Developing and Evaluating Ice Cloud Parameterizations for Forward Modeling of Radar Moments Using in situ Aircraft Observations

MAXIMILIAN MAAHN AND ULRICH LÖHNERT
Institute for Geophysics and Meteorology, University of Cologne, Cologne, Germany

PAVLOS KOLLIAS
Department of Atmospheric and Oceanic Sciences, McGill University, Montreal, Quebec, Canada

ROBERT C. JACKSON AND GREG M. MCFARQUHAR
Department of Atmospheric Sciences, University of Illinois at Urbana–Champaign, Urbana, Illinois

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ABSTRACT

Observing ice clouds using zenith pointing millimeter cloud radars is challenging because the transfer functions relating the observables to meteorological quantities are not uniquely defined. Here, the authors use a spectral radar simulator to develop a consistent dataset containing particle mass, area, and size distribution as functions of size. This is an essential prerequisite for radar sensitivity studies and retrieval development. The data are obtained from aircraft in situ and ground-based radar observations during the Indirect and Semi-Direct Aerosol Campaign (ISDAC) campaign in Alaska. The two main results of this study are as follows: 1) An improved method to estimate the particle mass–size relation as a function of temperature is developed and successfully evaluated by combining aircraft in situ and radar observations. The method relies on a functional relation between reflectivity and Doppler velocity. 2) The impact on the Doppler spectrum by replacing measurements of particle area and size distribution by recent analytical expressions is investigated. For this, higher-order moments such as skewness and kurtosis as well as the slopes of the Doppler spectrum are also used as a proxy for the Doppler spectrum. For the area–size relation, it is found that a power law is not sufficient to describe particle area and small deviations from a power law are essential for obtaining consistent higher moments. For particle size distributions, the normalization approach for the gamma distribution of Testud et al., adapted to maximum diameter as size descriptor, is preferred.

1. Introduction

Ice clouds are ubiquitous throughout Earth’s atmosphere and play a key role in Earth’s climate system (Waliser et al. 2009). Ice clouds influence Earth’s radiative budget, contribute to the dehydration of the upper atmosphere, and are crucial for the global hydrological cycle (Curry et al. 1996; Morrison et al. 2012). At the same time, they present one of the largest uncertainties in general circulation models because many of the processes of ice clouds are poorly understood, occur at subgrid scale, and observational constraints are limited (e.g., Zhang et al. 2005). Because of their ability to penetrate optically thick clouds and to detect even thin cirrus clouds, cloud radars have been increasingly used to study ice cloud microphysics (e.g., Brown et al. 1995; Benedetti et al. 2003; Matrosov et al. 2002).

One of the main challenges to measure ice clouds by remote sensing is that the transfer functions that relate the observables (e.g., radar Doppler spectrum) to cloud properties (e.g., ice water content IWC) are not uniquely defined (Szyrmer et al. 2012). To overcome this challenge, this study has two main objectives: 1) estimate a mass–size relation $m(D)$ from a combination of in situ and radar observations and 2) investigate the effect of describing particle area $A$ and number concentration $N$ as functions of size $D$ on moments of the radar Doppler spectrum. Both parts together result in a consistent set of equations to describe $m(D)$, $A(D)$, and $N(D)$ for...
simulating the radar Doppler spectrum. Such a set of equations is an essential prerequisite for sensitivity studies and the development of radar retrievals exploiting radar moments.

The first objective is investigated in this study because $m(D)$ introduces one of the greatest uncertainties when relating remote sensing observables to meteorological quantities. At the same time, $m(D)$ is particularly difficult to measure in situ. It can be obtained by collecting and melting single particles at the ground, but sample sizes are often very small (Magono and Lee 1966; Locatelli and Hobbs 1974; Mitchell et al. 1990). From aircraft, only ice water content IWC can be measured directly by probes (Korolev et al. 1998; Noone et al. 1988) and $m(D)$ has to be estimated from closure studies (Brown and Francis 1995; Heymsfield et al. 2004). Another, yet rather indirect, possibility is relating projected area (Schmitt and Heymsfield 2010) or particle shape (Jackson et al. 2012) as measured by particle imaging probes to $m(D)$. Most ice cloud retrievals using remote sensing instruments retrieve not $m(D)$ but IWC (e.g., Matrosov et al. 2002; Delanoë and Hogan 2008) and often need prior knowledge of $m(D)$ (e.g., Posselt and Mace 2014). In this study, we estimate $m(D)$ as a function of temperature by combining radar observations with aircraft in situ measurements. We use optimal estimation (Rodgers 2000) to find the pair of $m(D)$ coefficients such that the functional relation between effective radar reflectivity factor $Z_e$ and Doppler velocity $W$ observed by radar and forward modeled from in situ data match.

Knowledge of $m(D)$ enables us to investigate the second objective: similar to $m(D)$, $A(D)$ is commonly expressed by a power law. Gamma (Schneider and Stephens 1995) and exponential distributions (Gunn and Marshall 1958) are commonly used to describe $N(D)$ for snow and ice. More recently, Petty and Huang (2011) showed that they can be generalized for all particle size descriptors using the four-parameter modified gamma function. The use of lognormal distributions for describing cirrus clouds has also been described (Tian et al. 2010). The Indirect and Semi-Direct Aerosol Campaign (ISDAC) dataset with measured profiles of $A(D)$ and $N(D)$ allows us to investigate the impact of applying these functions on simulated radar observations. In addition, the sensitivity on assumed minimum and maximum particle size can be evaluated. For this, not only the standard moments $Z_e$, $W$, and Doppler spectrum width $\sigma$ are investigated but also the use of higher moments of the Doppler spectrum, such as skewness ($S_k$, third moment) and kurtosis ($K_4$, fourth moment) together with left slope (“slow” side of the spectrum) and right slope (“fast” side of the spectrum) of the peak ($S_l$ and $S_r$), is proposed as a proxy for the shape of the radar Doppler spectrum. For the following, we will call $S_k$, $K_4$, $S_l$, and $S_r$ “higher moments” even though the slopes are technically not moments. Consequently, $Z_e$, $W$, and $\sigma$ will be called “lower moments.” While the radar spectrum and higher radar moments have been used to estimate liquid water content (Babb et al. 1999) and precipitation rate (Atlas et al. 1973), to separate cloud drops from drizzle (Kollias et al. 2011b) and to locate supercooled liquid water (Shupe et al. 2004; Luke et al. 2010), no studies thus far have investigated the use of higher moments for studying ice clouds, and the radar spectrum has only been used for case studies (Verlinde et al. 2013).

To develop the set of parameters to describe $m(D)$, $mA(D)$, and $N(D)$, we combined observations of the millimeter-wavelength cloud radar (MMCR; Moran et al. 1998) at the Atmospheric Radiation Measurement (ARM) North Slope of Alaska site in Barrow with in situ aircraft data acquired during ISDAC (McFarquhar et al. 2011). To convert the in situ observations into higher radar moments, we use the passive and active microwave radiative transfer (PAMTRA) radar simulator (Kollias et al. 2011a; Mech et al. 2015, manuscript submitted to Atmos. Chem Phys.).

The observational dataset and the radar simulator are described in section 2. The novel method to estimate $m(D)$ is introduced and compared to other methods in section 3. Higher moments are used to evaluate parameterizations of $A(D)$ and $N(D)$ in section 4 and concluding remarks on the optimal set of ice cloud parameterization are provided in section 5. See appendix C for an overview of the most used abbreviations and symbols.

2. Dataset, forward operator, and instruments

In this section, the ISDAC dataset is introduced, and the forward operator PAMTRA is briefly described. In addition, the MMCR radar is presented and it is shown how both datasets are combined.

a. ISDAC

For this study, in situ aircraft data acquired during the ISDAC campaign in April 2008 (McFarquhar et al. 2011) are used. The ISDAC dataset contains observations of mostly stratocumulus ice and mixed-phase clouds featuring temperatures ($T$) between −40° and 5°C. It is available at the website of the U.S. Department of Energy ARM program (http://www.arm.gov/campaigns/aaf2008isdac). Best estimates of ice particle size distribution $N(D)$ and area versus size $A(D)$—with $D$ defined as the maximum dimension of the particle—are derived from a combination of the 2D stereo probe (2DS), 2D
cloud probe (2DC), cloud imaging probe (CIP), and 2D precipitation probe (2DP) using the methodology described in Jackson et al. (2012). The 2DC included a modified tip design to mitigate contamination due to shattered artifacts (Korolev et al. 2011, 2013a). The data are corrected for shattering effects using the interarrival time method (Field et al. 2006) and using the numbers, sizes, and gaps between fragments in a single image (Korolev and Isaac 2005). Although this approach reduces contributions from artifacts (Lawson 2011), shattered particles still contribute to $N(D)$ for $D < 0.5 \text{ mm}$ (Korolev et al. 2013b; Jackson et al. 2014). Jackson and McFarquhar (2014) showed for the 2DC that shattered particles made contributions of only about 15% to 20% to higher-order moments of $N(D)$, such as IWC and total area, when probes without modified tips are used. Thus, contributions of shattered artifacts to the moments of the $N(D)$ of the ISDAC in situ dataset are expected to be even less. As a consequence, particles with $D < 0.5 \text{ mm}$ are included for all approaches based on moments of $N(D)$ but removed for all least squares–based methods to avoid artifacts due to shattering. In this study, ice particles are assumed to be larger than 50 $\mu$m; hence, smaller particles are discarded because the uncertainty in their measured concentrations is high as shown by Jackson et al. (2012) for ISDAC.

Furthermore, the ISDAC dataset contains bulk measurements such as $T$, humidity ($q$), pressure ($p$), and IWC. Because of technical problems with the Nezzorov and CSI probes associated with electrical interference, IWC is derived from a combination of the size distributions measured by the optical array probes and shape distributions from the habit identification scheme of Jackson et al. (2012). Eddy dissipation rate $\varepsilon$, which is a measure of turbulence, was estimated from measurements of the Tropospheric Aircraft Meteorological Data Reports (TAMDAR) system (Moninger et al. 2010). TAMDAR reports $\sqrt{\varepsilon}$ at a low spatiotemporal resolution in only 27 discrete intervals. During ISDAC, $\varepsilon$ was—with very few exceptions at very low altitudes—reported as the lowest measurement interval “00,” which corresponds to $\sqrt{\varepsilon} < 0.1 \text{ m}^{2/3} \text{ s}^{-1}$ for both maximum and mean $\sqrt{\varepsilon}$ within the measurement increment. This highlights the low turbulence conditions during ISDAC around Barrow and, consequently, we assume an even lower, constant value of $\sqrt{\varepsilon} = 0.01 \text{ m}^{2/3} \text{ s}^{-1}$ for the complete dataset.

For this study, the integration time is increased from 1 to 10 s to ensure that the number of detected particles represents a statistically significant sample. The impact of averaging time on the results is discussed in appendix B, section c. Only particles classified as ice are considered in this study. In addition, ice particles obtained when more than 0.01 $\text{g m}^{-3}$ liquid water was detected by the King probe are not used in this study, representing 14% of the total dataset. Assuming $D = 20 \mu\text{m}$, 0.01 $\text{g m}^{-3}$ correspond to roughly $-38 \text{ dBZ}$, which is close to the lowest values observed for $Z_e$ observed in this study (section 4d), so we consider an impact of remaining liquid on the analysis unlikely.

b. PAMTRA radar simulator

To convert the ISDAC in situ microphysical measurements into radar observables (forward modeled ISDAC observation are called F-ISDAC herein), the passive and active microwave radiative transfer (PAMTRA) model, developed at the University of Cologne, is used (Mech et al. 2015, manuscript submitted to Atmos. Chem. Phys.). The active part of PAMTRA is based on Kollias et al. (2011a), but modifications to treat the microphysical and scattering properties of ice particles are necessary, as described below.

First, the backscattering cross sections for each size bin are calculated. For this, the $\mathbf{T}$-matrix method (Mishchenko 2000) is used assuming that the particles can be modeled as horizontally aligned, soft oblate spheroids with an aspect ratio (AR) of 0.6 (Hogan et al. 2012) defined as mixtures of ice and air (Petty 2001). To obtain the refractive index of soft spheres from the refractive index of solid ice [estimated using Warren and Brandt (2008)], the mixing formula of Maxwell Garnett (1904) is used, which depends on the effective particle density that follows from the mass-dimensional relationship and the specified spheroid. Even though databases based on realistic particle shapes using DDA (e.g., Liu 2008; Petty and Huang 2010; Tyynelä et al. 2011) are available, we prefer the $\mathbf{T}$-matrix method, because the databases publicly available do not cover all needed $A(D)$ and $m(D)$ relations. Several studies showed (e.g., Kneifel et al. 2011; Tyynelä et al. 2011; Hogan et al. 2012) that at 35 GHz, the $\mathbf{T}$-matrix approximation can be used at least for particles with $D < (5–10) \text{ mm}$. For the ISDAC dataset, larger particles are rare (see section 4c). The impact of assuming a fixed AR of 0.6 is discussed in appendix B, section a. As attenuation is expected to be low for snow and ice at 35 GHz (Matrosov 2007), attenuation effects are ignored in this study.

The effective radar reflectivity factor $Z_e$ can be obtained by integrating $\sigma(D) \times N(D)$ over $D$. If, however, higher radar moments need to be modeled, the Doppler spectrum has to be simulated, which requires application of a particle fall velocity versus diameter

\[ \text{1 A version of the McGill Radar Doppler Spectra Simulator (MRDSS) tuned for liquid particles is available at http://radarscience.weebly.com/software.html.} \]
relationship \( v(D) \). This velocity is a function of particle habit and is estimated using the method proposed by Heymsfield and Westbrook (2010), which depends on the particle shape by using \( D, m(D), \) and \( A(D) \) in addition to environmental conditions such as air density, \( T, \) and \( p \) as input. Here, positive values of \( v \) refer to particles that are falling toward the radar.

The contribution of the kinematic broadening to the radar Doppler spectrum is assumed to have a Gaussian distribution with standard deviation \( \sigma_k \) (m s\(^{-1}\)), which is convolved with the radar reflectivity spectrum (Gossard and Strauch 1989). The quantity \( \sigma_k \) is composed of

\[
\sigma_k^2 = \sigma_w^2 + \sigma_s^2 + \sigma_t^2,
\]

where \( \sigma_w^2 \) describes the contribution of the horizontal wind field to the radial velocity due to the finite radar beamwidth, \( \sigma_s^2 \) is the broadening term due to shear of the vertical wind, and \( \sigma_t^2 \) is the variance due to turbulence within the radar sampling volume. The turbulence variance \( \sigma_w^2 \) is estimated by \( \sigma_w^2 = \frac{U^2 \theta^2}{2.76} \) with \( U \) the horizontal wind as measured by the aircraft and \( \theta \) the half-power half-width one-way radar beamwidth in radians (Sloss and Atlas 1968; Nastrom 1997), which is 0.1558 for the MMCR in Barrow. The \( \sigma_t^2 \) broadening is neglected in this study, because it is expected to be smaller than the other terms and gradients of vertical winds are not available from aircraft observations. The quantity \( \sigma_t^2 \) depends on the Eddy dissipation rate \( \varepsilon \) as measured by the aircraft with (Shupe et al. 2008)

\[
\sigma_t^2 = \int_{k_s}^{k} a' \varepsilon^{2/3} k^{-5/3} dk = \frac{3a'}{2} \left( \frac{\varepsilon}{2\pi} \right)^{2/3} (L_s^{2/3} - L_{\lambda}^{2/3}),
\]

where \( k \) is the wavenumber, \( a' \) is the Kolmogorov constant [chosen as 0.5 according to Sreenivasan (1995)], \( L_{\lambda} = \lambda/2 \) is the smallest length scale, and \( L_s \) is the largest length scale observed by the radar. The latter is defined by \( L_s = Ut + \frac{2}{\varepsilon} \sin(\theta) \) with \( t \) the observation time and \( z \) is the height.

Finally, noise is added to the simulated radar Doppler spectrum using the methodology proposed by Zrnić (1975). Then, the spectra are averaged \( n_{\text{avg}} \) times, which reduces the noise variance by \( \sqrt{n_{\text{avg}}} \). The moments of the simulated Doppler spectrum are estimated similar to a real radar data processing scheme (e.g., Maahn and Kollias 2012): first, the noise is removed from the spectrum, then the moments of the most significant peak, defined as peak containing the bin with the greatest power, are determined.

PAMTRA estimates not only the lower, but also the higher moments: the skewness \( S_k \) indicates whether the peak has more weight on the left (\( S_k < 0 \)) or right side.

![Fig. 1. Examples of idealized radar Doppler spectra for different values of (a) skewness \( S_k \), (b) kurtosis \( K_u \) and (c) left \( S_l \) and right slope \( S_r \). Positive values of Doppler velocity refer to particles which are falling toward the radar.](image-url)
(S_k > 0) (Fig. 1a). The kurtosis $K_e$ describes the shape of the peak (Fig. 1b): while $K_e$ of a Gaussian is three, values below indicate a more round tip while larger values indicate a pointed tip. In addition to these higher moments, the slopes in $\text{dB s}^{-1}$ are estimated using the position of the left (right) edge of the spectrum, the mean noise power, and the position as well as power of the spectral peak (Fig. 1c). In contrast to $Z_e$, higher moments do not depend on the absolute calibration of the radar. Furthermore, the mean vertical air motion shifts only the Doppler spectrum and $W$ but does not change the higher moments. Higher moments and the slopes in particular are, however, strongly influenced by atmospheric turbulence and depend on the signal-to-noise ratio of the radar.

c. **MMCR**

As a reference, the MMCR of the U.S. Department of Energy ARM program located in Barrow (71.32°N, 156.62°W, 8 m MSL) at the North Slope of Alaska (NSA) is used. It is a vertical pointing pulsed radar system with a frequency of 35 GHz ($\lambda = 8.6$ mm) and can detect hydrometeors with $Z_e$ as low as $-50$ dBZ (Moran et al. 1998). For this study, the so-called boundary layer mode is used (Kollias et al. 2007), since it provides the best sensitivity and the highest Doppler velocity resolution covering a height range up to 6 km with a vertical resolution of 43.7 m. The boundary layer mode uses 10 spectral averages and is available with an effective time resolution of 1.4 s. Protat et al. (2011) found a calibration offset of $-9.8$ dB through comparisons of MMCR data obtained between March and October 2008 and the CloudSat satellite (Stephens et al. 2008), so the dataset is corrected accordingly.

The presence of supercooled liquid water can influence the radar Doppler spectrum (Shupe et al. 2004; Luke et al. 2010). Consequently, observations with a liquid water path measured by a microwave radiometer of more than 2010) are removed (15% of the radar dataset). Assuming a cloud depth of 2.5 km, this corresponds to a mean reflectivity due to liquid of $-32$ dBZ that can be neglected, because observations with $Z_e < -30$ dBZ were rare during ISDAC (section 4d). To reduce the impact of nonuniform beam filling effects near cloud edges, the MMCR observations at the boundaries of the observed hydrometeor layers are excluded in the analysis.

The synthetic radar Doppler spectra (forward modeled from the in situ measurements) and the MMCR recorded radar Doppler spectra (product nsamnerspecmomC1) are postprocessed with the same moment estimators to ensure consistency between the two data sources. Ghost peaks in the Doppler spectrum mirrored at 0 m s$^{-1}$, which occur for high $Z_e$, are filtered by processing the most significant peak only. To remove rare cases where ghost and real peak are not clearly separated, peaks with the left edge of the peak below $-0.5$ m s$^{-1}$ are also discarded.

Temperature profiles corresponding to the MMCR observations were obtained using the Merged Sounding product (Troyan and Jensen 1996), which is based on radiosonde observations interpolated with microwave radiometer measurements and model output of the European Centre for Medium-Range Weather Forecasts (ECMWF). Data above 0°C are discarded.

d. **Combination of the datasets**

Direct comparisons of aircraft and ground-based remote sensing instruments are challenging because of mismatches in time, space, and sampling volumes. Thus, MMCR observations and F-ISDAC calculations can only be compared statistically. For this, only aircraft measurements within a radius of 10 km around Barrow are used. The size of the radius is a tradeoff between the need for a sufficiently large dataset and the requirement to have in situ measurements that are representative for the conditions around the ARM site and capture the small-scale variability (see appendix B, section b for discussion of the 10-km radius). The MMCR dataset, in turn, is restricted to observations, where the aircraft was in a horizontal plane closer than 10 km, within 60 s, and within an altitude of $\pm 90$ m (2 MMCR height bins). After filtering, around 2100 10-s observations remained in the F-ISDAC and around 57200 observations remained in the MMCR dataset.

3. **Determination of the mass–size relation**

No direct measurements of particle mass versus diameter $m(D)$ are available for ISDAC; only IWC is included in the dataset obtained from shape analysis. Thus, we need to estimate $m(D)$, which is the first objective of this study. In this section, a novel approach for determination of the coefficients $a$ and $b$ of the power law mass–size relation $m(D)$:

$$m(D) = aD^b,$$

is presented and compared with other methods by application to the ISDAC dataset.

a. **Estimating the mass–size relation from the reflectivity–mean Doppler velocity relation**

The new method to estimate the mass–size relation $m(D)$ is based on the relation between $Z_e$ and $W$:

$$W = e(Z_e)^f,$$

where the coefficients $e$ and $f$ depend on microphysical conditions (Kalesse et al. 2013). Figure 2 shows that for
the MMCR dataset, there is a clear correlation between 
$Z_e$ and $W$ even though there is a high spread. Although 
vertical air motions also contribute to $W$, we expect that 
they do not influence the least squares fit to determine the 
coefficients $e$ and $f$ and cancel out over time. Extreme 
outliers are manually removed (see Fig. 2).

All parameters needed to estimate $Z_e$ and $W$ with 
PAMTRA are contained in the ISDAC dataset (see Fig. 3 
for flowchart diagram) except for $m(D)$. Consequently, we 
choose $m(D)$ such that the functional relation between $Z_e$ 
and $W$ is maintained and $e$ and $f$ of the MMCR observations 
and F-ISDAC agree. Using $m(D)$ as a closure leads to 
the fact that biases of the forward operator and especially 
the included $v(D)$ relation translate directly into biases of 
$m(D)$. As an advantage for remote sensing applications, 
$m(D)$ is estimated from the particle size range being most 
important most for radar observations (i.e., large particles). 
In situ observations of IWC are not required.

To find the optimal $m(D)$, we make use of the optimal 
estimation (OE) theory (Rodgers 2000). OE is a sim-
plicated Bayesian retrieval technique based on a Gaussian 
statistical model of the problem in addition to a priori 
information to estimate the state vector $\mathbf{x}$. The updated 
state vector $\mathbf{x}_{i+1}$ is obtained with

$$
\mathbf{x}_{i+1} = \mathbf{x}_i + (\gamma_i \mathbf{S}_a^{-1} + \mathbf{K}_i^T \mathbf{S}_e^{-1} \mathbf{K}_i)^{-1}\mathbf{K}_i \mathbf{x}_e^{-1} [\mathbf{y} - P(\mathbf{x}_i) \\
+ \mathbf{K}_i (\mathbf{x}_i - \mathbf{x}_e)],
$$

(5)

where $\mathbf{x}_e$ is the a priori assumption for $\mathbf{x}$, $\mathbf{S}_a$ is the a priori 
uncertainty expressed as the covariance matrix of $\mathbf{x}_a$, $\mathbf{S}_e$ 
is the uncertainty of $\mathbf{y}$ expressed as the measurement 
covariance matrix, and $\mathbf{K}_i$ is the Jacobian matrix of the 
forward model $P(\mathbf{x}_i)$; that is, $\mathbf{K}_i = \frac{\partial P(\mathbf{x}_i)}{\partial \mathbf{x}_i_{i=1...N}}$ 
with $x_i$ the elements of $\mathbf{x}_i$. The quantity $\gamma_i$ is an addi-
tional factor following Turner and Löhner (2014) to 
put a higher weight on $\mathbf{S}_e$ in the beginning of the re-
trieval in order to stabilize the retrieval in case of a bad 
first guess. Similarly, we use decreasing values of 10000, 
3000, 1000, 300, 100, 30, 10, 3, 1, 1, ... for $\gamma_i$. Because of 
the Bayesian concept, the uncertainty of the retrieval 
solution can be easily estimated from the covariance 
matrix $\mathbf{S}_i$ of the problem

$$
\mathbf{S}_i = \mathbf{B}_i^{-1} (\gamma_i^2 \mathbf{S}_a^{-1} + \mathbf{K}_i^T \mathbf{S}_e^{-1} \mathbf{K}_i)^{-1},
$$

(6)

where

$$
\mathbf{B}_i = (\gamma_i \mathbf{S}_a^{-1} + \mathbf{K}_i^T \mathbf{S}_e^{-1} \mathbf{K}_i).
$$

(7)

The iteration is started with $\mathbf{x}_1 = \mathbf{x}_a$ and is stopped when $\gamma_i = 1$ and the convergence criteria

$$
(\mathbf{x}_i - \mathbf{x}_{i+1})^T \mathbf{S}_i^{-1} (\mathbf{x}_i - \mathbf{x}_{i+1}) \ll N
$$

(8)

is met.

In this study, the retrieval is not applied to individual 
profiles but to the complete dataset. Consequently, $\mathbf{y}$ 
does not consist of radar observables such as $Z_e$ or $W$ but 
instead consists of the coefficients $e$, $f$ of the $Z_e - W$ 
relation, whereby $\mathbf{x}$ consists of the parameters of the 
$m(D)$ relation. Thus, parameters of the $m(D)$ relation can 
be found such that F-ISDAC features a similar $Z_e - W$ 
relation as the reference MMCR dataset. For the for-
ward model $P$, we use the PAMTRA forward operator, 
which is applied to all ISDAC profiles for every iteration 
step $i$. To ensure the stability of the retrieval, it is nec-
necessary to include median ($Z_e$) of the dataset as a third 
variable of $\mathbf{y}$. This ensures that OE does not solve the 
problem by iterating to very small values of $Z_e$ as a 
result of reducing $m(D)$ such that the majority of sim-
ulated $Z_e$ observations is below the sensitivity thresh-
old of the MMCR. In this case, the least squares fit of 
Eq. (4) could be easily solved because only few points 
would remain.

Equation (3) depends strongly on changes in $b$, which 
makes the retrieval unstable and makes convergence 
difficult to obtain. Hence, we used the normalization 
approach of Szyrmer et al. (2012); they defined a refer-
ence diameter $D^*$ with a corresponding reference mass 
m* given by

$$
m^* = m(D^*) = a(D^*)^b.
$$

(9)
Together with Eq. (3), this leads to

\[
m(D) = m^* (D/D^*)^b = \alpha C_0 (D/D^*)^b.
\]

(10)

Here, \(m^*\) is expressed as an arbitrary reference mass \(C_0\) and a dimensionless factor \(\alpha\). This form has the advantage that \(m(D)\) has a less dependence on \(b\) if the reference size is representative of the observations. In other words, most of the variability is expressed by \(\alpha\) while uncertainties in the estimate of \(b\) have little impact on derived particle mass. In accordance with Szyrmer et al. (2012), \(D^* = 1.2\) mm and \(C_0 = 3 \times 10^{-5}\) g are chosen. Hence, the state vector \(x\) of OE is composed of \(b\) and \(\log_{10}(\alpha)\). The logarithm of \(\alpha\) is used to prevent OE from iterating to negative values. OE requires the definition of an a priori value \(x_0\) for \(x\) and the uncertainties expressed as covariance matrices of a priori \(S_0\) and measurements \(S_e\). As an a priori values, we chose \(b = 2.1\), because it is a typical value for \(m(D)\) relations, and \(\alpha = 1\), since it corresponds to the \(m^*\) value proposed by Szyrmer et al. (2012). The quantity \(S_e\) is estimated from the uncertainty derived from the least squares fit to gain the coefficients \(e\) and \(f\) of the \(Z_e - W\) relation, 1 dB is assumed as the variance \(VAR\) of median \(Z_e\), and off-diagonal entries are neglected. For the diagonal of \(S_\alpha\), \(VAR(\log_{10}\alpha) = 1.0\) and \(VAR(b) = 0.5\) are chosen as estimates for the natural variability, and off-diagonal entries are neglected as well.

Figure 2 shows that there might be also a dependence of the \(Z_e - W\) relation on \(T\) (see Table 1 for coefficients). To investigate this, the retrieval is also applied to data subsets defined by \(T\) intervals between \(-40^\circ\) and \(0^\circ\) with \(5^\circ\) width. Only few observations are available for \(T > -30^\circ\) and thus only one interval from \(-40^\circ\) to \(-30^\circ\) was used.

b. Retrieval results

For the complete dataset, OE converges to \(b = 2.23 \pm 0.002\) and \(\alpha = 0.959 \pm (1.7 \times 10^{-6})\), which corresponds to \(a = 0.093 \pm 0.002\) kg m\(^{-2}\). These are similar to the values found by Mitchell et al. (1990) for aggregates of side planes \((a = 0.083, b = 2.2\) in SI units). Applied to the ISDAC dataset, the resulting \(Z_e - W\) distribution is presented in Fig. 2 and shows good agreement with the least squares fits between MMCR and F-ISDAC. Note that the retrieval errors are extremely small because 1) any forward model errors are neglected and 2) \(\alpha\) and \(b\) most likely vary from profile to profile, but only smoothed values are retrieved here. Hence, the values found for
**Table 1. Coefficients \(c\) and \(f\) of the reflectivity–fall velocity \((Z_m – W)\) relation [Eq. (4)] and retrieved coefficients \(a\), \(a\), and \(b\) of the (normalized) mass–size relation \(m(D)\) [Eqs. (3) and (10)]. All quantities are in SI units, only temperature is in °C and \(Z_m\) is in \(\text{mm}^6\text{m}^{-3}\).**

<table>
<thead>
<tr>
<th>Temp range</th>
<th>(-40) to (0)</th>
<th>(-40) to (-30)</th>
<th>(-30) to (-25)</th>
<th>(-25) to (-20)</th>
<th>(-20) to (-15)</th>
<th>(-15) to (-10)</th>
<th>(-10) to (-5)</th>
<th>(-5) to (0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z_m - W: e)</td>
<td>0.772</td>
<td>0.680</td>
<td>0.760</td>
<td>0.714</td>
<td>0.820</td>
<td>0.717</td>
<td>0.779</td>
<td>0.883</td>
</tr>
<tr>
<td>(Z_m - W: f)</td>
<td>0.102</td>
<td>0.119</td>
<td>0.091</td>
<td>0.096</td>
<td>0.036</td>
<td>0.076</td>
<td>0.129</td>
<td>0.065</td>
</tr>
<tr>
<td>(m(D): a)</td>
<td>0.935</td>
<td>0.568</td>
<td>0.812</td>
<td>0.997</td>
<td>1.282</td>
<td>0.913</td>
<td>0.837</td>
<td>1.195</td>
</tr>
<tr>
<td>(m(D): b)</td>
<td>0.093</td>
<td>0.053</td>
<td>0.042</td>
<td>0.013</td>
<td>0.001</td>
<td>0.020</td>
<td>0.074</td>
<td>0.120</td>
</tr>
<tr>
<td>(m(D): b)</td>
<td>2.228</td>
<td>2.223</td>
<td>2.134</td>
<td>1.933</td>
<td>1.476</td>
<td>2.004</td>
<td>2.215</td>
<td>2.234</td>
</tr>
</tbody>
</table>

\(a\) and \(b\) might be representative for the entire dataset, but their variability from profile to profile is not well represented by the retrieval errors.

The retrieval results for \(a\), \(b\), and derived \(a\) as a function of \(T\) and the corresponding errors (if large enough to be visible) are shown in Fig. 4. The values for \(a\), \(b\), and derived \(a\) vary with \(T\) from 0.57 to 1.28, 1.47 to 2.23, and from 0.001 to 0.120 in SI units, respectively (Table 1). The \(b\) and \(a\) follow an U-shaped curve with values larger than 2.1 for \(T < -25°C\) and \(T > -10°C\). We attribute this to the transition of the dominant shape of snow particles from compact, columnar polycrystals typical for cold temperatures to more fluffy particles, such as aggregated crystals or stellar single crystals, to heavy, rimed particles from compact, columnar polycrystals typical for cold temperatures.

**c. Reference methods for determination of \(m(D)\)**

The novel methods based on the \(Z_m – W\) relation with and without considering \(T\) (called m-ZW and m-ZWT, respectively, herein) are compared with various other methods to estimate \(m(D)\), which are described below.

The widely used \(m(D)\) relationship by Brown and Francis (1995) (m-BF herein) expresses \(m\) as a function of the mean particle diameter \(D_m\) (Heymsfield et al. 2012) converted it to a relation in terms of maximum particle dimension \(D\), where (in SI units)

\[
m(D) = 0.0121D^{1.9}.
\]

Heymsfield et al. (2004) proposed a method (constant \(\Lambda\) m-CL) to estimate the coefficients of the \(m(D)\) relation by testing different values for \(b\) from 1.7 to 2.7 in increments of 0.05 and estimated the corresponding \(a\) by comparison to the measured mean IWC:

\[
a = \frac{\text{IWC}}{\sum N(D)D^b \Delta D}.
\]

where \(\Delta D\) is the size bin width. To find \((a, b)\) most consistent with the measured IWC, the ratio between measured IWC and IWC estimated from \(a\) and \(b\) is calculated for every profile. The selected pair of \(a\) and \(b\) coefficients is the one that demonstrates the least sensitivity on the shape of \(N(D)\), which is characterized by the gamma distribution parameter \(\Lambda\) [see section 4b, Eq. (30) for definition]. Using this method, the following coefficients (in SI units) are found for the limited bulk mass data available in ISDAC:

\[
m(D) = 0.0428D^{2.1}.
\]

Jackson et al. (2012) estimated \(m(D)\) for ISDAC by shape estimation (m-SA). They determined particle...
Francis (1995) as well. For comparison, various $m(D)$ relations from literature are presented at arbitrary but typical temperatures: radiating assemblages of plates (Mitchell et al. 1990), densely rimed dendrites (Locatelli and Hobbs 1974), aggregates of side planes (Mitchell et al. 1990), stellar crystals with broad arms (Mitchell 1996), and aggregates of unrimed radiating assemblages of plates, side planes, bullets, and columns (Locatelli and Hobbs 1974), which is the $m(D)$ relation used by Brown and Francis (1995) as well. For comparison, various $m(D)$ relations from literature are presented at arbitrary but typical temperatures: radiating assemblages of plates (Mitchell et al. 1990), densely rimed dendrites (Locatelli and Hobbs 1974), aggregates of side planes (Mitchell et al. 1990), stellar crystals with broad arms (Mitchell 1996), and aggregates of unrimed radiating assemblages of plates, side planes, bullets, and columns (Locatelli and Hobbs 1974), which is the $m(D)$ relation used by Brown and Francis (1995) as well.

Schmitt and Heymsfield (2010) proposed a method based on fractal geometry (m-FG) that makes it possible to derive the exponent of the power law $m(D)$ relation for every profile individually. They found for midlatitude regions that the exponent of the area–size relation

$$A = cD^d$$  \hspace{1cm} (15)

is related to the exponent of the $m(D)$ relation with $b = 1.25d$. To obtain the coefficients of the $A(D)$ relation, the least squares fit presented in section 4a is used. After obtaining $b$, $a$ is chosen in such a way that the measured IWC is conserved similar to Eq. (13) but for every profile individually. This method has the drawback that it is only applicable to profiles where a good fit of Eq. (15) to the observed values can be made. Hence, profiles with less than five size bins with $N(D)\Delta D > 10\,m^{-3}$ or with an explained variance $R^2$ below 0.7 are excluded from the analysis.

Based on this method, Heymsfield et al. (2013) developed a temperature $T$ dependent ($m$-TD) estimates for $a$ and $b$ from 10 different field campaigns obtained in different regions from $-12^\circ$ to $71^\circ$ latitude. They found

\[
b = 2.31 + 0.0054T \quad \text{and} \quad a = 0.0081e^{0.01317T}10^{2b-3}.
\]  \hspace{1cm} (16)  \hspace{1cm} (17)

**Comparison to MMCR**

To investigate the impact of the various $m(D)$ parameterizations on the radar moments, F-ISDAC is calculated for each method and compared against MMCR observations with histograms and medians of the radar moments $Z_e$, $W$, and $\sigma$ (Fig. 5, columns 1–3). To compare the distributions more quantitatively, we use the Kolmogorov–Smirnov statistic ($d_a$; Massey 1951). Lower values indicate better agreement with $d_a=0$ for samples originating from equal distributions. Note that we use $d_a$ only as a degree of agreement and not to test whether the two underlying probability distributions differ. Here and for all following comparisons, the MMCR dataset is also restricted to periods where the F-ISDAC data are available (i.e., where $N(D)\Delta D > 10\,m^{-3}$ and $R^2 > 0.7$). Consequently, the size of the MMCR dataset is different for each comparison.

The new methods m-ZW and m-ZWT of finding the coefficients of the $m(D)$ relation based on the $Z_e–W$ relation lead to a high agreement of MMCR and F-ISDAC observations. Since only $Z_e$ and $W$ are used to derive the coefficients of the $m(D)$ relation, agreement of $\sigma$ is as an independent test, because $Z_e$, $W$, and $\sigma$ depend all differently and nonlinearly on $m(D)$. This and the good agreement to reported values of $a$ and $b$ lead to the conclusions that uncertainties in the forward model or vertical air motion do not contribute to the bias in the retrieved $a$ and $b$ coefficients. For m-ZWT, $d_a$ is smaller than 0.09 for all moments. The offset of $Z_e$ is $-1.9\,dB$, which is within the uncertainty range of $Z_e$, given the large calibration offset of the MMCR of $-9.8\,dB$ found by Protat et al. (2011). For the retrieval without considering temperature m-ZW, agreement for $Z_e$ decreases to $-2.7\,dB$ and the offset for
\( \sigma \) increases significantly from 0.002 to 0.02. As a consequence, it is concluded that one \( m(D) \) relation is not sufficient for forward modeling the complete dataset and variability due to temperature has to be accounted for. The quantity \( Z_e \) of MMCR shows a bimodal distribution and the larger peak around 10 dB cannot be fully reproduced by F-ISDAC when using \( m-ZWT \). A reason for that could be the reduced accuracy of the \( T \)-matrix approach for larger particles leading to decreased backscattering. Some studies suggested also that the exponent of the \( m(D) \) relation is different for larger particles (e.g., Matrosov 2007; Mitchell 1996). This, however, cannot be investigated with the given dataset.

The \( Z_e \) is correlated to \( W \) and \( \sigma \) and it is investigated in Fig. 5 (columns 4–5) whether this functional relationship can be reproduced by F-ISDAC. For \( m-ZW \) and \( m-ZWT \) it can be seen that we can reproduce the \( Z_e - W \) relation, which is the basis for the retrieval used. In addition, the \( Z_e - \sigma \) relation—which follows a similar power law—is reproduced with less difference for \( Z-ZWT \) compared to \( Z-ZW \). Similar to the 1D histogram for \( \sigma \), this is an important closure indicating that the derived \( m(D) \) is consistent with the observations.

Comparison to the reference methods for \( m(D) \) reveals that the other methods cannot reproduce the MMCR observations as well as \( m-ZWT \). Even though the comparison of 1D histograms reveals only minor offsets for \( m-BF \) (Fig. 5c), comparison of the \( Z_e - \sigma \) relation shows that \( \sigma \) is negatively correlated to \( Z_e \) instead of positively (Fig. 5c, last column). The performance of \( m-CL \) is similar to \( m-ZWT \) with respect to the functional relations and to the offsets (Fig. 5d), but \( d_a \) is larger for all moments. Even though agreement of \( Z_e \) is almost perfect for \( m-SA \), the other moments are highly biased for \( m-SA \) (Fig. 5e). Similarly, \( m-FG \) and \( m-TD \) lead to relatively large offsets for \( W \) and \( \sigma \) (Figs. 5f,g). Note that the methods \( m-CL \) and \( m-FG \) depend on IWC, which had to be partly recovered from the \( m-SA \) method (Jackson et al. 2012) because of technical problems. Hence, the performance of \( m-CL \) and \( m-FG \) might be better for other campaigns with higher-quality IWC measurements.

**Fig. 5.** Normalized 1D (columns 1–3) and 2D (columns 4–5) histograms of radar moments: (columns 1, 4, 5) effective reflectivity factor \( Z_e \), (column 2, 4) Doppler velocity \( W \), and (column 3, 5) spectrum width \( \sigma \) of MMCR observations [gray lines (1D) and filled contour plot (2D), respectively] and forward modeled ISDAC observations [F-ISDAC, black lines (1D) and contour plot with lines (2D), respectively] for (top to bottom) miscellaneous mass parameterizations presented in section 3. The \( N(D) \) and \( A(D) \) of the particles are taken from measurements. For each dataset, the number of cases is given. For 1D plots, the vertical lines denote the median of the distributions, the difference between both medians is denoted in the upper-left corner, and the Kolmogorov–Smirnov statistic \( d_a \) is presented in the upper-right corner of each panel. For 2D plots, a least squares power fit is shown for MMCR (gray) and F-ISDAC (black).
The fact that all methods for \( m(D) \) are able to reproduce \( Z_e \) distributions very similar to the MMCR dataset, even though some show offsets for \( W \) and even stronger for \( \sigma \), underlines that investigation of \( Z_e \) alone is not sufficient to evaluate \( m(D) \) relations. The results for m-BF show that even separate investigation of \( W \) and \( \sigma \) is not sufficient and the functional relationships among \( Z_e \), \( W \), and \( \sigma \) have to be investigated as well.

4. Evaluating parameterizations for projected area and particle size distribution using higher moments

With the novel method for \( m(D) \), the ISDAC dataset contains all variables required by PAMTRA. In this section, the impact of replacing measurements of \( A(D) \) and \( N(D) \) by corresponding parameterizations is evaluated using lower and higher moments of the Doppler spectrum (see Fig. 3 for schematic overview). Finally, the impact of truncation effects is investigated.

a. Methods to parameterize projected area

Particle area is usually parameterized using an exponential law (e.g., Mitchell 1996) as shown in Eq. (15). The uncertainty of measured \( A(D) \) follows from the uncertainty of the total cross section area of all measured particles \( A_{\text{tot}}(D) = A(D)N_{\text{true}}(D) \), which is described with \( 1/\sqrt{N(D)_{\text{true}}} \) (Hallett 2003) where \( N_{\text{true}}(D) \) is the number of particles detected by the cloud probe not normalized by measurement volume and size bin width at a certain \( D \).

First, a least squares fit is applied to the complete dataset to estimate the coefficients of Eq. (15) (A-GLS). To avoid biases due to shattering effects on the probe tips, only particles larger than 0.5 mm are taken into account. The derived \( A(D) \) relation is (in SI units)

\[
A = 0.282D^{1.949}. \tag{18}
\]

Second, the least squares fit is applied to every profile of the dataset individually (A-LS), with the uncertainty of \( A(D) \) still considered. Particles smaller than 0.5 mm and size bins where \( N(D)\Delta D < 10 \text{ m}^{-3} \) are excluded from the fit. Profiles are completely discarded if \( R^2 \) of fit and observations is less than 0.7.

The fall velocity relation of Heymsfield and Westbrook (2010) used by PAMTRA depends on both \( m \) and \( A \). If both relations are expressed by power laws in terms of \( D \), the degrees of freedom of the fall velocity relation is reduced by one. To investigate whether this has any impact on the forward operator, random noise is added to the result of Eq. (15) as a third method (A-LSN):

\[
A = cD^d10^e(D), \tag{19}
\]

where \( r \) is a random number dependent on particle size. For every profile, different random numbers are applied. The \( r \) follows a Gaussian distribution with mean value of 0 and a spread derived from the standard deviation of \( \log_{10}[A(D)_{\text{fit}}/A(D)_{\text{meas}}] \). As already discussed, shattering effects can occur for particles smaller than 0.5 mm. Consequently, for this part, the mean of the standard deviation for \( D > 0.5 \text{ mm} \) is used. Values of the standard deviation vary between 0.08 and 0.51 with a mean value of 0.24.

b. Methods to parameterize particle size distribution

Similar to the area, the particle size distribution \( N(D) \) is a direct measurement of the in situ probes. Hence, a reference is available to investigate the performance of the various methods. The uncertainty of the measurement of \( N(D) \) is estimated with \( 1/\sqrt{N_{\text{true}}} \) (Hallett 2003).

Most of the formulas used to describe \( N(D) \) can be related to the modified gamma distribution, which is usually expressed with

\[
N(D) = N_0D^\mu\exp(-\Lambda D^\gamma), \tag{20}
\]

where \( N_0 \) describes the overall scaling and \( \mu, \Lambda, \gamma \) control the shape. The gamma distribution, which has only three parameters, can be obtained by setting \( \gamma = 1 \). If, additionally, \( \mu = 0 \) is applied, the result is the two-parameter exponential distribution. An extensive review of the modified gamma distribution and its derivatives is given by Petty and Huang (2011).

Alternatively, a lognormal distribution can be used to describe \( N(D) \), which is defined as

\[
N(D) = \frac{N_T}{\sqrt{2\pi sD}}\exp\left[-\frac{\ln^2(D/D_T)}{2s^2}\right], \tag{21}
\]

where \( N_T \) describes the overall scaling and \( s \) and \( D_T \) are the shape parameters.

The parameters of Eqs. (20) and (21) can be obtained using miscellaneous methods, either by fits or by moment matching. In this section, various approaches are presented. For all presented methods, only profiles where the \( R^2 \) of fit and measurement is greater than 0.7 are investigated.

As an independent reference, which does not depend on the hydrometer measurements of ISDAC, the method of Field et al. (2005) is presented (herein N-FI), which was obtained from measurements of stratiform ice cloud around the British Isles and is used widely in
models. They showed that \( N_0 \) of the exponential distribution can be given by

\[
N_0 = 5.65 \times 10^5 \exp(-0.107T),
\]

(22)

where \( T \) is the ambient air temperature \( T \) in °C. Thereafter, \( \Lambda \) of the exponential distribution is derived from the total number of particles \( N_{\text{tot}} \) with \( \Lambda = N_0/N_{\text{tot}} \).

The easiest way to find the coefficients of a modified gamma distribution is to apply a least squares fit (N-LSMG). Here, it is only applied to the modified gamma distribution, since gamma and exponential distribution are special cases.

The concept of normalizing \( N(D) \) was introduced by Testud et al. (2001) for liquid clouds and by Delanoë et al. (2005, 2014) for ice clouds (N-NG). The normalized \( N(D) \) is defined as

\[
N(D) = N_0^g F(D/D_m),
\]

(23)

where \( D_m \) is the “mass–weighted” scaling parameter for the particle size and \( F(D/D_m) \) is the normalized function. This formulation has the advantage that scaling of the distribution in direction of \( N \) is clearly separated from scaling in the direction of \( D \) and a change of \( N_0^g \) has only an effect on the total number of particles, whereas a modified \( D_m \) changes only the size of the particles. Delanoë et al. (2014) found that normalization works best on the basis of modified and (nonmodified) gamma distributions with only little difference between both variants. Thus, we use the gamma distribution for simplicity and to keep the number of variables at a minimum. In contrast to Delanoë et al. (2005, 2014), we chose the particle’s maximum dimension \( D \) instead of the equivalent melted diameter \( D_{\text{melt}} \) as the size descriptor. Therefore, we have to rephrase most the formulas used for the normalization. In appendix A, it is described in detail how defining \( D_m \) as “mass–weighted” mean diameter and \( N_0^g \) with

\[
D_m = \frac{M_{b+1}}{M_b}
\]

and

\[
N_0^g = \frac{M_b^{b+2}}{M_b^{b+1} C}
\]

leads to a conservation of IWC and to the definition of the normalization function

\[
F(D/D_m) = \frac{(b + \mu + 1)^{b+\mu+1} \Gamma(b+1)}{\Gamma(b+\mu+1)(b+1)^{b+1}} \left( \frac{D}{D_m} \right)^{\mu} e^{-(b+\mu+1)D/D_m},
\]

(26)

where \( b \) is the exponent of the \( m(D) \) relation, \( M_j \) is the \( j \)th moment of \( N(D) \), \( \mu \) describes the shape of the distribution, and \( C \) is an arbitrary constant. With \( b = 3 \) like for a water drop, these equations collapse into the corresponding ones of Testud et al. (2001) and Delanoë et al. (2005). Using Eqs. (24) and (25), \( D_m \) and \( N_0^g \) can be directly calculated from the measured, truncated moment \( M_j \) as a replacement for the theoretical \( M_j \) defined from 0 to \( \infty \). For \( \mu \), Delanoë et al. (2005) suggested a value of 3, but they used the equivalent melted diameter as the size descriptor and thus we cannot transfer that value to our study. Instead, we use a least squares fit to find a best estimate of \( \mu \) for every measured \( N(D) \). Defining \( D_m \) and \( N_0^g \) such that they depend on the exponent \( b \) of the \( m(D) \) relation means that their definition might change with changing \( b \). At first sight, this might appear counterintuitive, but it is important to note that the original studies by Delanoë et al. (2005, 2014) had a similar drawback by using the melted equivalent diameter as a size descriptor; that is, the \( m(D) \) relation was implicitly included in the size parameter.

Tian et al. (2010) presented another approach for normalizing \( N(D) \) for exponential, gamma, and log-normal distributions. Here, only the latter (N-LD) is investigated, because it performed best for the dataset used by Tian et al. (2010) containing convective cirrus clouds. For N-LD, they used the first, second, and fourth moment to define the parameters of

\[
D_T = \frac{(M_2/M_1)^2}{\sqrt{M_4/M_2}},
\]

(27)

\[
s = \sqrt{\ln \left( \frac{(M_4/M_1)^{1/3}}{(M_2/M_1)^{1/3}} \right)}, \quad \text{and}
\]

(28)

\[
N_T = M_1 \left( \frac{M_4/M_1}{(M_2/M_1)^{1/3}} \right)^{1/3}.
\]

(29)

The characteristic feature of the parameterization by Tian et al. (2010) is that they force all distributions to collapse onto the same function. The presented equations are based on the assumption that the corresponding particle size functions describe all particles from zero size to infinity. In reality, however, particle sizes are limited from a certain minimum to maximum diameter (\( D_{\text{min}} \) and \( D_{\text{max}} \), so that the real measured moment \( M_j \) is different from the moment \( M_j \) defined from zero to infinity. The proposed correction by Tian et al. (2010) is used here.

We also use the discrete incomplete gamma fitting (N-DIGF) introduced by Freer and McFarquhar (2008) and extended by McFarquhar et al. (2015). They
normalize the gamma distribution partly by introducing \( D_\ast \), which ensures that \( N_0 \) has the same unit as \( N(D) \):

\[
N(D) = N_0 \left( \frac{D}{D_\ast} \right)^\mu \exp(-\Lambda D). \tag{30}
\]

Because SI units are chosen in this study, we chose \( D_\ast = 1 \) m. The coefficients \( N_0, \mu, \) and \( \Lambda \) are found by minimizing

\[
\chi^2 = \sum_{j=1,2,6} \left( \frac{M_j - \bar{M}}{\sqrt{M_j M_j}} \right), \tag{31}
\]

using the method of Byrd et al. (1995), where \( j = 1, 2, 6 \) indicates the three moments used in the normalization. The advantage of the DIGF, the provision of an uncertainty range for the found parameters, is not exploited in this study.

c. Intercomparison of parameterizations

Here, the impact of the different parameterizations of \( A(D) \) and \( N(D) \) on the derived radar moments is examined. Only in situ data obtained within a 10-km radius around Barrow are analyzed (see section 2d for details). To investigate the numerical stability of the derived parameterizations, they are not only applied to the size bins where the parameterizations are derived [i.e., measurements are available and \( N(D) > 0 \)], but also to the full range of \( D \) from 0.05 to 12.8 mm as determined by the in situ particle probes.

Frequency distributions of \( A \) versus \( D \) are presented in Fig. 6. For A-MEAS, the step in spread of the distribution due to the different probes can be clearly seen around 0.3 mm. For particles larger than 0.3 mm, a power law with two coefficients seem sufficient to describe \( A(D) \) because the spread (a factor of \( \sim 2.5 \)) is evenly distributed around the A-GLS line. The scatter is slightly reduced if A-LS instead of A-MEAS is used. Application of A-LSN, instead, leads to scatter of the same order as for A-MEAS for medium size particles but leads to overestimation of scatter for larger particles. Additional scatter on the order of up to one magnitude occurs for larger particles if \( A(D) \) is extrapolated until the maximum diameter of 12.8 mm. Note that application of the least squares fit reduces the number of profiles from 2122 to 1437. The reduction mainly occurs because many profiles with too few measurements with \( N(D) \Delta D > 10 \) m\(^{-3} \) are available for the least squares fit; that is, mostly profiles with very low number concentrations are affected.

For \( N(D) \), frequency distributions of the various methods are presented in Fig. 7. In contrast to \( A(D) \), N-MEAS does not follow a straight line and has a much wider spread of two to three orders of magnitude indicating high variability of the observed \( N(D) \). Similar to A-MEAS, the transition between the different in situ probes is clearly visible around 0.3 and 0.8 mm. Application of N-FI and N-LSMG leads to strong differences to N-MEAS in the shape of the 2D distribution. In particular, particles larger than 1 mm and extrapolation to even higher diameters lead to artificially high particle size distributions that might cause biases in \( Z_t \) owing to the strong scattering of large particles. As an advantage, the total number of observations is constant for N-FI and is reduced only to 1698 and 1762 cases for N-LSMG and N-NG. Application of N-DIGF and N-LN, instead, leads to a stronger reduction of the number of observations to 1260 and 830 cases, respectively, due to application of the fits and the \( R^2 \) test. Consequently, the
latter, N-LN, is excluded from further analysis. Regarding the frequency distribution, N-NG and N-DIGF show highest similarities to N-MEAS with the latter having more outliers. Both methods lead to small \( N(D) \) if extrapolated to \( D \) up to 12.8 mm.

d. Comparison to MMCR

In this section, F-ISDAC and MMCR data are compared using the various methods to describe \( A(D) \) and \( N(D) \). The evaluation is carried out in two steps: the method, which performs best for \( A(D) \), is also used during the evaluations of the methods describing \( N(D) \). Similar to section 3d the Kolmogorov–Smirnov statistic \( d_a \) (Massey 1951) and the median are used as a measure of similarity. For the analysis, we consider not only the standard, lower moments but also the higher moments of the Doppler spectrum.

For \( A(D) \), Fig. 8 shows comparisons between MMCR and F-ISDAC for A-MEAS and the corresponding parameterizations. m-ZWT is used for \( m(D) \) and hence the data shown in Fig. 8a is identical to Fig. 5b. The lower moments \( Z_r, W, \) and \( \sigma \) show only little variation between A-MEAS and the other parameterizations. Hence, higher moments of the Doppler spectrum are also used for evaluation. Note that functional relations between higher moments and \( Z_r \) are not as pronounced as for lower moments and hence not analyzed here. Figure 8a shows that for A-MEAS the agreement of higher moments is also high when using m-ZWT for \( m(D) \). A major offset is only visible for \( S_r \), which is shifted toward steeper slopes and has a wider spread. This is most likely related to sampling problems of fast-falling, large, and rare particles that are insufficiently sampled by the in situ sensors (Korolev and Isaac 2005). In addition, the accuracy of the \( T \)-matrix approximation of the scattering properties is decreasing with increasing particle size as a result of resonance effects reducing the backscattered power (Kneifel et al. 2011). Consequently, the right, fast tail of the Doppler peak is cut and the slope is too steep. Because the slopes are derived from the Doppler spectrum in log scale, the impact of small perturbations at the borders of the peak on the slopes is rather high even though these perturbations hardly influence the moments that are derived using linear units. Only \( S_k \) is shifted toward negative values in comparison to MMCR by 0.17 owing to the truncation of the right, fast tail of the Doppler spectrum.

A-GLS and A-LS lead to very similar results, indicating the low variability of the \( A(D) \) relation during ISDAC; \( d_a \) values are even slightly better for A-GLS.
than for A-LS for all moments but $\sigma$ and $S_k$. For all least squares fits, particles with $D < 0.5\, \text{mm}$ are ignored to remove shattering artifacts (section 2a), but from the fact that agreement of $S_l$ (which is partly determined by small particles) is not decreasing from A-MEAS to A-GLS and A-LS, it can be assumed that representation of $A$ for particles $<0.5\, \text{mm}$ is good even though they are excluded from the least squares fit or that they have only minor impact on the radar moments.

Strikingly, the offsets of $S_k$ and $K_u$ are increased for A-GLS and A-LS in comparison to A-MEAS and MMCR. This is most likely—again—related to problems at the right, fast side of the Doppler spectrum because the offset of $S_l$ increases from $-23\, \text{dB s m}^{-1}$ to $-40$ to $-38\, \text{dB s m}^{-1}$ as well. Since at the same time, $W$ and $\sigma$ change only little, the peak becomes more asymmetric and $S_l$ is biased. Methods to express $A(D)$ by a gamma relation [similar to Eq. (20), but with $\gamma = 1$] or by two different sets of power law coefficients depending on $D$ did not lead to a significant improvement (not shown).

To investigate whether the offset of $S_k$ is caused by a certain part of the size spectrum, A-LS is applied only to parts of the spectrum and measurements are used otherwise. As a result, the offset of $S_k$ is reduced to $-0.12$ for $D < 0.3\, \text{mm}$ and to $-0.14$ for $0.3 < D < 1\, \text{mm}$ but is increased to $-0.22$ for $D > 1\, \text{mm}$ (not shown). Hence, the effect is caused by particles larger than $1\, \text{mm}$.

However, if noise (A-LSN) is added to $A(D)$ estimated with A-LS so that the scatter of $A(D)$ around the A-GLS line is increased (see Fig. 6c), the offsets are reduced and $d_a$ values for $S_k$ and $K_u$ are in the same order as for A-MEAS (Fig. 8d). Only $\sigma$ is slightly increased—especially for larger $Z_r$ values (not shown). We conclude that small, random deviations from the power law are mandatory to reproduce realistic distributions of $S_k$. A variation of the coefficients $c$ or $d$, instead, is not sufficient because, otherwise, the difference between A-GLS and A-LS would be larger. A possible reason could be that if both $A$ and $m$ are expressed by power laws, the degrees of freedom of the fall velocity relation is reduced because both $A$ and $m$ can be expressed in terms of $D$. This could lead to a reduced variability of $v(D)$, which apparently becomes most evident for larger particles. Consequently, we use A-LSN as the preferred method for $A(D)$ in the following.

Figure 9 shows the comparison of F-ISDAC and MMCR using lower and higher moments for the various methods to describe $N(D)$. For all methods, m-ZWT and A-LSN are used to estimate $m(D)$ and $A(D)$, respectively.

For N-FI (Fig. 9b), agreement of the lower moments is similar to N-NG, but the offsets for $S_k$, $K_u$, and $S_l$ are increased. This highlights the need for also investigating higher moments when comparing $N(D)$ parameterizations. N-LSMG leads to increased offsets for all moments (Fig. 9c). N-NG shows (Fig. 9d) that this is most likely related to the fit type instead of being related to the use of a gamma distribution to describe $N(D)$. For all moments, offsets and $d_a$ values are almost identical for N-MEAS and N-NG. We attribute this to the combination of the least squares fit and the conservation of PSD moments of the N-NG method. N-DIGF, which is also designed to conserve moments, leads to increased offsets especially for the lower moments (Fig. 9e).
addition, the number of profiles are reduced by 1/3 for this method, mostly because of the $R^2 > 0.7$ condition.

e. Truncation effects

Finally, the best methods for $A(D)$ and $N(D)$ are used to investigate whether the minimum and/or maximum detected particle diameter $D_{\text{min}}$ and $D_{\text{max}}$ have to be treated as additional parameters or whether a fixed value for $D_{\text{min}}$ and $D_{\text{max}}$ is sufficient (Fig. 10).

To investigate how truncation effects have to be handled, the minimum and maximum observed diameters of the distribution $D_{\text{min}}$ and $D_{\text{max}}$ are estimated. Then, (i) the method applied so far [i.e., application of the parameterizations for $m(D)$, $A(D)$, and $N(D)$ only if measured $N(D) > 0$ to ensure equal treatment of measurements and parameterizations] is compared with (ii) applying $D_{\text{min}}$ and $D_{\text{max}}$ instead (i.e., interpolating gaps in between), (iii) applying only $D_{\text{max}}$ and using a fixed value of 0.05 mm for $D_{\text{min}}$, and (iv) using also a fixed value of 12.8 mm for $D_{\text{max}}$ (i.e., extrapolation), which is the maximum of the measurement range of the in situ probe. The difference of F-ISDAC of (ii) and (iii) to (i) are negligible. Omission of $D_{\text{max}}$ leads to only slightly increased (i.e., steeper) values for $S_t$. Hence, we conclude that smaller particles and gaps of the particle spectrum have little impact on the radar moments. For (iv), agreement of F-ISDAC to MMCR is actually better than for (i): the offset of $S_k$ is decreased from −0.2 to −0.08 and—even more strongly pronounced—the offset of $S_r$ is reduced from −31 to −6 (even though the shape of distribution still cannot be captured correctly and $d_a$ is only 0.21). This indicates that the problems with the right, fast side of the spectrum were partly caused by problems with measuring rare, fast particles with in situ probes. As a disadvantage, agreement of $\sigma$ and $S_r$ decrease slightly.

The fact that omission of $D_{\text{max}}$ does not lead to decreased results can be explained by the shape of $N(D)$: for N-NG, $N(D)$ is strongly decreasing for larger particles and consequently $N(D)$ is small, even for a fixed $D_{\text{max}} = 12.8$ mm. If a method without this property would be used for $N(D)$; for example, N-FI, $Ze$ would increase by around 10 dB for a fixed $D_{\text{max}}$ of 12.8 mm (not shown). In conclusion, for the presented dataset with the chosen parameterizations, it is not necessary to keep $D_{\text{min}}$ and/or $D_{\text{max}}$ as additional parameters in the forward model. However, an investigation is recommended for every dataset.

5. Conclusions

In situ aircraft and MMCR radar observations obtained during the ISDAC campaign around Barrow, Alaska, were used to develop a consistent dataset of stratocumulus ice cloud properties such as mass $m(D)$, area $A(D)$, and particle size distribution $N(D)$. For $m(D)$, a novel method to derive the power law coefficients based on the functional relation between effective reflectivity factor $Ze$ and Doppler velocity $W$ was developed and compared to other methods. Then, the effect of applying various parameterizations to $A(D)$ and $N(D)$ on radar simulated observations was investigated. For this, higher moments such as skewness $S_k$, kurtosis $Ku$ and the slopes ($S_t$, $S_r$) were also used as a proxy for the radar Doppler spectrum. The principal conclusions of this study are as follows:

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**Fig. 9.** As in Fig. 8, comparison of MMCR observations and various methods to represent $N(D)$ for ISDAC data. For $A(D)$, the least squares with noise (A-LSN) method is used.
1) $m(D)$ can be successfully obtained from a combination of in situ and radar observations by choosing $m(D)$ with optimal estimation such that the $Ze$-$W$ relations of radar and forward modeled in situ data match (Fig. 2). The found, temperature dependent, $m(D)$ relations are in high agreement with relations reported in literature (Fig. 4) and also lead to a high agreement of $Ze$ to Doppler spectrum width ($s$) relations between observation and model (Fig. 5). From this, we conclude that the potentially large errors in $m(D)$ due to forward model uncertainties can be neglected.

2) Even though other methods to describe $m(D)$ lead to similar $Ze$ as observed by the MMCR, they cannot reproduce the functional relations $Ze - W$ and—even more pronounced—$Ze - \sigma$. From this, it is concluded that analyzing $Ze$, $W$, and $\sigma$ independently is not sufficient when analyzing $m(D)$ relations (Fig. 5).

3) The presented forward model PAMTRA is able to simulate lower and higher moments consistently on the basis of aircraft in situ observations and the novel method to derive $m(D)$ (Fig. 8a). We attribute a small offset for $S_k$ (−0.17) and larger offsets for $S_s$ (−22 dB m$^{-1}$) to in situ sampling issues of rare, large, and fast-falling particles as well as to the decreasing accuracy of $T$-matrix scattering calculations for larger particles.

4) Even though power law fits are well suited to describe $A(D)$ (Fig. 6) and lead to a high agreement of lower moments, consistent results for higher moments can be only obtained if an additional, size dependent noise factor is applied to the result of the power law (Fig. 8). We conclude that small deviations from the power law for single parts of $A(D)$ are essential to obtain realistic Doppler spectra.

5) For the description of $N(D)$, most consistent results are found for a gamma distribution if the coefficients are estimated using a moment preserving approach (Fig. 9). Of these, the normalized gamma distribution approach by Testud et al. (2001), which was modified to work with maximum dimension as size descriptor, is identified to work best for the ISDAC dataset. The use of higher moments was also found to be a valuable addition.

6) For the investigated dataset, the minimum and maximum measured particle size do not have to be passed to the radar simulator as additional parameters (Fig. 10) and fixed boundaries can be assumed instead. Extrapolating the particle size distribution to the maximum diameter of 12.8 mm removes the offset for $S_k$ found before and improves the agreement of $S_s$.

These results are robust with respect to uncertainties in the aspect ratio as shown in appendix B, section a, but the large calibration offset of the MMCR of −9.8 dB prevents more detailed analysis of this. While the estimation of scattering properties with $T$ matrix is sufficient for this study, more sophisticated methods need be used for observations of larger snowflakes and/or with radars using smaller wavelengths. The dataset investigated in this study is rather small and consists of only 1690 profiles. In this study, the dataset cannot be extended.
because no other datasets with observations of polar ice clouds are available to the authors knowledge in the vicinity of a cloud radar and where modified probe tips were used that reduce the impact of shattering. Thus, more aircraft campaigns in the vicinity of super sites instrumented with cloud radars are desirable to extend the dataset, to confirm the results, and to investigate how the found results depend on temperature or geographical location. In addition, the maximum distance between aircraft and radar for the comparison could be decreased to enhance agreement of cloud characteristics. A larger dataset would also potentially increase the spread of observed $A(D)$, which does show only small variability in this study and would allow a more thorough investigation of the temperature dependence of the mass–size relation. Better measurements of ice water content IWC would be also an additional, independent test of the found $m(D)$ relation. Analysis of $S_r$ showed that better measurements of rare, large particles by aircraft in situ probes are also desirable for the future.

The presence of more modern cloud radars such as the $K_a$-band ARM zenith radar (KAZR; Lamer et al. 2014) or the MIRA-35 cloud radar (Melchionna et al. 2008), which are available at numerous ground-based sites today, can also lead to better possibilities for the exploration of higher moments and the Doppler spectrum because of their better sensitivity and larger spectral resolution. The question of why deviations from the $A(D)$ power law for particles greater 1 mm are essential to obtain consistent results for the higher moments has to be investigated in greater detail in the future. The impact of supercooled liquid water on the Doppler spectrum, for example, via riming of ice particles, also has to be included in future studies.

This study shows that higher radar moments are a valuable addition to the set of radar observables and should be exploited more often in the future. The advantage of using higher moments is—besides increasing the number of observables—that they are neither affected by calibration issues (as $Z_e$) nor biased by vertical air motions (as $W$). Furthermore, higher moments can be also used as an independent closure test for evaluating numerical weather models or ice cloud retrievals by forward modeling their output and comparing modeled and observed higher moments. Turbulence was very low during ISDAC and had hence only little impact on the Doppler spectrum via kinematic broadening. Because turbulence tends to make radar Doppler spectra more Gaussian, sensitivity studies are needed to investigate whether higher moments can be still exploited in more turbulent conditions.

This study developed a consistent set of parameterizations to describe $m(D)$, $A(D)$, and $N(D)$ that can be used in combination with the presented PAMTRA model (Mech et al., 2015) to investigate the potential for a retrieval of microphysical ice cloud properties based on lower and higher moments. Assuming that the ability to simulate higher moments indicates also the ability to simulate the full Doppler spectrum consistently (except for the fastest particles), this also allows for investigating the potential of using the full Doppler spectrum for retrievals. Then, the information content, which higher moments or the full radar Doppler spectrum can provide to ice cloud retrievals, can be estimated using inverse retrieval methods, such as optimal estimation (Rodgers 2000). In addition, the impact of using an optimized radar configuration or the addition of additional observation frequencies can be investigated.

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APPENDIX A

Normalization of the Gamma Distribution with Maximum Dimension $D$ as Size Descriptor

According to Testud et al. (2001) and Delanoë et al. (2005), the normalized $N(D)$ is defined as

$$N(D) = N_0^* F(D/D_m) \quad (A1)$$

or with $X = D/D_m$:

$$N(X) = N_0^* F(X), \quad (A2)$$

APPENDIX A
where $N_0^b$ is the scaling parameter in the direction of the concentration axis and $D_m$ is the scaling parameter for the particle size.

In contrast to Delanoë et al. (2005, 2014), we chose the ice particle’s maximum dimension instead of the equivalent melted diameter as the size descriptor. Therefore, we have to rephrase most the formulas used for the normalization.

We begin with the constraint that the normalized $N(D)$ is supposed to be mass consistent:

$$IWC = \int m(D)N(D)\,dD.$$  \hspace{1cm} \text{(A3)}

Assuming a $m(D)$ relation of $m = a D^b$ and using Eq. (A2) we get

$$IWC = \int a D^b N(D)\,dD = a D_m^{b+1} N_0^b \int X^b F(X)\,dX.$$ \hspace{1cm} \text{(A4)}

The term $\int X^b F(X)\,dX$ is supposed to be constant to put all the variability of IWC into $D_m$ and $N_0^b$. Hence, we define

$$D_m = \frac{M_{b+1}^{1/m}}{M_b^{1/m}},$$ \hspace{1cm} \text{(A5)}

where $M_j$ is the $j$th moment of the distribution defined as $M_j = \int N(D)D^j\,dD$. This yields to

$$D_m = \frac{\int N(D)D^{b+1}\,dD}{\int N(D)D^b\,dD} = \frac{M_{b+1}^m}{M_b^m} \int F(X)X^{b+1}\,dX.$$ \hspace{1cm} \text{(A6)}

using Eq. (A2). Since this equation has to be valid for all $X$ and $F(X)$, we can follow

$$\int F(X)X^{b+1}\,dX = \int F(X)X^b\,dX = C,$$ \hspace{1cm} \text{(A7)}

where $C$ is an arbitrary constant. Using $C$, Eq. (A4) becomes

$$IWC = a D_m^{b+1} N_0^b C,$$ \hspace{1cm} \text{(A8)}

$$\Rightarrow N_0^b = \frac{IWC}{a D_m^{b+1} C}.$$ \hspace{1cm} \text{(A9)}

Using Eq. (A5) and knowing that $IWC = a M_b$, $N_0^b$ becomes

$$N_0^b = \frac{M_{b+2}^b}{M_{b+1}^b C}.$$ \hspace{1cm} \text{(A10)}

In principle, $C$ can be chosen arbitrarily and so far the derived equations are valid for all kinds of distributions. We follow Delanoë et al. (2005), which chose $C$ in such a way that $N_0^b$ is equal to $N_0$ of the exponential distribution. The $i$th moment of an exponential distribution is defined as

$$M_i = N_0 \frac{\Gamma(j+1)}{\Lambda^{j+1}}.$$ \hspace{1cm} \text{(A11)}

Inserting this in (A5) yields to

$$D_m = \frac{\Gamma(b+1)}{\Lambda}. $$ \hspace{1cm} \text{(A12)}

This is used to rephrase Eq. (A10), which results in

$$N_0^b = N_0 \frac{\Gamma(b+1)}{(b+1)^{b+1} C}.$$ \hspace{1cm} \text{(A13)}

Requiring $N_0^b = N_0$, we get

$$C = \frac{\Gamma(b+1)}{(b+1)^{b+1}}.$$ \hspace{1cm} \text{(A14)}

However, Delanoë et al. (2005) showed that an exponential distribution is not sufficient to give a realistic approximation of $N(D)$ for ice; instead, they suggested a modified or nonmodified gamma distribution. Since they found only little difference between both variants, we decided to use a gamma distribution for simplicity and to keep the number of variables smaller. The definition of $C$ is simply overtaken from the exponential distribution.

For the gamma distribution, the $i$th moment is defined as

$$M_i = N_0 \frac{\Gamma(j+\mu+1)}{\Lambda^{j+\mu+1}}.$$ \hspace{1cm} \text{(A15)}

Inserting this in (A5) yields to

$$D_m = \frac{b+\mu+1}{\Lambda}.$$ \hspace{1cm} \text{(A16)}

These two equations are used to rephrase Eq. (A10):

$$N_0^b = \frac{M_{b+2}^b (b+1)^{b+1}}{M_{b+1}^b \Gamma(b+1)} = \frac{N_0 D_m^b (b+\mu+1)^{b+\mu+1} \Gamma(b+1)}{(b+1)^{b+1}}.$$ \hspace{1cm} \text{(A17)}

Therefore, the gamma distribution

$$N(D) = N_0 D^\mu \exp(-\Lambda D)$$ \hspace{1cm} \text{(A18)}

can be normalized by inserting Eq. (A17) for $N_0$ and Eq. (A16) for $\Lambda$:
If the $m(D)$ relation of water spheres is used ($\alpha = \pi \rho/6$ and $b = 3$), these equations collapse into the corresponding ones of Testud et al. (2001) and Delanoë et al. (2005).

APPENDIX B

Sensitivity to Model and Data Processing Assumptions

Here, we motivate why we 1) use an aspect ratio (AR) of 0.6 for estimating the scattering properties, 2) limit the aircraft to a radius of 10 km around the ARM NSA site in Barrow, and 3) why we average aircraft data to 10 s. For this, F-ISDAC data are statistically compared with MMCR using different AR, maximum radii, and averaging times (Fig. B1). The quantity $m(D)$ is estimated using $Z_e - W$ optimal estimation with temperature (m-ZWT), and measurements are used for $A(D)$ and $N(D)$.

$$N(D) = N_0^b (b + \mu + 1)^{b+\mu+1} \Gamma(b + 1) \Gamma(b + \mu + 1)(b + 1)^{b+1} \left( \frac{D}{D_m} \right)^\mu \exp[-(b+\mu+1)D/D_m].$$  \hspace{1cm} (A19)

a. Sensitivity to aspect ratio

Unfortunately, we cannot estimate AR from ISDAC data directly, because $D_{\text{short}}$, defined as the particle extent perpendicular to $D$ (see Hogan et al., 2012), is not available in the dataset. Other studies found typical mean AR in ice clouds of 0.6 (Korolev and Isaac 2003; Hogan et al. 2012) or 0.65–0.7 (Tyynelä et al. 2011). If we approximate AR from $A(D)$ and $D$ of the ISDAC dataset, we find $AR = 0.5$ on average. With this method, however, we underestimate AR because $A(D)$ is usually smaller than the corresponding cross section of an ellipsoid (Hogan et al. 2012) that would be required to estimate the correct AR. Consequently, the measurements indicate that AR is $>0.5$ also for the ISDAC dataset.

To investigate the sensitivity of the higher moments to AR, values between 1.0 and 0.4 are applied to the ISDAC dataset in steps of 0.2 and compared to MMCR measurements (Fig. B1). For 1.0, scattering is estimated with...
Mie instead with $T$ matrix. With decreasing AR, the offset of $Z_e$ is also decreasing because smaller AR lead to larger particle density because of the smaller particle volume. The secondary peak of $Z_e$ can only be reproduced by $\text{AR} \leq 0.6$. For all other moments, the impact of AR is surprisingly low. Apparently, AR has only minor influence on all radar moments except $Z_e$. Results for AR of 0.4 and 0.6 are most consistent with MMCR observations, but the value of 0.6 is chosen for the rest of the study to be in accordance with literature and observations.

b. Sensitivity to maximum distance to barrow

If the maximum distance between the aircraft and the radar observations is chosen too large, the datasets are not comparable owing to the spatial variability. X-band radar observations of the stratocumulus sampled during ISDAC showed small-scale structure and inhomogeneities in the microphysical structure on scales on the order of 2 to 3 km (McFarquhar et al. 2011). Choosing a too-small distance, instead, leads to a too-little number of observations prohibiting statistical comparisons. Fig. B1 presents histograms of the radar moments of F-ISDAC and MMCR observations for a maximum radius of 10, 5, and 1 km. A reduction of the radius from 10 to 5 km leads to a decrease of the number of observations by three quarters. However, a significant change in the distribution of the moments cannot be seen. A further reduction to 1-km radius reduces the number of observations to only 29. Slightly increases offsets for $s$ and $K_u$ are most likely related to the sample size.

c. Sensitivity to in situ data averaging time

Averaging of the 1-s ISDAC data is required to improve the representation of rare, large particles. A too-long averaging time, however, might smooth out small-scale ice cloud variability. To investigate which averaging time is appropriate, averaging times of 5, 10, and 30 s are compared (Fig. B1). The number of observations does, as expected, scale with the inverse of the averaging time. The better representation of larger particles can be seen from the better agreement of $\sigma$ and $K_u$ with the sample size.

APPENDIX C

List of Symbols

- $A$: (Projected) particle area
- $a$: Prefactor of mass–size relation
- A-GLS: General least squares fit applied to the complete dataset
- $\alpha$: Mass–size reference factor
- $A$-LS: Least squares fit applied to every profile individually
- $\beta$: Exponent of mass–size relation
- $A$-LSN: Least squares fit applied to every profile individually with additional noise factor
- $c$: Prefactor of area–size relation
- $A$-MEAS: Area from in situ measurements
- $D$: Maximum particle dimension
- AR: Aspect ratio
- $d$: Exponent of area–size relation
- $\Delta D$: Particle size bin width
- $e$: Prefactor of reflectivity–Doppler velocity relation
- $f$: Exponent of reflectivity–Doppler velocity relation
- $\delta$: Kolmogorov–Smirnov statistic
- $IWC$: Ice water content
- $K_u$: Kurtosis
- $\Lambda$: Shape coefficient of exponential and (modified) gamma distribution
- $m$: Particle mass
- $m$-CL: Constant $\Lambda$ method by Heymsfield et al. (2004)
- $m-FG$: Fractal geometry method by Schmitt and Heymsfield (2010)
- $m-SA$: Shape analysis data of Jackson et al. (2012)
- $m-ZW$: Optimal estimation method using $Z_e - W$ relation
- $m-ZWT$: Optimal estimation method using $Z_e - W$ relation and temperature
- $M_j$: $j$th moment of a distribution
- $M_j^\prime$: $j$th measured moment of a distribution
- $N$: Particle size distribution
- $M$: Mass–size relation
- N-DIGF: Discrete incomplete gamma fitting technique by Freer and McFarquhar (2008)
- N-FI: Method by Field et al. (2005) to estimate $N(D)$ from temperature
- N-LD: Normalized lognormal distribution approach by Tian et al. (2010)
- $N$-LSMG: Least squares method to estimate coefficients of the modified gamma distribution
- $N$-F: Field et al. (2005) to estimate $N(D)$ from temperature
- $N$-LD: Normalized lognormal distribution approach by Tian et al. (2010)
- $N$-LSMG: Least squares method to estimate coefficients of the modified gamma distribution
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N-MEAS Particle size distribution from in situ measurements

N-NG Normalized gamma distribution approach by Testud et al. (2001)

\( N_{true} \) Sample size detected by cloud probe

\( R^2 \) Correlation coefficient

\( \delta_k \) Skewness

\( \Delta_l \) Left slope

\( \delta_r \) Right slope

\( \sigma \) Doppler spectrum width

\( T \) Temperature

\( W \) Mean Doppler velocity (positive means toward the ground)

\( Z_e \) Effective radar reflectivity factor


