scales of less than \( 4 \Delta x \) must be filtered out of the horizontal wind fields prior to retrieving the vertical velocity field. However, \( 2 \Delta x \) scales will be represented (although most likely in a damped fashion) in the solution for \( w \) obtained from the filtered horizontal wind fields through the nonlinear interactions in \( u, v, \) and \( \xi \).

**3. Using the vorticity equation as a weak constraint in dual-Doppler analyses**

Recall that specification of the vertical velocity boundary condition(s) is a prominent source of vertical velocity errors when mass continuity is used in dual-Doppler analyses. Conveniently, decomposing \( w \) in a manner consistent with the form of the mass continuity constraint allows the vorticity equation to be used to help specify the boundary condition field(s).

**a. Using the vorticity equation as a weak constraint with mass continuity as a strong constraint**

If anelastic mass continuity is imposed as a strong constraint the vertical velocity can be parsed as

\[
\rho w(x, y, z) = \rho_0 w_0(x, y) - \int_0^z \frac{\partial u}{\partial x} \frac{\partial w}{\partial y} \, dz' \quad \text{or}
\]

\[
w^*(x, y, z) = w^*_0(x, y) + w^*_1(x, y, z), \quad (6)
\]

where \( \rho = \rho(z) \) is the mean density profile, \( z_0 = z_0(x, y) \) is the level where the boundary condition is specified, and \( w_0 = w(z_0) \) is the unknown vertical velocity boundary condition field. The presentation of this technique is simplified by working with the vertical mass flux \( w^* \) instead of the vertical velocity \( w \) \( (w^*=\rho w) \). The quantity \( w^*_1 \) represents that portion of the total vertical mass flux field known from horizontal divergence. Equation (6) is valid regardless of whether \( z_0 \) is taken to be the bottom, top, or somewhere in the middle of the analysis domain.

In this approach we seek to minimize the cost functional:

\[
J = \int_V \left[ \alpha^2 \left( \frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y} + \frac{w^*}{\rho} \frac{\partial \xi}{\partial z} \right)
\right.
\]

\[
\left. + (\xi + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{\rho} \left( \frac{\partial w^*}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial w^*}{\partial y} \frac{\partial u}{\partial x} \right) \right] \, dz^*, \quad (7)
\]

over the analysis domain \( V \). Provision for the weighting
function $\alpha^2$ in the cost functional could lead to improved results through preferential weighting of areas expected to have smaller retrieval errors. However, determination of the most appropriate weighting function will not be addressed in this study. Taking the first variation of this cost functional with respect to $w_0^g$, setting the result to zero, and integrating by parts yields the Euler–Lagrange equation for the vertical mass flux boundary field:

$$\begin{align*}
a \frac{\partial^2 w_0^g}{\partial x^2} + 2b \frac{\partial^2 w_0^g}{\partial x \partial y} + c \frac{\partial^2 w_0^g}{\partial y^2} + d \frac{\partial w_0^g}{\partial x} + e \frac{\partial w_0^g}{\partial y} + f w_0^g + g &= 0, \\
&\text{where} \\
a &= -\int_\epsilon \left[ \frac{\alpha^2}{\rho^3} \frac{\partial^2 u}{\partial z^2} \right] dz' \\
b &= \int_\epsilon \frac{\alpha^2}{\rho^3} \frac{\partial u \partial v}{\partial z} dz' \\
c &= -\int_\epsilon \left[ \frac{\partial^2 u}{\partial x^2} \right] dz' \\
d &= \int_\epsilon \frac{\alpha^2}{\rho^3} \left[ \frac{\partial u \partial \xi}{\partial z} + \frac{\partial u \partial \eta}{\partial z} + \frac{\partial \eta \partial \xi}{\partial z} + \frac{\partial \eta}{\partial z} \right] dz' \\
e &= \int_\epsilon \frac{\alpha^2}{\rho^3} \left[ \frac{\partial u \partial \xi}{\partial z} + \frac{\partial u \partial \eta}{\partial z} + \frac{\partial \eta \partial \xi}{\partial z} + \frac{\partial \eta}{\partial z} \right] dz' \\
f &= \int_\epsilon \frac{\alpha^2}{\rho^3} \left[ \frac{\partial u \partial \xi}{\partial z} + \frac{\partial u \partial \eta}{\partial z} + \frac{\partial \eta \partial \xi}{\partial z} + \frac{\partial \eta}{\partial z} \right] dz' \\
g &= \int_\epsilon \frac{\alpha^2}{\rho} \left[ \frac{\partial u \partial \xi}{\partial z} + \frac{\partial u \partial \eta}{\partial z} + \frac{\partial \eta \partial \xi}{\partial z} + \frac{\partial \eta}{\partial z} \right] dz' \\
\sigma &= \frac{1}{\alpha^2} \left( \frac{\partial u \partial \xi}{\partial z} + \frac{\partial u \partial \eta}{\partial z} + \frac{\partial \eta \partial \xi}{\partial z} + \frac{\partial \eta}{\partial z} \right) \\
\text{VE}_k &= \frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y} + w_k \frac{1}{\rho} \frac{\partial \xi}{\partial z} \\
&\quad + (\xi + f) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w_k}{\partial x} \left( \frac{1}{\rho} \frac{\partial u}{\partial z} \right) \\
&\quad - \frac{\partial w_k}{\partial y} \left( \frac{1}{\rho} \frac{\partial u}{\partial z} \right). \\
\end{align*}$$

Equation (8) is a linear second-order PDE for $w_0^g$. It can be shown to be elliptic through application of Schwarz’s inequality (as in Shapiro and Mewes 1999). The natural boundary conditions for this approach are the Robin conditions:

$$\begin{align*}
-a \frac{\partial w_0^g}{\partial x} - b \frac{\partial w_0^g}{\partial y} + w_0^g &= \int_\epsilon \frac{\alpha^2 \partial u \partial \xi}{\rho^3 \partial z \partial z} dz' \\
+ \int_\epsilon \frac{\alpha^2 \partial u}{\rho \partial z} (\text{VE}_k) dz' &= 0 \\
\text{on the east and west boundaries, and} \\
b \frac{\partial w_0^g}{\partial x} + c \frac{\partial w_0^g}{\partial y} + w_0^g &= \int_\epsilon \frac{\alpha^2 \partial u \partial \xi}{\rho^3 \partial z \partial z} dz' \\
+ \int_\epsilon \frac{\alpha^2 \partial u}{\rho \partial z} (\text{VE}_k) dz' &= 0
\end{align*}$$

on the north and south boundaries. Alternatively, if $w_0^g$ is known on the lateral boundaries it could be specified as a boundary condition. The vertical limits of integration in (9)–(11) can vary horizontally, which makes the method well-suited for the irregular upper and lower data boundaries encountered with real data. This method can be applied to either method U or D of MB98, depending upon whether the vertical velocity boundary condition field is chosen to be valid at the bottom or top of the data volume, respectively.

There are several advantages to applying the vorticity equation in this manner. By using it to obtain a vertical velocity boundary condition, we are able to take advantage of the error-filtering effects of the vertically integrated coefficients in (9)–(11). The problems associated with misplaced characteristic curves and the orphaned regions mentioned in section 2 are avoided altogether (we have an elliptic equation, which has no characteristic curves). Further, minimization of the cost functional is naturally dominated by regions that exhibit large vertical shear and/or large magnitudes of $\alpha \partial u \partial z$, since the cost functional is punished most by an inappropriate $w$ field in these regions. Since these regions naturally play the most significant role in determining the appropriate boundary condition field, it is not necessary to have significant shear and/or $\alpha \partial u \partial z$ present throughout the entire depth of a vertical column.

It should be noted that, in general, the variable sign of the coefficient $f$ in (9) prevents solution uniqueness from being guaranteed except for “sufficiently small areas” that depend upon the form of that coefficient, that is, situation dependent (Petrovsky 1991). This situation applies to both of the methods presented within this work. Fortunately, in practice, the solutions appear to have been unique in all of our experiments. We have tested for uniqueness by supplying a randomly changing initial field to the elliptic solver, then comparing the solutions obtained from each initial field.

This technique can easily be adapted into Brander’s (1977) iterative technique. To proceed we must first construct a first-guess estimate of the horizontal wind field and the “known” portion of the vertical mass flux field.
\[ w(x, y, z) = \frac{\rho_a}{\rho}(\frac{z_r - z}{z_r - z_b}) w_b(x, y) + \frac{\rho_r}{\rho}(\frac{z - z_r}{z_r - z_b}) w_r(x, y) + \frac{1}{\rho}(\frac{z - z_b}{z_r - z_b}) \int_{z_b}^{z} \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz' \]

Equation (8) can then be used to solve for the optimal \( w_b(x, y) \), after which the total three-dimensional wind field can be constructed by reapplying the Brandes technique with the new boundary condition field and dividing through by \( \rho \). In theory, the technique is insensitive to the level at which the boundary condition is applied, since noise from accumulating divergence errors will be distributed to areas where disagreement with the vorticity equation constraint is minimized. However, imperfect filters applied to \( u \) and \( v \) damp the response of the technique to 2\( \Delta x \) scales, limiting the technique’s ability to distribute noise. Therefore, downward integration is often still possible, although to a lesser degree than with traditional methods of boundary condition specification.

b. Using both the vorticity equation and mass continuity as weak constraints

In this section the vorticity equation is used to obtain a set of two vertical velocity boundary fields that minimize the sum of the weighted, squared errors in both the vorticity and anelastic mass continuity equations. In this case we parse the vertical velocity field into a “weak-constraint form” in terms of upper and lower boundary fields \( (w_r \) and \( w_b) \):

\[ a_{b,T} = -\int_{z} \alpha^2 F_{b,T} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} dz' \quad b_{b,T} = \int_{z} \alpha^2 F_{b,T} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) dz' \quad c_{b,T} = -\int_{z} \alpha^2 F_{b,T} \frac{\partial u}{\partial z} \frac{\partial u}{\partial z} dz' \]

\[ d_{b,T} = \int_{z} \alpha^2 F_{b,T} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \right) dz' \]

\[ e_{b,T} = \int_{z} \alpha^2 F_{b,T} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \frac{\partial u}{\partial z} + \frac{\partial \nu}{\partial z} \right) dz' \]

\[ f_{b,T} = \int_{z} \alpha^2 F_{b,T} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \frac{\partial u}{\partial z} + \frac{\partial \nu}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) + \frac{\partial \nu}{\partial z} \right) dz' \]

\[ g_{b,T} = \int_{z} \alpha^2 F_{b,T} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) + \frac{\partial \nu}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right) + \frac{\partial \nu}{\partial z} \right) dz' \]

where \( \rho = \rho(z) \) is the mean density profile, \( w_b \) and \( \rho_b \) \( (w_r \) and \( \rho_r \) are the vertical velocity and mean density at a lower (upper) boundary located at \( z_b(x, y) \) \( \{z_r(x, y) \}, \) and \( w_k \) represents that portion of the vertical velocity known from horizontal divergence. Substituting (12) into the cost functional,

\[ J = \int \left\{ \alpha^2 \left[ \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} \right] + \left( \zeta + f \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right\} \right\} dz', \]

and setting the first variations of \( J \) with respect to \( w_b \) and \( w_r \) separately equal to zero, yields the Euler–Lagrange equations:

\[ a_{b} \frac{\partial^2 w_b}{\partial x^2} + b_{b} \frac{\partial^2 w_b}{\partial x \partial y} + c_{b} \frac{\partial^2 w_b}{\partial y^2} + d_{b} \frac{\partial w_b}{\partial x} + e_{b} \frac{\partial w_b}{\partial y} + f_{b} w_b + g_{b} = 0, \]

\[ a_{r} \frac{\partial^2 w_r}{\partial x^2} + b_{r} \frac{\partial^2 w_r}{\partial x \partial y} + c_{r} \frac{\partial^2 w_r}{\partial y^2} + d_{r} \frac{\partial w_r}{\partial x} + e_{r} \frac{\partial w_r}{\partial y} + f_{r} w_r + g_{r} = 0, \]
The equations (14) comprise a system of coupled linear elliptic second-order PDEs for the vertical velocity boundary fields, $w_B$ and $w_T$. The coupling appears in the inhomogeneous terms $g_B$ and $g_T$. The natural boundary conditions of (14) are the Robin conditions, 

$$
\sigma = \frac{1}{\alpha^2} \left( \frac{\partial u}{\partial x} \frac{\partial^2 \zeta}{\partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 \zeta}{\partial x^2} \right)
$$

$$
\text{VE}_{B,T} = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (w_k + F_{T,B} w_{T,B} \frac{\partial \zeta}{\partial z}) + (\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial (w_k + F_{T,B} w_{T,B})}{\partial y} \frac{\partial u}{\partial z}.
$$

(15)

The equations (14) comprise a system of coupled linear elliptic second-order PDEs for the vertical velocity boundary fields, $w_B$ and $w_T$. The coupling appears in the inhomogeneous terms $g_B$ and $g_T$. The natural boundary conditions of (14) are the Robin conditions, 

$$
-a_{B,T} \frac{\partial w_{B,T}}{\partial x} - b_{B,T} \frac{\partial w_{B,T}}{\partial y} + c_{B,T} \int_{z} \alpha^2 F_{R,T} \frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} dz' = 0,
$$

(16)

on the east and west boundaries, and 

$$
b_{B,T} \frac{\partial w_{B,T}}{\partial x} + c_{B,T} \frac{\partial w_{B,T}}{\partial y} + w_{B,T} \int_{z} \alpha^2 F_{R,T} \frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} dz' = 0.
$$

(17)

on the north and south boundaries. Once again, if $w$ is available on the lateral boundaries it can be specified in lieu of the natural boundary conditions.

Note that anelastic mass continuity has been added as a separate weak constraint in the cost functional (13). This optional constraint limits the degree to which mass continuity is violated during the process of finding the upper and lower boundary fields. In regions where the vorticity equation contains significant errors, absence of this extra weak constraint could yield vertical velocity fields that severely violate mass continuity (when viewed in concert with the first-guess horizontal wind fields). This extra constraint was unnecessary in the previous method, since mass continuity was implicitly used as a strong constraint through the form of $w$ assumed.

This technique is also straightforward to implement. A simple dual-Doppler analysis technique can first be used to estimate $u$ and $v$ (and $w_k$). Using a provisional value of $w_T = 0$, (14) can then be solved for $w_B$. Using this $w_B$ field, we can solve for $w_T$. The system as a whole can be solved by continuing to iterate back and forth between interim solutions of $w_B$ and $w_T$ in (14) until no further significant adjustments are made to either solution. Using these $w_B$ and $w_T$ fields, the total vertical velocity field can be constructed from (12). A particularly useful modification of this technique arises when one of the boundary fields can be estimated reliably using another technique (e.g., applying the impermeability condition at the ground level). In this case, (14) can still be used to retrieve the opposing boundary condition. For example, if the data region spans downward to the ground but ends significantly below the storm top, the latter equation of (14) could be used to retrieve an optimal vertical velocity boundary condition for the upper boundary of the data region while the impermeability condition is applied at the ground. In this situation we are no longer solving a coupled system of equations, so there is no need to iterate back and forth between interim solutions for $w_B$ and $w_T$. In any case, the three-dimensional wind field at this point in the multistep process generally will not satisfy the radial wind observations or the anelastic mass continuity equation without an extra correction step to $u$ and $v$, since mass continuity has been applied as a weak constraint. A final correction step may therefore be desirable upon retrieval of the vertical velocity field.

One means of correcting $u$ and $v$ (for fixed $w$) is to apply the radial wind observations as strong constraints, in which case anelastic mass continuity will not be satisfied exactly in the final analysis fields. Alternatively, it is possible to correct $u$ and $v$ by applying the radial wind observations as weak constraints while imposing mass continuity as either a weak or strong constraint, with $w$ fixed from the previous step (Shapiro and Mewes 1999). A new correction step designed for application with this particular multistep technique is outlined in the appendix. It is similar in spirit to the $u$ and $v$ correction technique of Ray et al. (1980), with the exceptions that the adjustments here vary with height and that the minimization is performed against the actual radial wind observations (as opposed to first guesses of the $u$ and $v$ fields).

### 4. Tests on numerically simulated data

#### a. Procedural notes

The new techniques presented in section 3 were tested on simulated data sampled from the Advanced Regional Prediction System (ARPS; Xue et al. 2000; 2001). The first set of experiments involve data sampled from a simulation of the 20 May 1977 Del City, Oklahoma, supercell thunderstorm (henceforth referred to as the Del City storm). Data for the second set of experiments were sampled from an ARPS simulation of a thunderstorm...
The simulations were performed on a 101 × 101 × 35 point vertically stretched Arakawa C grid with uniform 500-m horizontal grid spacing. Vertical grid spacing ranged from 25 m near the ground up to 1000 m near the top of the domain. Wind fields were interpolated from the staggered and stretched ARPS grid to a nonstaggered, nonstretched analysis grid with 500-m grid spacing in all directions prior to performing dual-Doppler analyses. The vertical structures of the simulation and analysis grids are therefore very different.

Doppler radars were assumed to lie in the lower left (SW) and upper left (NW) corners of the analysis grid. Radial wind observations were simulated by projecting the interpolated Cartesian wind fields at each grid point in the directions of the radars. Errors due to inaccurate radar reflectivity/terminal velocity parameterizations are therefore absent from these experiments. However, random errors with standard deviation of 1.0 m s⁻¹ and magnitudes ranging up to 3.0 m s⁻¹ were explicitly introduced into the simulated observations at each analysis point. No bias or trend errors were explicitly introduced.

The practical data availability problems resulting from the earth's curvature, ground clutter contamination, and scan geometry limitations were addressed in our tests by limiting the spatial availability of the data. In addition, observations were made available only at discrete intervals of 90 and 300 s, intended to be representative of the volume scan periods of the Dopplers on Wheels (DOWs) research radars (Wurman et al. 1997) and the WSR-88D network (Klazura and Imy 1993). No attempts were made to mimic the nonsimultaneous nature of radar observations in real data.

The labeling system in our experiments is as follows: U, D, and C represent upward, downward, and constrained integration techniques (in U and D, mass continuity is applied as a strong constraint; in C it is applied as a weak constraint while retrieving \(w\)—although it can later be applied as a strong constraint when correcting \(u\) and \(v\)); I, E, and V represent application of impermeability, extrapolated divergence (followed by impermeability), or the vorticity equation, respectively, to obtain the required vertical velocity boundary condition(s) at the top (t) and/or bottom (b) of the data domain; and 90 and 300 represent the volume scan periods (in seconds) used with the vorticity equation. For example, a technique labeled C-Ib-Vt-90 would utilize mass continuity as a weak constraint in the step where \(w\) is obtained, with impermeability applied as the lower boundary condition and the vorticity equation (using 90-s volume scan periods) used to obtain the upper boundary condition. The natural boundary conditions are imposed on the lateral boundaries for the vorticity-based techniques in the following results. It will be useful to note that techniques U and D in this work are identical to the U and D techniques of MB98, and technique C is similar to MB98’s technique B.

The weight function \(\alpha^2\) in cost functionals (7) and
(13) have been treated as constants in the following experiments (1.0 in U and D, and 250.0 in C). The actual value of the constant is irrelevant in techniques U and D (any constant value yields the same solution). However, in method C, this constant determines the relative weights put on the mass continuity and vorticity equations, and does affect the results. A series of experiments (not shown) suggested that the method C retrievals are only weakly dependent on the value of this weight for a large range of weights. The value of 250.0 was chosen arbitrarily but generally caused the value of the vorticity portion of the cost functional to dominate the mass continuity portion by a 5:1 ratio, meaning that the vorticity equation has about five times as much influence as mass continuity in determining opposing vertical velocity boundary conditions. It is noteworthy that if \( \alpha^2 \) is set too low we effectively enforce mass continuity to be met exactly in a technique where mass continuity is intended to be held as a weak constraint during construction of the \( w \) field.

For comparison, control analyses were constructed using common traditional dual-Doppler analysis techniques that would be appealing in each situation. For example, the impermeability condition is applied as a vertical velocity boundary condition at the ground and/or at storm top whenever data extend to those levels. When data fail to extend downward to the ground, extrapolated divergence followed by application of the impermeability condition was used to determine a lower boundary condition.

Recall that the vorticity equation requires filtering of \( 2\Delta x \) and \( 3\Delta x \) scales in the first-guess wind fields prior to retrieval of the boundary condition fields. However, the retrieved boundary fields do contain information in the \( 2\Delta x \) scale (although in a damped fashion, as pointed out in section 3a). In order to retain as much small-scale detail as possible in the analyses, we have chosen to completely discard the highly filtered first-guess wind fields used in obtaining the boundary condition. The full three-dimensional wind field was then reconstructed using only the radial wind observations, the newly obtained boundary condition fields, a 1–2–1 Shuman filter (Shuman 1957; Ray et al. 1975), and the analysis framework noted for each experiment (i.e., one of techniques U, D, or C). Application of the 1–2–1 Shuman filter to the horizontal wind fields is a common practice for limiting the accumulation of divergence errors. This same filtering technique was applied in all analyses (for both the control and vorticity techniques).

b. Analyses of the Del City storm

The primary storm of this simulation was initiated by releasing a thermal bubble over the southeastern corner of the domain. As the induced updraft evolves, it is also advected northwestward, becoming established and nearly stationary just south of the center of the domain by \( t = 1800 \) s (Fig. 2, top left). The precipitation core and associated downdraft become organized around this time, and the resulting cold pool begins to spread northwestward from the primary updraft. A series of weaker and less steady storms form along the advancing boundaries of this cold pool. Between \( t = 1800 \) s and \( t = 4200 \) s the primary storm remains nearly stationary while a quasi-steady rotating updraft and rear flank downdraft typical of a supercell thunderstorm evolve. Finally, between \( t = 4200 \) s and \( t = 7200 \) s the storm begins to propagate southeastward into the low-level inflow, nearly reaching the southeastern boundary of the domain by \( t = 7200 \) s. Therefore, although the average storm motion through the 7200-s simulation was removed from the sounding, the storm actually traverses from the southeast corner of the domain to the center of the domain, then back again, over the course of the 2-h simulation.

Recall from section 2 that the accuracy of the discretized estimates of \( \partial \mathbf{Q} / \partial t \) can be increased by performing the analyses in a reference frame in which the effects of advection are minimized. Since the average motion of the primary storm in these experiments has been subtracted from the velocity field at the outset, the Gal-Chen (1982) or related techniques have not been utilized in the following results. Unfortunately, the secondary storms and other features present in the simulation domain propagate at significantly different speeds (up to 15 m s\(^{-1}\)) and in different directions than the primary updraft. Although retrievals of the vertical velocity in these secondary storms could in principal be improved by redefinition of the radial wind observations in reference frames moving with each storm of interest, for the purposes of this work we will instead limit our focus to a 50 \( \times \) 50 gridpoint subdomain surrounding the quasi-stationary primary storm of each simulation (i.e., the same applies for the upcoming microburst storm).

The first three experiments using the Del City storm assume that the volume coverage pattern (VCP) of the scanning radars provides data for elevation angles of up to 20\(^\circ\). This is considered to be a typical situation. For example, in its severe weather VCP mode the WSR-88D network can complete sweeps of 14 elevation angles in 5 min, with the top of the beam just reaching 20\(^\circ\) in the final sweep. For research radars such as the DOWs, complete coverage of twenty 90\(^\circ\) azimuthal sector scans with a 1\(^\circ\) beamwidth would require a 20\(^\circ\) s\(^{-1}\) azimuthal scan rate to completely encompass the volume below 20\(^\circ\) elevation in 90 s. This is, in fact, comparable to the current scan rate capabilities of the DOWs (Wurman 1999). In the first of these three experiments, case I, all data between 0\(^\circ\) and 20\(^\circ\) elevation for both radars were included within the analysis. In the second and third experiments (cases II and III), data below 1\(^\circ\) (\( \sim500 \) m) and 2\(^\circ\) (\( \sim1500 \) m), respectively, were rejected. These values were intended to represent typical and extreme cases of beam blockage and/or data contamination at low levels by terrain and other ground clutter. The 20\(^\circ\) maximum elevation angle falls far below
(3–5 km) the actual storm top for all but the first 20 min of the simulation, leaving the upper vertical velocity boundary condition unknown. Historically, this lack of boundary condition information would likely prevent application of the downward (D) or constrained (C) integration techniques. Therefore, the researcher encountering this situation would probably resort to upward integration of the mass continuity equation (technique U) from the lower boundary, where \( w \) appears to be known with greater accuracy. As mentioned previously, this technique can suffer severely from accumulating divergence errors in some circumstances.
When data are available down to the surface (case I) the primary concern is finding a way to negate the effects of accumulating divergence errors. Conveniently, a vorticity equation constraint can be very useful in this regard. This can be seen in Fig. 3a by comparing curve U-Ib (control) with curves C-Ib-Vt-90 and C-Ib-Vt-300. Averaged across all 24 retrieval times, the rms vertical velocity error of control technique U-Ib is 2.51 m s$^{-1}$. This is a relatively low rms error for technique U, given the considerable integration depth, and is likely attributable to the fact that the Shuman filter is particularly efficient at removing purely random errors in the horizontal divergence field. The control technique would likely have performed much worse were nonrandom errors applied (as noted in MB98). In any case, when constraining the vertical velocity field using an upper
Fig. 3. (left) Root-mean-square vertical velocity retrieval errors as a function of simulation time from experiments on the Del City supercell thunderstorm simulation using volume scan geometries providing data between (a) 0° and 20°, (b) 1° and 20°, (c) 2° and 20°, and (d) 1° and 30°. Errors are expressed in m s⁻¹. Each control case is indicated with a solid bold line. Average rms errors for each technique across all 24 times are indicated in parentheses. (right) Shown are (e) the median normalized error and (f) median correlation coefficient (as measured against the true field) of the vertical velocity fields retrieved from the control and best vorticity–based technique for each of these four scan geometry cases.
boundary condition retrieved from the vorticity equation, the rms errors are reduced to 1.89 and 2.26 m s\(^{-1}\) with techniques C-Ib-Vt-90 and C-Ib-Vt-300, respectively (up to a 25% reduction). Results from techniques C-Vb-Vt-90 and C-Vb-Vt-300 are shown for completeness, as the impermeability condition would likely be applied at ground level whenever data are available to the ground. However, technique C-Vb-Vt-90 is still able to show improvement over the control technique even without utilizing impermeability. This is not the case with technique C-Vb-Vt-300, which shows a nearly 50% increase in error relative to C-Vb-Vt-90. Rapid collection of volume scans is clearly desirable for the vorticity techniques in this case. Recall that the C techniques of this work are similar to MB98’s technique B, which was shown to be much more robust than technique U in the presence of bias errors. Thus, we speculate that the error reductions noted here (relative to the control run) may be understated in cases where bias error is present.

If data below 1\(^{\circ}\) are rejected due to ground clutter or other data contamination (case II), we are faced with the common situation of not having reliable knowledge of either an upper or lower boundary condition. In this case it is common to apply technique U using the extrapolated divergence technique to estimate the lower boundary condition. However, the vorticity equation can offer information regarding one or both of the required boundary conditions for common techniques in this situation. For example, we can use extrapolated divergence to estimate the lower boundary condition while using the vorticity equation to retrieve an upper boundary condition in technique C, controlling accumulating divergence errors. We could also use the vorticity equation alone to retrieve both the upper and lower boundary conditions in technique C. The benefits of the vorticity equation in this second experiment are explored in Fig. 3b. Method U-Eb (control) yields an average rms error of 3.34 m s\(^{-1}\) under these circumstances. Methods C-Eb-Vt-90 and C-Eb-Vt-300 show up to a 23% improvement over the control method, with 2.57 and 2.84 m s\(^{-1}\) rms errors, respectively. Method C-Vb-Vt-90 also shows a 21% improvement over the traditional technique, with an rms error of 2.64 m s\(^{-1}\). However, similar to case I, method C-Vb-Vt-300 shows a nearly 50% increase in error relative to C-Vb-Vt-90. Once again, the ability to rapidly collect volume scans is clearly beneficial to the vorticity techniques.

In extreme cases where data below 2\(^{\circ}\) (or even higher) are unavailable, such as in case III, there has traditionally been little hope of retrieving accurate vertical velocity fields since reliable estimation of the boundary condition was exceedingly difficult. This is reflected in Fig. 3c by curve U-Eb (control), which now exhibits an average rms vertical velocity error of 5.01 m s\(^{-1}\). Although applying extrapolated divergence in method C with the vorticity equation providing the top boundary condition does show some improvement over the traditional technique, the most significant improvement is evident in curves C-Vb-Vt-90 and C-Vb-Vt-300, which show up to a 44% improvement over the control technique with 2.79 and 3.91 m s\(^{-1}\) rms errors, respectively. Note that the trend of a 50% increase in error for technique C-Vb-Vt with 300-s volume scans (as opposed to 90-s volume scans) is continued. It is also interesting to note that the rms errors from the C-Vb-Vt technique grew only slightly between cases I and III, from 2.43 to 2.79 m s\(^{-1}\) for 90-s volume scans and from 3.59 to 3.91 m s\(^{-1}\) for 300-s volume scans, in spite of the fact that the lower boundary was moved approximately 1500 m away from the surface. This fact bodes well for use of method C-Vb-Vt in cases of significant scan geometry limitations due to blockage, contamination, or other factors.

In case IV we assume data are available between 1\(^{\circ}\) and 30\(^{\circ}\) for each radar. This scan geometry provides data to above storm top, allowing for the application of a downward integration technique (D). Recall that with technique D the noise resulting from accumulation of mass divergence errors is realized at lower levels, where density stratification causes the resulting \(w\) errors to be significantly smaller than those observed with the upward integration technique. For this case we have chosen a lower elevation limit of 1\(^{\circ}\) instead of 0\(^{\circ}\), so that the impermeability condition cannot be applied at both the upper and lower boundaries. The vorticity equation would not be necessary were data available to 0\(^{\circ}\), since it is not likely to provide better information than the impermeability condition on either of these boundaries.

Since the accumulation of divergence errors should be significantly reduced, and since we can safely assume \(w = 0\) at the upper data boundary here, the vertical velocity retrievals from the control technique should be much improved in case IV. In Fig. 3d we see that this is indeed the case, as the average rms error from the control technique (D-It) has dropped to 1.60 m s\(^{-1}\). 36% better than the control technique in any of the upward-integration control cases. Reduction of \(w\) errors through use of the vorticity equation proves to be difficult in this case. Only technique C-Vb-It-90 is able to show any improvement over the control technique, with a slight 2% error reduction to 1.57 m s\(^{-1}\). Notice that using extrapolated divergence from 1\(^{\circ}\) (~500 m) in combination with a \(w = 0\) storm-top boundary condition in technique C (curve C-Eb-It) actually produces worse results than control technique D-It. In other words, specifying a second (reasonable) boundary condition actually deteriorates results. Based upon MB98, this finding is expected only when the degree of error randomness is high, as is the case here. In the presence of systematic errors MB98 showed that specification of the second boundary condition can be very beneficial, even over the control downward-integration technique used here. Again, based on this finding we would expect further improvement over the control technique when nonrandom errors are present, although this has not yet been verified experimentally.
If viewed in terms of vertical mass flux \( (\rho w) \) instead of vertical velocity, the error reduction with technique C-Vb-It-90 is a slightly more significant 9% (from 0.93 to 0.85 kg m\(^{-2}\) s\(^{-1}\)). Recall from the discussion in section 1 that, unlike vertical velocity errors, vertical mass flux errors should be relatively insensitive to the direction of integration. Unfortunately, none of the cases presented here span the entire depth from 0° to 30° to permit fair comparison of techniques U and D with known \( w = 0 \) boundary conditions. However, average vertical mass flux errors of 1.13 and 0.93 kg m\(^{-2}\) s\(^{-1}\) from the control techniques in cases I (U-Ib) and IV (D-It), re-
spectively, are statistically similar results, considering that the average density within the data region of case I is higher (so that \( r_w \) errors should be larger as well).

Figures 3e and 3f explore the normalized errors and correlations with the true field of the vertical velocity fields retrieved from the control and best vorticity–based technique in each scan geometry scenario. The normalized errors were calculated by dividing the rms error at each retrieval time by the variance (around \( w = 0 \)) of the true wind field at that time. Note that median values over the 24 retrieval times are shown in lieu of mean values. The median statistic was chosen because it was less sensitive to the abnormally large (small) values of the normalized error (correlation coefficient) that occur during the earliest portions of the simulation, when the true vertical velocities are small. Notice from
Fig. 3 that introduction of the vorticity equation allows for nearly universal improvement over the control technique for both error measures. Also, note that the presence of significant areas of near-zero vertical velocity within the subdomain of the primary storm cause the variance of the true vertical velocity field to be relatively small. Since this variance was used in the denominator when calculating normalized errors, the normalized error values are inflated to deceivingly large values.

c. Analyses of the microburst storm

The apparent sensitivity of the vorticity equation to the magnitude of the vertical wind shear was established in section 2. Given this sensitivity, it is not surprising that the vorticity equation appears to be a useful constraint in a high-shear case such as the Del City storm. However, to be widely applicable the vorticity equation must hold promise of improving vertical velocity retrievals under much more common environmental shear profiles.

To examine this potential we test the new techniques using a simulation of a typical dry microburst-producing thunderstorm. This storm was initiated by releasing a thermal bubble into an environment similar to the composite dry microburst environment of Brown et al. (1982), depicted in Fig. 1b. The environmental shear profile for this case is much weaker than that of the Del City storm. This type of storm is also attractive because it often occurs in the western United States, where beam blockage in complex terrain and the low-precipitation nature of the storms often lead to sparse radar data at low elevation angles. Since the divergence associated with microbursts may be strongest in the lowest few hundred meters of the atmosphere, sparse data at low levels may be especially problematic for extrapolated divergence techniques that estimate the low-level divergence using data at the lowest radar data level (Ray 1976).

To initiate the storm, a thermal bubble was released over the center of the simulation domain. A thunderstorm rapidly develops, with the heaviest precipitation and first significant microburst occurring at approximately $t = 1800$ s (see Fig. 4, top left). The storm continues to evolve and produce weaker microbursts between $t = 1800$ s and $t = 7200$ s while exhibiting a gradual decrease in intensity. Although much of the average storm motion was subtracted from the sounding at the outset, the storm still drifts slowly southward at a rate of approximately 5 km h$^{-1}$ throughout the course of the simulation.

The retrieval experiments for this storm used the same four VCPs used in the Del City storm retrievals. As in the Del City storm, the 20$^\circ$ maximum elevation angle used falls well below the actual storm top for all but the first 20 min of the simulation, and therefore necessitates an upward integration technique in cases I, II, and III. In case IV, data were assumed to span from 1$^\circ$ to 30$^\circ$ (above storm top), so that a downward integration technique using a zero vertical velocity at storm top could be safely applied.

Recall that in case I we can safely apply the impermeability condition at the ground, so our primary concern is limiting the detrimental effects of accumulating divergence errors. In spite of the weaker shear profile of this storm, technique C-Ib-Vt again appears to be the preferred analysis method for case I (see Fig. 5a). This technique yields rms vertical velocity errors of 1.20–1.24 m s$^{-1}$ for 90- and 300-s volume scan periods, respectively, when averaged across all 24 retrievals from the microburst storm. This amounts to an approximately 45% error reduction relative to control technique U-Ib for case I, which exhibits an average rms error of 2.17 m s$^{-1}$. Comparison of Figs. 3a and 5a reveals that the rms vertical velocity errors are lower for all techniques in this storm as compared to the Del City storm. However, it can be noted that the vorticity equation techniques all show substantial improvement over the Del City storm retrievals, in spite of the weaker shear profile, whereas the conventional technique shows only limited improvement. Also interesting is the increased similarity between the retrieval quality with 90- and 300-s volume scans in this case as compared to the Del City storm, likely indicating more linear local evolution in $\zeta$ with this storm.

The vorticity equation continues to show promise for improving vertical velocity retrievals in cases II and III. Figures 5b and 5c illustrate the dangers of extrapolating divergence in microburst situations, as the rms errors for any techniques applying extrapolated divergence (denoted “Eb”) increase dramatically after the storm first begins producing microbursts, shortly after $t = 1500$ s. In cases II and III technique C-Vb-Vt yields the best results, with rms errors ranging from 1.67 to 1.73 m s$^{-1}$ with 90-s volume scan periods, and between 1.92 and 1.98 m s$^{-1}$ with 300-s volume scan periods. This amounts to as much as a 44% reduction in rms errors relative to control technique U-Eb for case II, and up to a 61% reduction in rms errors for case III. The techniques utilizing only the vorticity equation to obtain the boundary condition(s) are again relatively insensitive to movement of the lower data boundary away from the earth’s surface. This behavior is similar to that noted with the Del City storm and suggests that these techniques may provide reliable vertical velocity retrievals even in the most severe cases of beam blockage. It is also interesting that longer volume scan periods are not as detrimental to the vorticity techniques in the microburst storm as they were with the Del City storm. However, during the rapid evolution phase of this microburst storm (the period surrounding $t = 1800$ s), significant error reductions are again noted for technique C-Vb-Vt-90 relative to C-Vb-Vt-300.

In case IV we again apply a downward integration technique with $w = 0$ at storm top as the control technique (technique D-It). As in the Del City case, this is
Fig. 5. (left) Root-mean-square vertical velocity retrieval errors as a function of simulation time from experiments on the microburst storm. The volume scan geometries for each experiment provided data between (a) 0° and 20°, (b) 1° and 20°, (c) 2° and 20°, and (d) 1° and 30°. Errors are expressed in m s⁻¹. Each control case is indicated with a solid bold line. Average rms errors for each technique across all 24 times are indicated in parenthesis. (right) Shown are (e) the median normalized error and (f) median correlation coefficient (as measured against the true field) of the vertical velocity fields retrieved from the control and best vorticity–based technique for each of these four scan geometry cases.
**Fig. 6.** Horizontal wind vectors and contours of $w$ from case III with the microburst storm at $t = 3600$ s at (left) $z = 1.5$ and (right) 10.0 km. Fields shown are (a), (b) the true wind field, (c), (d) control technique U-Eb, (e), (f) technique C-Eb-Vt-90, (g), (h) technique C-Vb-Vt-90, and (i), (j) technique C-Vb-Vt-300. Contour interval is 2.5 m s$^{-1}$. Maximum contours shown are ±25 m s$^{-1}$.
a more difficult technique for the vorticity equation to improve upon (see Fig. 5d). However, with this storm the C-Vb-It technique is able to show improvement over the control technique with either 90- or 300-s volume scans. Technique C-Vb-It also applies $w = 0$ at storm top but uses a constrained vertical velocity profile with the lower boundary condition determined with the vorticity equation. Error reductions are between 11% and 13%, with 1.02 and 1.01 m s$^{-1}$ errors for 300- and 90-s volume scan periods, respectively, versus 1.15 m s$^{-1}$
average rms errors for the control technique. Although this improvement is still only modest, it is more significant than the improvement noted with the Del City storm. Note that curve C-Eb-It is by far the least desirable of the methods explored for case IV. Clearly, the detrimental effects of errors in lower boundary condition specification often outweigh the effects of accumulating divergence errors in this case. Based on MB98’s results, this is to be expected when systematic errors in divergence are not present.

Figures 5e and 5f explore the normalized errors and correlation coefficients of the vertical velocity fields retrieved from the control and best vorticity-based technique in each scan geometry scenario (similar to Figs. 3e and 3f for the Del City storm). In this storm, introduction of the vorticity equation appears to allow for improvement over the control technique in every case. The normalized errors from all of the techniques show an increase over the Del City storm (in spite of generally lower rms errors), due to the generally weaker updraft and downdrafts of the microburst storm. The case-by-case trends in the correlation coefficients for the two storms are nearly identical, although the correlations are slightly higher in the Del City storm.

Since more than 1700 dual-Doppler analyses were performed for this work, it is unfortunately not feasible to thoroughly examine the spatial structures of all the retrieved vertical velocity fields. Nonetheless, it is desirable that at least a cursory examination of the spatial structure of the analyses be presented. Toward this end, Fig. 6 shows the vertical velocity fields at \( t = 3600 \) s from some of the techniques discussed for case III (data coverage from 2° to 20° elevation) of this microburst storm. This is one of many cases in which the vorticity equation offers a clear benefit to the analyses. The wind fields at \( z = 1.5 \) and 10 km are examined in the left- and right-hand panels, respectively, of Fig. 6. The lower level (1.5 km) represents the first level above ground for which data were available across most of the subdomain shown. The upper level (10 km) is the highest level for which most significant vertical motions associated with the storm are present in the dual-Doppler analyses. Above \( z = 10 \) km the 20° elevation angle (marked by the missing data regions at 10 km) passes through the updraft, and data become unavailable. Comparison of Fig. 6a with Figs. 6c and 6e shows that the vertical velocity field at \( z = 1.5 \) km is very poorly retrieved when the lower boundary fields are determined through extrapolation of divergence. Whereas the true field shows a large downdraft area and only insignificant updrafts, the fields from extrapolated divergence show large, spurious updrafts and practically no downdraft. On the other hand, the low-level vertical velocity fields are retrieved much better with the techniques using the vorticity equation to estimate the lower boundary condition (cf. Fig. 6a with Figs. 6c and 6e). When combined with the effects of accumulating divergence errors, the poor estimation of the low-level vertical velocity field in control technique U-Eb yields very poor results at \( z = 10 \) km (cf. Figs. 6b and 6d). If the vorticity equation is used to determine both lower and upper boundary conditions, we gain both a better representation of the
Fig. 7. Vertical profiles of the rms vertical velocity retrieval error from the control and best vorticity–based techniques for each storm simulation given (a) case I scan geometry, (b) case II scan geometry, (c) case III scan geometry, and (d) case IV scan geometry. The profiles shown represent averages over all 24 retrieval times for each storm.

low-level vertical velocity field and a means of controlling accumulating divergence errors. As a result, the vertical velocity fields at $z = 10$ km from techniques C-Vb-Vt-90 and C-Vb-Vt-300 (Figs. 6h and 6j, respectively) show significant improvement over the control technique (and even over technique C-Eb-Vt-90, shown in Fig. 6f).

Although the vorticity techniques correctly retrieve the bulk features of the vertical velocity field in Fig. 6, it is evident that they generally underestimate the true updraft and downdraft velocity magnitudes. Slight underestimation of magnitudes of vertical velocity maxima is a persistent problem with the vorticity techniques in this work. However, the problem seems to be attributable to imperfect filters rather than a property of the techniques. Although removal of only $2\Delta x$ and $3\Delta x$ scales is intended, the filtering schemes used have a significant damping effect on scales of $4\Delta x$ and larger as well. Since wave interactions on the smallest remaining scales are responsible for creating the small-scale features in the boundary condition fields, these features are inevitably damped in magnitude. This problem is closely related to the discussion at the end of section 3a.

Figure 7 presents an alternative view of the spatial structure of the retrieval errors. The average vertical profiles of the rms error from the control and best vorticity–based technique for each scan geometry scenario in each storm are explored in Figs. 7a–d. These profiles represent the averages over the entire length of the respective simulations. The benefits of applying the vorticity equation to reduce the accumulation of divergence errors during vertical integration of the mass continuity equation are immediately apparent in Fig. 7a, as the improvement over the control techniques grows substantially as the height above the fixed (from impermeability) lower boundary condition increases. In Figs. 7b and 7c, the additional benefits of using the vorticity equation to specify both boundary conditions become apparent. This is evidenced by the reduction in rms errors over the entire depth of the analysis domains, including the lower boundary (where extrapolating divergence has traditionally been the preferred option of boundary condition specification). Figure 7d shows how application of the vorticity equation is still able to yield a reduction in the accumulation of divergence errors, even when downward integration is used as the control technique.

d. Limiting the data region to the storm volume

In all of the experiments discussed thus far data have been limited only by radar elevation angle, leaving data
available over a significant region where no hydrometeors would be available to provide radar observations. These experiments were useful because they provide a clear picture of the results from each technique that is not dependent upon storm shape, presence of precipitation, or other logistical factors complicating the analyses. However, for real data applications it is important to show that the methods proposed here continue to work well under more realistic data coverage scenarios. To investigate this situation it is useful to revisit cases I and IV of the microburst storm, but limiting data to regions containing hydrometeors (as defined by a 0.1 g kg\(^{-1}\) total mixing ratio of rainwater, ice, snow, and hail). From Srivastava (1985) we may deduce that this region should correspond roughly to a 20–25 dB\(\text{Z}\) reflectivity threshold. The east–west cross-sectional area of the new restricted analysis region is slightly smaller than the region contained within the 0.1 g kg\(^{-1}\) mixing ratio contour of Figs. 4b, 4d, 4f, and 4h, which includes cloud-water mixing ratio in addition to those previously mentioned.

Technique U-Ib was again chosen as the control technique for case I. However, unlike the experiments of section 4c, data were not available to ground level in all areas since hydrometeors were not present down to ground level in every column. Given this fact, technique U-Eb would appear to have been a logical control technique. However, the rms errors from the two techniques favored technique U-Ib. This result was likely a combination of the shortcomings of extrapolating divergence in microburst situations and also the need to extrapolate divergence from varying heights dependent upon the hydrometeor distribution, which can create considerable noise in the vertical velocity field. Since we are not restricted to either of these modes of “guessing” a boundary condition with technique C-Vb-Vt, it was chosen as an ideal vorticity technique for comparison. The results of this experiment (Fig. 8a) continue to be encouraging. Even when faced with a much more limited data region, technique C-Vb-Vt, with either 90- or 300-s volume scan periods, is still preferable over the control technique. Techniques C-Vb-Vt-90 and C-Vb-Vt-300 show error reductions of as much as 27% over the control technique, comparable to the 36% reduction in error noted for the same case and technique (but with a much larger data region) in section 4c. Although this same experiment was not carried out for cases II and III, we can likely infer from the results of section 4c that the vorticity-based techniques would become even more advantageous when data at low elevation angles is lacking.

Case IV continues to be more problematic for the vorticity techniques given this limited data region. Only technique C-Vb-It-90 was able to show any improvement over technique D-It, with a small 3% error reduction from 1.85 to 1.78 m s\(^{-1}\) (see Fig. 8b). Technique C-Vb-Vt-90, the best technique for case I, showed slightly larger errors than technique D-It. It should be noted that the true vertical velocity at the storm top (as defined by the presence of hydrometeors in this simulation) is very near zero. However, it has been noted in the literature that this is not always the case with the “storm top” as observed by radar (Protat and Zawadzki 1999). When the vertical velocity at storm top is suspected of differing significantly from zero, the vorticity-based techniques may hold more promise for improvement. Also, all previous discussion pertaining to the potential for more significant improvements over the control technique when systematic errors are present is still valid with this case.

5. Summary and conclusions

In this work we revisit the long-standing problem of dual-Doppler analysis of the vertical velocity field. Our approach focuses on the use of the anelastic vertical

---

**Fig. 8.** Root-mean-square vertical velocity retrieval errors as a function of simulation time from experiments on the microburst storm with the data region limited to areas where hydrometeors are present. The volume scan geometries for these experiments provided data within the storm between (a) 0° and 20°, and (b) 1° and 30°. Errors are expressed in m s\(^{-1}\). The control cases for each experiment are indicated with a solid bold line. Average rms errors for each technique across the 22 times shown in the graphs are indicated in parentheses. The data region before \(t = 900\) s was too small to permit comparisons.
vorticity equation as a new analysis constraint. Several new multistep dual-Doppler analysis methods are proposed, each building upon traditional analysis frameworks but incorporating the anelastic vertical vorticity equation as a weak constraint to aid in the retrieval of appropriate boundary condition(s).

A series of four data coverage scenarios were investigated. Each was designed around the scanning capabilities of current operational and research radars. The scenarios were implemented in two different simulations, one of a supercell thunderstorm and another of a microburst-producing thunderstorm.

Based on the results of retrievals using these simulated datasets the vorticity equation appears to hold significant potential for improving vertical velocity retrievals in datasets where data are not available to storm top. When data were available to storm top, the vorticity equation was only capable of lending modest improvements to the traditional downward-integration technique in our experiments. However, we have applied only random errors to the data in our experiments. Techniques relying upon direct integration of mass continuity (such as techniques U and D) have been shown by MB98 to be very susceptible to bias errors, while those constraining the vertical velocity field with two boundary conditions (such as MB98’s technique B) were shown to be much more robust in the presence of these systematic errors. This suggests that the vorticity equation applied with our technique C might provide more substantial benefits than downward integration from storm top in other datasets that exhibit nonrandom errors, although this has yet to be verified experimentally.

The reader may notice that no results from the technique suggested in section 3a have been shown. This technique was applied on each of the cases investigated within this work, but the results were generally inferior to those obtained from technique C (the framework of which was presented in section 3b). We believe that this is because the technique of section 3a (which can be either technique U or D, with the boundary condition estimated with the vorticity equation) suffers from accumulating divergence errors, as in the control techniques of this work. Results of these experiments were generally inferior to those of the control technique in cases I and IV with both storms (i.e., the vorticity equation was inferior to the impermeability condition). However, results were similar to the control technique in case II and superior to those of the control technique in case III.

Overall, the results of this work indicate that the vorticity equation will generally not be preferable to application of the impermeability condition as a means of determining an appropriate boundary condition in cases where the impermeability condition can be applied with confidence. However, it does appear that the vorticity equation may be preferable to extrapolating divergence for a lower boundary condition in many cases. The wide adaptability of the vorticity techniques is highlighted when applied to retrieve a second opposing boundary condition when one boundary condition is already known through other means, as it can help limit accumulating divergence errors in this case. This may prove to be one of the most widely applicable scenarios for using the vorticity equation, since datasets frequently fail to span from ground level to storm top (so that impermeability cannot be applied safely at both boundaries of the data region). In cases where data fail to extend to any level where impermeability could be applied, the vorticity equation still provides significant potential for reliable vertical velocity retrievals. This appears to be true even in situations that traditionally would have prohibited such analyses.

It was noted by Protat and Zawadzki (2000) that inclusion of the vorticity equation in their analysis framework led to little improvement in the three-dimensional wind field, although it did significantly improve temperature and pressure retrievals. It is our belief that the vastly different nature of the approach taken here is responsible for the disparity in results.

Two important findings have come out of this work. First, we have shown that retrievals performed with 90-s volume scan periods provide results widely superior to retrievals performed with 300-s volume scan periods, especially under rapidly evolving storm conditions. The high-temporal-resolution data allow the vorticity techniques to outperform traditional methods of boundary condition specification in most cases. In fact, with 90-s volume scan periods the vorticity equation was able to improve upon the control analyses in each of the four scan geometry cases, at virtually all retrieval times, for both storms. This fact argues strongly for the deployment of more rapid scanning research radars in meteorological field experiments and (eventually) in operational radar networks. Another significant finding is that, although the vorticity equation is sensitive to shear magnitude, the vertically integrated nature in which it has been applied here seems to negate the need for strong vertical wind shear and, additionally, provides an error-filtering effect. Results were actually superior with the lower shear environment of the microburst storm.

Future work on our retrieval methods will explore methods for weighting function specification to further improve results, investigations with nonrandom errors, and on application of these techniques to real data.

Acknowledgments. We thank John Albert at the University of Oklahoma Department of Mathematics and David Dowell at the University of Oklahoma School of Meteorology for useful discussions. This research was supported by the United States Department of Defense (Office of Naval Research) through an Augmentation Award for Science and Engineering Research Training (AASERT Grant N00014-97-1-0763, a supplement to Grant N00014-96-1-1112), by the Center for Analysis and Prediction of Storms under Grant ATM91-20009.
Setting the first variations of (A1) with respect to velocity of scatterers within the observation volume. The radar beams observing a point \((x, y, z)\) where \(x, y, z\) are the direction cosines of the two radar beams observing a point \((x, y, z)\) and \(V_1, V_2, Z_1, Z_2\) are the direction cosines of the two radar beams observing a point \((x, y, z)\) and \(W = (w + V_t)\), where \(V_t\) is negative and represents the terminal velocity of scatterers within the observation volume. Here we minimize the cost functional:

\[
J = \int_V \left[ \alpha^2(V_{11} - X_t u - Y_t u - Z_t W)^2 + \beta^2(V_{22} - X_t u - Y_t u - Z_t W)^2 + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial (\rho w)}{\partial z} \right) \right] dV, \quad (A1)
\]

where \(\lambda\) is a Lagrange multiplier, \(\vec{R}_1 = (X_t, Y_t, Z_t)\) and \(\vec{R}_2 = (X_t, Y_t, Z_t)\) are the direction cosines of the two radar beams observing a point \((x, y, z)\), and \(W = (w + V_t)\), where \(V_t\) is negative and represents the terminal velocity of scatterers within the observation volume. Setting the first variations of (A1) with respect to \(u, v\), and \(\lambda\) to zero and simplifying yields the system:

\[
u = 1 \frac{\Theta - A_2 \frac{\partial \lambda}{\partial x} + A_1 \frac{\partial \lambda}{\partial y}}{\Theta}
\]

\[
\frac{\partial^2 \lambda}{\partial x^2} - (A_1 + B_1) \frac{\partial^2 \lambda}{\partial x \partial y} + A_1 \frac{\partial^2 \lambda}{\partial y^2} + \left( \frac{\partial B_1}{\partial x} - \frac{\partial A_2}{\partial y} - B_2 \frac{\partial \Theta}{\partial x} + A_2 \frac{\partial \Theta}{\partial y} \right) \frac{\partial \lambda}{\partial x}
\]

\[
\frac{\partial \lambda}{\partial x} + \frac{\partial \Phi}{\partial x} - \Phi \frac{\partial \Theta}{\partial x} - \Omega \frac{\partial \Theta}{\partial y} = 0,
\]

where

\[
A_1 = 2(\alpha^2 X_1 + \beta^2 X_2)
\]

\[
B_1 = 2(\alpha^2 Y_1 + \beta^2 Y_2)
\]

\[
C_1 = 2[\alpha^2 X_2(Z_1 W - V_{11}) + \beta^2 X_2(Z_1 W - V_{11})]
\]

Notice that \(\Theta\) is proportional to the square of the \(k\) component of the cross product between \(\textbf{R}_1\) and \(\textbf{R}_2\). This term will be nonzero and the system (A2) will be defined only if (i) \(\alpha\) and \(\beta\) are nonzero everywhere, (ii) the two Doppler radars are not collocated, and (iii) observational points directly over the baseline are not included in the analysis domain. Under these same conditions the second-order PDE for \(\lambda\) in (A2) can be shown to be elliptic. Conveniently, the natural boundary conditions for this PDE are \(\lambda = 0\). After retrieving the \(\lambda\) field, the \(u\) and \(v\) fields can be obtained using the first two equations of (A2). Note that it is important to use a discretization of the mass continuity equation in this correction step that is consistent with the discretization used in (13) of the main text of this article and in constructing the vertical velocity field.

To implement this correction step, the system (A2) must be solved separately on each vertical level of the analysis grid. Since the anelastic mass continuity equation provides only \(nz-1\) relationships between the wind fields on the \(nz\) grid levels, the horizontal wind field on one vertical grid level must be left unadjusted or must be corrected using alternative assumptions (e.g., accepting the radial wind observations as strong constraints).

REFERENCES


Gal-Chen, T., 1982: Errors in fixed and moving frame of references:


