Error Structure of Multiparameter Radar and Surface Measurements of Rainfall
Part I: Differential Reflectivity

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ABSTRACT

Fluctuations in the radar measurements of $Z_{DR}$ are due to both signal power fluctuations and the cross-correlation between the horizontal and vertical polarized signals. In Part I of this study, these signals are simulated for an S-band radar for backscatter from rain media, which is characterized by a gamma model of the raindrop size distribution (RSD). The parameters $N_0$, $D_0$, $m$ of the gamma RSD are then varied over the entire range found in natural rainfall. Thus, the radar simulations contain the effects of both statistical fluctuations and physical variations. We also simulate sampling of raindrops by disdrometer. The sampling errors are related to the Poisson statistics of the total number of drops in the fixed sample volume and to the statistics that govern the gamma distribution of drops as a function of size. We simulate disdrometer RSD samples over the entire range of $N_0$, $D_0$, $m$ values found in rainfall, so that the effects of statistical fluctuations and physical variations are introduced.

It is shown that $Z_{DR}$, computed from disdrometer RSD samples, is correlated with $Z$ and with other moments of the RSD when the same disdrometer data is used. This correlation is purely statistical and is independent of the physical correlation. We use the radar and disdrometer simulations to intercompare the rain rate as derived by the radar $Z_{DR}$-method with the rain rate estimated by the disdrometer. Our simulation results are used to explain the correlation and error structure of radar/disdrometer-derived rain rate intercomparison data reported in the literature.

1. Introduction

The differential reflectivity ($Z_{DR}$) technique was first introduced by Seliga and Bringi (1976) to improve the accuracy of radar estimates of rainfall rate over conventional $Z-R$ methods. This improvement is achieved by making two nearly simultaneous measurements of horizontally and vertically polarized reflectivities ($Z_H$, $Z_V$) of the rain medium from which differential reflectivity (in decibels) is derived as $Z_{DR} = 10 \log(Z_H/Z_V)$. From $Z_H$ and $Z_{DR}$ two parameters of the raindrop size distribution (RSD) are estimated (Ulbrich and Atlas 1984). Considerable progress has been made by researchers in this respect, and comparative case studies of rainfall rate ($R$) measurements with $Z_{DR}$ radars and ground-based disdrometers or rain gauges have produced very encouraging results (Bringi et al. 1982; Seliga et al. 1986; Goddard et al. 1982; Goddard and Cherry 1984; and Direskeneli et al. 1986).

Even though researchers have achieved improvements by using $Z_{DR}$-based rain rate estimates rather than estimates using reflectivity ($Z$) alone, however, the actual degree of improvement is not conclusive and is still being debated. Goddard and Cherry (1984), for example, concluded that they obtained an improvement using the $Z_{DR}$-method rather than $Z-R$ relations. Based on disdrometer data, they obtained a reduction in the standard error of $R$ using the $Z_{DR}$ method to 14% from the 33% obtained using the $Z-R$ relationship. Similar results using radar measurements of $Z$ and $Z_{DR}$ resulted in a reduction in the standard deviation to only 32% from the 40% obtained using $Z-R$ relations. Ulbrich and Atlas (1984) analyzed a large number of disdrometer-based RSD samples (moderate-to-high rain rates) and showed that $Z_{DR}$ methods could theoretically decrease the standard error in rain rate estimates to 14% from the 30% to 50% standard error associated with $Z-R$ relations. Direskeneli et al. (1986) used radar data to show that the $Z_{DR}$-method improved rain rate estimates by reducing the standard error to 31% from the 40% obtained using a local $Z-R$ relationship. A cursory inspection of these results would appear to imply that the $Z_{DR}$ method did not perform significantly “better” than $Z-R$ relations, at least not to the extent shown by Ulbrich and Atlas (1984). Since fluctuations in the radar and surface (e.g., disdrometer) observations of rainfall are governed by two totally different stochastic processes, the fluctuations and mutual correlations in these processes must be considered before firm conclusions can be drawn.

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In a previous paper, Chandrasekar and Bringi (1987, henceforth referred to as CB), used simulations to study the error structure of radar reflectivity and disdrometer measurements of rainfall. In this paper we continue this analysis of the error structure of rainfall measurements with an emphasis on differential reflectivity. The analysis is done in two stages, i.e., using analytical techniques and large-scale simulations. Analytical results of variance in the measurements and correlation between different estimates (e.g., Z and Z_{DR} from disdrometer data) are given to provide some physical insight into the nature of fluctuations in the measurements. But we cannot proceed very far analytically when we start considering the physical fluctuations in natural rainfall as well as the statistical fluctuations that occur in the measurements. Simulations offer a powerful method of studying this problem, since natural variations in rainfall can be introduced separately from statistical fluctuations.

Section 2 describes the raindrop size distribution model, while section 3 discusses the shape of raindrops and their influence on dual-polarized measurements. In section 4 we analytically derive the accuracy of Z_{DR} when calculated using disdrometer data and in section 5 the correlation between Z_{DR} and other RSD moments. Section 6 discusses radar measurements of Z and Z_{DR} and their simulations. Section 7 discusses the Z_{DR} method for rainfall estimates and their accuracies when obtained from both radar and disdrometer. Section 8 presents a detailed simulation study of rain rate estimates using dual-polarized techniques for both radar and disdrometer measurements and uses the simulation results to explain some features of radar/disdrometer intercomparisons conducted using the Z_{DR} technique. Section 9 summarizes the key results.

2. Raindrop size distribution

The distribution of raindrop sizes is of central importance in determining such properties of the rain medium as reflectivity (Z), liquid water content, and rainfall rate (R). The space-time evolution of the RSD is typically due to a variety of physical processes, e.g., evaporation, collision-coalescence, collisional breakup, drop sorting. Both cloud models and measurements of RSDs at the surface and aloft show that a gamma model of the RSD can describe many of the natural variations in the RSD (Ulbrich 1983):

\[
N(D) = N_0 D^m e^{-\Delta D},
\]

where \( N(D) \) is the number of raindrops per unit volume per unit size interval (\( D \) to \( D + \Delta D \)). In terms of the conventional gamma probability density function (pdf), \( N(D) \) can be written in the equivalent form,

\[
N(D) = \frac{N_T}{\Gamma(\alpha)b^\alpha} D^{\alpha-1} e^{-D/b^\alpha},
\]

where \( \alpha > 0, \beta > 0, D \geq 0, \Gamma(\cdot) \) represents gamma function and \( N_T \) is the total number of drops. We note that

\[
N_0 = \frac{N_T}{\Gamma(\alpha)b^\alpha}; \quad m = \alpha - 1, \quad \Delta = \frac{1}{b}.
\]

A physically meaningful parameter known as the median volume diameter \( D_0 \) can be defined by

\[
\int_0^{D_0} D^3 N(D) dD = \int_0^\infty D^3 N(D) dD, \quad (3)
\]

where \( D_0 \) is such that all drops with diameter \( \leq D_0 \) contribute to one-half the total liquid water content. Ulbrich (1983) has shown that \( \Delta D_0 \approx 3.67 + m \). Reflectivity (\( Z_{sd} \)) and rain intensity (\( R_{sd} \)) can be formulated as various moments of \( N(D) \):

\[
Z_{sd} = \int_0^\infty D^4 N(D) dD, \quad \text{mm}^6 \text{m}^{-3}, \quad (4)
\]

\[
R_{sd} = \frac{\pi}{6} \int_0^\infty D^3 v(D) N(D) dD, \quad \text{mm h}^{-1}, \quad (5)
\]

where \( v(D) = C_v D^{0.67} \) is the raindrop still-air fall speed (Atlas and Ulbrich 1977). The typical value for \( C_v \) is 17.67 m s\(^{-1}\) cm\(^{-0.67}\).

RSDs can be estimated by using surface instruments, e.g., disdrometers (drop-size meters with sampling volume dependent on drop size), or 2D-PMS probes mounted on instrumented aircraft (Cooper et al. 1983). A number of authors have used \( N(D) \) data from disdrometers to compute derived quantities like Z, R, etc. (Ulbrich 1983; Joss and Gori 1978; Bringi et al. 1982; Goddard et al. 1982; Seliga et al. 1986).

3. Shape of raindrops

The equilibrium shape of a raindrop falling at its terminal fall speed is that shape for which forces due to surface tension, hydrostatic pressure, and aerodynamic pressure (due to airflow around the drop) are in balance (Pruppacher and Pitter 1971). For drops greater than 1 mm in diameter the shapes are nonspherical and can be approximated by oblate spheroids. Pruppacher and Pitter (1971) show that the approximate axis ratio (\( b/a \)) of a drop and the diameter \( D_e \) (of an equivalent spherical raindrop) are related by

\[
\frac{b}{a} = r = 1.03 - 0.062 D_e, \quad (6)
\]

where \( D_e \) is in millimeters, and \( a \) and \( b \) are the major and minor axes of the drop, respectively. The \( Z_{DR} \) technique exploits this fact by measuring the polarization-dependent backscatter from the raindrops at two polarization states that nominally coincide with the horizontal and vertical directions. Seliga and Bringi (1976) defined \( Z_{DR} \) in terms of the RSD as,
where $\sigma_H$ and $\sigma_V$ are the radar cross sections of the oblate raindrops at horizontal ($H$) and vertical ($V$) polarizations, respectively. In this paper we use the shape-versus-size relation of raindrops to compute $Z_{DR}$, which is similar to the procedure used by Seliga and Bringi (1976). The conclusions presented in this paper will not be significantly altered if different nonequilibrium relations are used (Jameson 1983).

4. Statistics of $Z_{DR}$ estimated from disdrometer

Gertzman and Atlas (1977) have shown that, in raindrop sampling devices such as disdrometers, the measurement variability is due to both statistical sampling errors and real fine-scale physical variations that are not readily separable from the statistical ones. Sasyo (1965) and Cornford (1967) have shown that, for a constant mean rain intensity, the total number of raindrops observed will be distributed about its mean according to the Poisson distribution. This property has been used by Joss and Waldvogel (1969), Gertzman and Atlas (1977) and Wong and Chidambaram (1985) to obtain the fractional standard deviation (FSD) of higher-order moment estimators (some of which correspond to radar measurements) of RSDs. FSD is defined as the standard deviation divided by the mean of the estimator. Chandrasekar and Bringi (1987) obtained the correlation between the various moments of the RSD calculated from raindrop samples. Those results were limited to radar reflectivity. In this section we extend the results to include a study of the accuracy of differential reflectivity estimates from the disdrometer. We rewrite (2) in the form of a gamma probability density function (pdf) as

$$f(D) = \frac{\Lambda^{m+1}}{\Gamma(m+1)} D^m e^{-\Lambda D}.$$  

In the following development, we assume that the sampling volume, $V$, is constant and does not vary with raindrop size. If $V$ does vary with $D$, as is the case with disdrometers, it alters the sampled RSD. This dependency can be introduced by multiplying $f(D)$ by the sample volume function $V(D)$.

If $n$ raindrops are observed within a fixed sample volume $V$ with diameters $D_1, D_2, \ldots, D_n$, then this RSD is a composite distribution of the total number of raindrops (or equivalently, the concentration of drops within any interval $D$ to $D + \Delta D$) and the drop diameter, where the diameters are distributed according to the gamma pdf, and the total number of raindrops ($n$) is distributed according to the Poisson distribution. Conventional estimators of higher-order RSD moments are expressed as follows:

$$Z = \frac{1}{V} \sum_{i=1}^{n} D_i^6 \text{ mm}^6 \text{ m}^{-3}$$  

$$R = \frac{7.12 \times 10^{-3}}{V} \sum_{i=1}^{n} D_i^{3.67} \text{ mm hr}^{-1}.$$

The above estimators can be written in general as

$$\hat{P}_a = \frac{C_a}{V} \sum_{i=1}^{n} D_i^a$$  

where $P_a = C_a \int D^a N(D) dD$.

Gertzman and Atlas (1977) and Chandrasekar and Bringi (1987) have shown that $\hat{P}_a$ is an unbiased estimator and its FSD is given by

$$\text{FSD} \left( \hat{P}_a \right) = \left[ \frac{\Lambda^{m+1}}{\Gamma(m+1)} \right]^{1/2} \Gamma^{1/2}(m+\alpha+1) \Gamma(m+\alpha+1).$$

In this section we present a formulation to obtain the variance of differential reflectivity estimates obtained from disdrometer measurements.

When the sample volume of raindrops does not change with the size of the raindrop, $Z_{DR}$ estimates from RSD measurements can be defined by

$$\hat{Z}_{DR} = \frac{\sum_{i=1}^{n} \sigma_{HH}(D)}{\sum_{i=1}^{n} \sigma_{VV}(D)} = \frac{X}{Y},$$

where $\sigma_{HH}$ and $\sigma_{VV}$ are the horizontal and vertical backscatter cross sections of individual oblate spheroidal raindrops given as a function of drop equivalent diameter $D$; $Z_{DR}$ is defined as $10 \log_{10} \hat{Z}_{DR}$. Equation (11) represents $Z_{DR}$ as the ratio of two random sums of random variables. Taking expectation on both sides of (11), we get (Mood et al. 1974)

$$E(\hat{Z}_{DR}) \approx \frac{E(X)}{E(Y)} - \frac{1}{[E(Y)]^2} \text{ cov}[X, Y] + \frac{E(X)}{[E(Y)]^3} \text{ var}(Y),$$

where $E(\ )$ stands for expected value. The above expectations are taken over random sums. Using conditioning techniques similar to those used by CB we get

$$E[\sum_{i=1}^{n} \sigma_{HH}(D)] = E\{E[\sum_{i=1}^{n} \sigma_{HH}(D) | n]\},$$

where $E(-|-)$ refers to conditional expectation. It is easily shown that (13) can be written as

$$E[\sum_{i=1}^{n} \sigma_{HH}(D)] = E(n)E(\sigma_{HH}),$$

where $E(n)$ is the expectation with respect to the total
number of drops and \( E(\sigma_{HH}) \) is the expectation with respect to the gamma pdf.

\[ E(n) \text{ can be obtained as } VN_{T} \text{ and is given by} \]
\[ E(n) = \frac{VN_{0} \Gamma(m + 1)}{A^{m+1}}. \]  

(15)

Hence, we can write
\[ E(X) = E(\sigma_{HH}(D)) \cdot E(n) \]  
\[ E(Y) = E(\sigma_{VV}(D)) \cdot E(n). \]  

(16a)

(16b)

Using principles similar to those used to obtain (14),
\[ \text{cov}(X, Y) = E \left\{ \text{cov}(\sum_{i=1}^{n} \sigma_{HH}(D), \sum_{i=1}^{n} \sigma_{VV}(D) | n) \right\} \]
\[ + \text{cov} \left\{ E(\sum_{i=1}^{n} \sigma_{HH}(D) | n), E(\sum_{i=1}^{n} \sigma_{VV}(D) | n) \right\}. \]

(17)

Similarly, from (16) and (17) we can write
\[ \text{var}(X) = E(n) [ \text{var}(\sigma_{HH}(D)) + E^2(\sigma_{HH}(D))] \]  
\[ \text{var}(Y) = E(n) [ \text{var}(\sigma_{VV}(D)) + E^2(\sigma_{VV}(D))]. \]  

(18)

(19)

The analytical details of the above equations are given in CB.

The variance of \( Z_{DR} \) can be written as (Mood et al. 1974)
\[ \text{var}(Z_{DR}) \approx \left[ \frac{E(X)}{E(Y)} \right]^2 \times \left[ \frac{\text{var}(X)}{E^2(X)} + \frac{\text{var}(Y)}{E^2(Y)} - \frac{2 \text{cov}(X, Y)}{E(X)E(Y)} \right]. \]  

(20)

Substituting (13) through (18) into (20) we get
\[ \frac{\text{var}(Z_{DR})}{Z_{DR}} \approx \frac{1}{E(n)} \times \left[ \frac{E(\sigma_{HH})}{E^2(\sigma_{HH})} + \frac{E(\sigma_{VV})}{E^2(\sigma_{VV})} - \frac{2E(\sigma_{HH}\sigma_{VV})}{E(\sigma_{HH})E(\sigma_{VV})} \right]. \]  

(21)

The expectation values in (21) are to be computed as integrals over the RSD for the shape of raindrops given by (6). It turns out that (21), as written, is the difference of two almost equal quantities and needs to be computed in slightly different way from what is given in (21). This same feature occurs in several subsequent equations, and the method of computing them is discussed in appendix A. Figure 1 shows the fractional standard deviation of \( Z_{DR} \) estimates in dB obtained from disdrometer measurements multiplied by the constant \( C = 0.1 \ln 10V_{T} \); FSD \( (Z_{DR}) \) is plotted as a function of \( D_{0} \) with \( m \) as a parameter. The contribution to \( Z_{DR} \) increases with raindrop size, since the larger drops are more oblate than the smaller ones. This means that the variance contribution due to sampling increases with increased \( D_{0} \) of the distribution. This value is normalized with respect to \( Z_{DR} \), which also increases with \( D_{0} \). Thus, we have an increasing trend for FSD for a given \( m \) that is later suppressed by increasing \( Z_{DR} \). The variability of FSD \( (Z_{DR}) \) with \( m \) for a given \( D_{0} \) can be explained easily from the above arguments. The increasing \( m \) of the distribution makes it less asymmetric about the median. This can easily be observed, since \( m = 0 \) is an exponential distribution without any symmetry, whereas very large \( m(m \rightarrow \infty) \) corresponds to a Gaussian distribution that is symmetric. This implies that for a constant \( D_{0} \) the large drop contribution is less for higher \( m \), which means a reduced standard deviation in \( Z_{DR} \), while the mean \( Z_{DR} \) does not change as much with \( m \). As a result, we can see the FSD of \( Z_{DR} \) reducing with increased \( m \). The rate of decrease falls quickly with increasing \( m \), since the distribution approaches symmetry quickly.

Figure 1 can be used to obtain the standard deviation of \( Z_{DR} \) estimates when it is obtained from disdrometer samples. In the above analysis we have assumed that the sample volume does not change with drop size. If the sample volume does change with drop size, the technique for handling this situation is described in appendix B.

5. Correlation of RSD moments with \( Z_{DR} \)

The estimates of \( Z \) and \( R \) are obtained from the moments of the RSD. When samples are taken from
one RSD, the fluctuations in the samples give fluctuations in the estimates of the moments of the RSD. If we compute the moments from the samples of the same RSD then these estimates are correlated. This correlation has been discussed extensively in CB; however, their formulation cannot be used directly for correlations involving $Z_{DR}$, since it is a ratio of two quantities.

In the following development we study the correlation of $Z_{DR}$ with $Z$; the correlation of $Z_{DR}$ with other RSD moments can be similarly obtained. We repeat (11), which defines the estimate of $Z_{DR}$:

$$
\hat{Z}_{DR} = \frac{\sum_{i=1}^{n} \sigma_{HH}^i(D)}{\sum_{i=1}^{n} \sigma_{VV}^i(D)} = \frac{X}{Y}.
$$

(22a)

The above can be expanded in a two-dimensional Taylor Series about $E(X)$ and $E(Y)$ up to first order terms as (Mood et al. 1974)

$$
\hat{Z}_{DR} = \frac{E(X)}{E(Y)} \left[ 1 + \frac{(X - E(X))}{E(X)} - \frac{(Y - E(Y))}{E(Y)} \right].
$$

(22b)

The estimate of $Z$ from disdrometer RSD samples can be defined as

$$
\hat{Z} = \frac{\lambda^4}{\pi^3 |K|^2} \frac{1}{V} \sum_{i=1}^{n} \sigma_{HH}^i(D),
$$

(23)

where $V$ is the sampling volume, $\lambda$ is the wavelength and $K = (\mu^2 - 1)/(\mu^2 + 2)$, where $\mu$ is the complex refractive index of water. For simplicity we assume $V$ to be fixed even though for a disdrometer the sampling volume is a function of drop size (please see appendix B). With the above formulations, the covariance between $\hat{Z}$ and $\hat{Z}_{DR}$ can be written as

$$
cov(\hat{Z}, \hat{Z}_{DR}) = C^* \times \cov \left[ X, \frac{E(X)}{E(Y)} \left[ 1 + \frac{X - E(X)}{E(X)} - \frac{Y - E(Y)}{E(Y)} \right] \right].
$$

(24)

where $C^* = \frac{\lambda^4}{\pi^3 |K|^2}$.

Equation (24) can be simplified into

$$
cov(\hat{Z}, \hat{Z}_{DR}) = C^* \frac{E(X)}{E(Y)} \left[ \frac{\var(X)}{E(X)} - \frac{\cov(X, Y)}{E(Y)} \right].
$$

(25)

The correlation between $\hat{Z}$ and $\hat{Z}_{DR}$ can be written as

$$
\rho_{\hat{Z}, \hat{Z}_{DR}} = \frac{\cov(\hat{Z}, \hat{Z}_{DR})}{\sqrt{\var(\hat{Z}) \var(\hat{Z}_{DR})}}.
$$

(26)

The superscript “dis” on $\rho$ indicates that it refers to the correlation between disdrometer estimates. Using results from (18), (21) and (25) in (26), the correlation between $\hat{Z}$ and $\hat{Z}_{DR}$ can be written as

$$
\rho_{\hat{Z}, \hat{Z}_{DR}} = \frac{\left\{ \frac{E(\sigma_{HH}^i)}{E(\sigma_{HH})} - \frac{E(\sigma_{HH} \sigma_{VV})}{E(\sigma_{VV})} \right\}}{\left\{ \frac{E(\sigma_{HH}^2)}{E^2(\sigma_{HH})} + \frac{E(\sigma_{VV}^2)}{E^2(\sigma_{VV})} - \frac{2E(\sigma_{HH} \sigma_{VV})}{E(\sigma_{HH})E(\sigma_{VV})} \right\}^{1/2}}.
$$

(27)

and the $Z_{DR}$ method based on estimates of $\hat{Z}$ and $\hat{Z}_{DR}$ is not given here, since the analytical steps get very complicated; however, this correlation effect is displayed through simulations in section 8. Thus, we see from these results that various quantities, such as rainfall rate, reflectivity and differential reflectivity, are correlated if obtained from the same set of raindrop samples. The significance of this correlation in interpreting experimental observations is deferred to section 8.

6. Dual-polarized radar measurements

Radar measurements of reflectivity $Z$ and differential reflectivity $Z_{DR}$ involve measurement of mean (time averaged) backscattered power $(P)$ from a given resolution volume at two polarizations, namely, horizontal and vertical. Signal power fluctuations are well known and were treated by Marshall and Hitschfeld (1953). Fluctuations in $Z_{DR}$ depend on both signal power fluctuations and cross-correlation between the
7. ZDR-method for rainfall rate

The radar measurables used in the dual polarized (or dual-parameter) estimates of rainfall rate are Z and ZDR. Ulbrich and Atlas (1984) suggested an expression for rain rate in terms of Z and ZDR:

$$ R = F_m Z \cdot Z_{DR}^{-1.5}. $$

Equation (30) has to be adjusted for different gamma distributions by varying $F_m$. In this work the parameters $N_0$, $D_0$ and $m$ of the gamma distribution are varied over the entire range suggested by Ulbrich (1983) for natural rainfall. The value of $N_0$ is varied between $10^{4.2} \exp(2.8 m)$ m$^{-3}$ mm$^{-1}$ and $10^{5.5} \exp(3.57 m)$ m$^{-3}$ mm$^{-1}$, $D_0$ is varied between 0 and 2.5 mm, and $m$ (which may have noninteger values) is varied between -1 and 4. The variations in these parameters are done randomly. Z, ZDR and R are computed for each triplet of points ($N_0$, $D_0$, $m$) and a nonlinear regression is performed to fit R in the form

$$ R = \tilde{F} \cdot Z^\alpha Z_{DR}^\beta. $$

We find the best-fit values to be $\tilde{F} = 2.397 \times 10^{-3}$, $\alpha = 0.94$ and $\beta = -1.08$. It is also of interest to consider single-parameter methods such as the Marshall-Palmer (1948) $Z$–$R$ relation,

$$ Z = 200 R^{1.6}, $$

which is a particular form of the general relation $R = C_1 Z^\alpha$. Equations like (31) and (32) are used with both disdrometer and radar measurements to compute rainfall rates. We now obtain the accuracy of such estimates.

Using analytical techniques similar to those used in sections 4 and 5 we can write for single parameter methods

$$ \text{var}(R) \approx a^2 C_1^2 \text{var}(Z). $$

Thus, the fractional standard deviation of R is given by

$$ \text{FSD}(R) \approx a \text{FSD}(Z). $$

The FSD(Z) for disdrometer estimates of Z is given by CB as

$$ \text{FSD}(Z) = \left[ \frac{1}{N_0 V} \sum \frac{(Z_i - \bar{Z})^2}{\bar{Z}^{m+1}} \right]^{1/2} \frac{\Gamma^{1/2}(m + 13)}{\Gamma(m + 7)}, $$

whereas for radar estimates it depends on the number of samples averaged and the Doppler spectrum width.

For dual-parameter methods, the variance of $R$ can be approximated using techniques similar to those used in sections 4 and 5 as (Mood et al. 1974)
\[
\frac{\text{var}(R)}{E^2(R)} \approx \alpha^2 \frac{\text{var}(Z)}{E^2(Z)} + \beta^2 \frac{\text{var}(Z_{\text{DR}})}{E^2(Z_{\text{DR}})} + 2\alpha\beta \frac{\text{cov}(Z, Z_{\text{DR}})}{E(Z)E(Z_{\text{DR}})}.
\]

Note that the covariance term in the above expansion makes a significant difference in the case of radar and disdrometer measurements. In the case of radar measurements, we have seen in section 6 that Z and \(Z_{\text{DR}}\) are nearly uncorrelated and as a result,

\[
\frac{\text{var}(R)}{E^2(R)} \approx \alpha^2 \frac{\text{var}(Z)}{E^2(Z)} + \beta^2 \frac{\text{var}(Z_{\text{DR}})}{E^2(Z_{\text{DR}})}.
\]

From the above equation we can see that the statistical fluctuation of \(R\) has increased by adding the extra \(Z_{\text{DR}}\) measurement, because that extra measurement adds its own fluctuations to the fluctuations in \(R\). Thus \(Z_{\text{DR}}\) must be measured as accurately as possible. In contrast to the added "noisiness," the ability of mean \(Z_{\text{DR}}\) in tracking the mean drop size of the RSD leads to more accurate \(R\) estimates (Ulbrich and Atlas 1984).

In the case of disdrometer measurements the covariance term in (35) cannot be dropped; however, we need to note an important advantage here, because \(\beta\) is negative. Since \(\beta\) is negative and the correlation between \(Z\) and \(Z_{\text{DR}}\) is positive, the variance in \(R\) is less than what it would be if \(Z\) and \(Z_{\text{DR}}\) were uncorrelated. Rewriting (35) in terms of correlation coefficient we get

\[
\frac{\text{var}(R)}{E^2(R)} \approx \alpha^2 \frac{\text{var}(Z)}{E^2(Z)} + \beta^2 \frac{\text{var}(Z_{\text{DR}})}{E^2(Z_{\text{DR}})} + 2\alpha\beta \frac{\text{var}(Z)\rho_{Z,Z_{\text{DR}}}}{E^2(Z)E^2(Z_{\text{DR}})}. \tag{36}
\]

We know from section 5 that \(\rho_{Z,Z_{\text{DR}}}\) is typically 0.8. Using values for \(\alpha\) and \(\beta\) we can write (36) as

\[
\frac{\text{var}(R)}{E^2(R)} \approx \left\{ 0.94 \frac{\text{var}(Z)^{1/2}}{E^2(Z)^{1/2}} - 1.08 \left[ \frac{\text{var}(Z_{\text{DR}})^{1/2}}{E^2(Z_{\text{DR}})^{1/2}} \right]^2 \right\} \rho_{Z,Z_{\text{DR}}} + (1 - \rho_{Z,Z_{\text{DR}}}) \left[ \frac{\text{var}(Z)^{1/2}}{E^2(Z)^{1/2}} \right] \left[ \frac{\text{var}(Z_{\text{DR}})^{1/2}}{E^2(Z_{\text{DR}})^{1/2}} \right]. \tag{37}
\]

We can see from the above equation that the first bracketed term is independent of \(\rho\) while the second term increases with decreasing \(\rho\). Reducing the variance of an estimate based on the correlation between two quantities is a standard phenomenon in statistics that is sometimes introduced deliberately in the design of experiments.

8. Radar/disdrometer simulations for intercomparison

We now define the following variables:

\(R^{sd}(Z, Z_{\text{DR}})\) Dual-Parameter estimate of rainfall rate obtained from theoretical values (based on size distribution) of \(Z\) and \(Z_{\text{DR}}\) using (31).

\(R^{sd}(Z)\) Single-Parameter estimate of rainfall rate obtained from the theoretical value of \(Z\) using (32).

\(R^{rd}\) Theoretical rain rate obtained as the 3.67th moment of the gamma RSD.

Figure 3 shows a scatterplot of \(R^{sd}(Z, Z_{\text{DR}})\) versus \(R^{sd}\) in which each point represents the triplet \((N_0, D_0, m)\) from a gamma RSD, with the gamma parameters varied over the whole range prescribed by Ulbrich (1983). Figure 3 shows that \(R^{sd}(Z, Z_{\text{DR}})\) is a good estimator of \(R^{sd}\) as first proposed by Seliga and Bringi (1976). Figure 4 depicts a scatter plot of \(R^{sd}(Z)\) versus \(R^{sd}\), which shows that the \(Z-R\) method estimates rain rate reasonably well up to rain rates of about 20 mm h\(^{-1}\) but that \(R^{sd}(Z)\) is biased for high rain rates and the scatter between \(R^{sd}\) and \(R^{sd}(Z)\) increases significantly. This shows that \(Z-R\) relations cannot track the variability in the drop size distribution, as was clearly established by Atlas and Ulbrich (1974), and Ulbrich and Atlas (1978).

In sections 4 and 5 we have derived analytical formulas for \(\text{var}(Z_{\text{DR}})\) and the correlation between \(Z\) and \(Z_{\text{DR}}\), in which each estimate is based on disdrometer samples of a gamma RSD. The analytical studies get very complicated if we need to study the correlation between quantities that are expressed in the form of power laws, e.g., the correlation between \(R\) of (30) and \(R\) of (9b). In order to overcome this problem CB have simulated disdrometer samples of a gamma RSD. In this section we use simulations (for both radar and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Scatter plot of \(R^{sd}(Z, Z_{\text{DR}})\) versus \(R^{sd}\) for gamma RSDs. Each point represents a triplet of values \((N_0, D_0, m)\) that vary over a wide range. This figure shows the ultimate accuracy of \(R^{sd}(Z, Z_{\text{DR}})\) in measuring rain rate. The abscissa values correspond to the exact rain rate for gamma RSDs.}
\end{figure}
simulate radar $Z_{sm}$ and radar $Z_{DR}$, and the corresponding values of $R_{sm}(Z)$ and $R_{sm}(Z, Z_{DR})$. The various data points in the sequence of Figs. 5-8 correspond to varying $N_0$, $D_0$, $m$ over the entire range of naturally occurring values (see Ulbrich 1983). Thus, these figures incorporate physical fluctuations onto which statistical fluctuations are superimposed.

The first group of Figs. 5 and 6 have along the abscissa values of $R$ (the "exact" rain rate) and $R_{dis}$ (disdrometer-sampled rain rate), respectively. Figure
rain rate more accurately than $Z$ alone. Figures 8a and 8b show scatter plots that are simulations of those typically observed in radar/dirdrometer intercomparison experiments. Figure 8a shows the plot of $R^{\text{sm}}(Z)$ versus $R^{\text{dis}}$, whereas Fig. 8b shows the plot of $R^{\text{sm}}(Z, Z_{\text{DR}})$ versus $R^{\text{dis}}$. The scatter in Fig. 8b is only marginally less than that in Fig. 8a (especially for $R \leq 25$ mm h$^{-1}$). We have seen from Figs. 5b and 7b that $R^{\text{sm}}(Z, Z_{\text{DR}})$ “performs” as well as $R^{\text{dis}}(Z, Z_{\text{DR}})$ when the standard for comparison is $R^{\text{ad}}$. However, a comparison of Figs. 8b and 6b might wrongly indicate that $R^{\text{sm}}(Z, Z_{\text{DR}})$ is not as good as it should be for measuring rain rate when the standard for comparison is $R^{\text{dis}}$. Figures 8b and 6b show the typical type of observables available from experiments (Goddard and Cherry 1984; Ulbrich and Atlas 1984).

In Table 1 we show standard deviations obtained using dirdrometer and radar data by Goddard and Cherry (1984), Seliga et al. (1986) and Direskeneli et al. (1986). The percentage standard deviations are determined for $R^{\text{dis}}(Z)$ and $R^{\text{dis}}(Z, Z_{\text{DR}})$ versus $R^{\text{dis}}$, and for $R^{\text{sm}}(Z)$ and $R^{\text{sm}}(Z, Z_{\text{DR}})$ versus $R^{\text{dis}}$. We continue to use our notation to describe the experimental data for ease of comparison with our simulation results depicted in Figs. 5–8. The approximate factor of 2 reduction in standard deviation between the first two rows of Table 1 seems good as shown by our simulation results in Figs. 6a and 6b and the related discussion. In contrast, the reduction in standard deviation between the last two rows of Table 1 is deceptively “poor,” again as shown by our simulation results in Figs. 8a and 8b and the related discussion. To the extent that natural raindrop size distributions can be modelled by the gamma form, the experimental data of Goddard...
and Cherry (1984) and Direskeneli et al. (1986), along
with our radar and disdrometer simulation results,
show that the use of \( Z \) and \( Z_{DR} \) can better charac
terize the physical fluctuations of the gamma RSD pa
rameters than \( Z \) alone, thereby allowing an accurate es
timation of rain rate, especially for \( R \geq 20 \text{ mm h}^{-1} \).
Radar-disdrometer intercomparisons performed so far
do not conclusively prove that \( Z_{DR} \) can assist in track
ing the physical variability of the raindrop size dis	ribution, since statistical fluctuations appear to contrib-
ute to a significant portion of the observed features.
We recognize that physical factors such as those enu
merated by Zawadski (1984) will also contribute to the scatter between \( R_{mm}(Z, Z_{DR}) \) and \( R_{dis} \).

9. Conclusions

We have studied the error structure of rainfall in
tercomparison experiments done using radar and dis-

Fig. 7. a. As in Fig. 5a except ordinate values are obtained from radar simulations of reflectivity using \( Z = 200 R^{1.6} \); see section 6. \( R^{ad} \) is the exact rain rate. b. As in Fig. 7a except ordinate values are obtained from radar simulations of \( Z \) and \( Z_{DR} \) using (31).

Fig. 8. a. As in Fig. 7a except abscissa values are obtained directly from disdrometer samples, i.e., abscissa is the same as in Fig. 6a. This figure shows a typical radar/disdrometer simulation that can be used for intercomparison studies. Statistical fluctuations of \( R_{mm}(Z) \) and \( R_{mm} \) are independent. Compare with the scatter in Fig. 6a. b. As in Figure 8a except ordinate values are obtained from radar simulations of \( Z \) and \( Z_{DR} \), i.e., ordinate same as in Fig. 7b. Compare with the scatter in Fig. 6b. Also note that the scatter in this figure is not significantly less than in Fig. 8a; however, the bias has been removed, especially for \( R \approx 25 \text{ mm h}^{-1} \).
drometer measurements for both single and dual-parameter methods. We have obtained expressions for the accuracy of \(Z_{\text{DR}}\) estimates using disdrometer samples, and also the correlation of \(Z_{\text{DR}}\) with other RSD moments, e.g., reflectivity. The analysis shows that \(Z_{\text{DR}}\) estimates from disdrometers are correlated with estimates of \(Z\) with the correlation coefficient typically \(\sim 0.8\). This correlation is purely statistical and is independent of the physical correlations, which is in direct contrast with radar measurements, where \(Z\) and \(Z_{\text{DR}}\) are nearly uncorrelated. We have then obtained approximate expressions for the standard errors in rainfall rate in terms of standard error in \(Z\) and \(Z_{\text{DR}}\) and their correlations. We show that, in the case of radar (where the measurements are uncorrelated) the fluctuations of \(Z\) and \(Z_{\text{DR}}\) add up cumulatively and contribute to the statistical fluctuations in rainfall rate; however, disdrometer estimates of \(Z\) and \(Z_{\text{DR}}\) are positively correlated, and, combined with a negative exponent on \(Z_{\text{DR}}\) in the expression for \(R\), help to reduce the fractional standard error of the dual-parameter rainfall rate estimates. We have also shown that disdrometer-derived \(R\) (3 \(\cdot\) 67th moment) is highly correlated with dual-parameter estimates of \(R\) obtained using \(Z\) and \(Z_{\text{DR}}\) from the same disdrometer samples. This feature is important since these two quantities (\(R_{\text{dis}}\) and \(R_{\text{dis}}(Z, Z_{\text{DR}})\)) are often compared to each other. Again, we note that the correlations referred to are purely statistical in nature and arise because the same disdrometer samples are used to compute \(R\), \(Z\), and \(Z_{\text{DR}}\).

We have simulated radar measurements of \(Z\) and \(Z_{\text{DR}}\) as well as disdrometer samples of gamma RSDs. This simulation is computationally intensive, since the physical fluctuations of rainfall (i.e., gamma RSD parameters \(N_0\), \(D_0\), \(m\)) have to be varied along with the statistical fluctuations. Thus, we have had complete control over the physical and statistical variables.

We have applied our radar and disdrometer simulations to study the effect of the various correlations between the measurable often used in the intercomparison of rain rate. From the simulation results we can see that the scatter between \(R_{\text{dis}}\) and \(R_{\text{dis}}(Z, Z_{\text{DR}})\) is significantly smaller than the scatter between \(R_{\text{dis}}\) and \(R_{\text{dis}}(Z)\) for three reasons: 1) \(Z\) and \(Z_{\text{DR}}\) together track the RSD better than \(Z\) alone, 2) the standard error of \(R_{\text{dis}}(Z, Z_{\text{DR}})\) is small because of the correlation between \(Z_{\text{dis}}\) and \(Z_{\text{DR}}\), and 3) though \(R_{\text{dis}}\) and \(R_{\text{dis}}(Z, Z_{\text{DR}})\) are both measured quantities they are highly correlated.

Thus, in disdrometer measurements, the statistical fluctuations (being correlated) cause a compression of the scatter, while in the case of radar/disdrometer intercomparison studies the fluctuations are uncorrelated, thus increasing the scatter between the two measurements of rainfall rate. Only the physical contribution is common to both radar and disdrometer measurements. This simulation study explains to some extent the varied degrees of accuracy reported in the literature regarding experiments conducted to verify the dual-parameter method of rain rate estimation. We need to note here that our simulations were performed for an ideal physical situation, namely, radar and disdrometer sampling of exactly the same rain medium, which is assumed to be homogeneous. If there is even a slight difference between the rain medium sampled by the radar and disdrometer (e.g., due to physical causes, as enumerated by Zawadzki 1984) it will only make the case worse as compared to what we have shown in our simulations.

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APPENDIX A

Computation of FSD (\(Z_{\text{DR}}\))

Fractional Standard deviation of \(Z_{\text{DR}}\) is given by

\[
\frac{\text{var}(Z_{\text{DR}})}{Z_{\text{DR}}^2} \approx \frac{1}{E(n)} \left[ \frac{E(\sigma_{HH}^3)}{E^2(\sigma_{HH})} + \frac{E(\sigma_{VV}^3)}{E^2(\sigma_{VV})} - 2 \frac{E(\sigma_{HH}\sigma_{VV})}{E(\sigma_{HH})E(\sigma_{VV})} \right].
\]

In the above equation, FSD is obtained as the difference of two almost equal quantities. This can be rewritten by multiplying numerator and denominator by \(E^2(\sigma_{HH})E^2(\sigma_{VV})\) as

\[
[FSD(Z_{\text{DR}})]^2 = \frac{1}{E(n)} \left[ E^2(\sigma_{VV})E(\sigma_{HH}^3) + E^2(\sigma_{HH})E(\sigma_{VV}^3) - 2E(\sigma_{HH})E(\sigma_{VV})E(\sigma_{HH}\sigma_{VV}) \right] \times [E^2(\sigma_{HH})E^2(\sigma_{VV})]^{-1}. \quad (A1)
\]

In the above equation \(E(\sigma_{HH}), E(\sigma_{VV})\) can be computed separately. We can then take these constants in-
side the expectation, and (A1) can be rewritten as (Mood et al. 1974)

\[
\text{FSD}(\hat{Z}_{\text{DR}}) = \frac{1}{E(n)} \left[ E\left( E^2(\sigma_{\text{HH}})\sigma_{\text{HH}}^2 + E^2(\sigma_{\text{HH}})\sigma_{\text{VV}}^2 \right) - 2E(\sigma_{\text{HH}})E(\sigma_{\text{HH}}\sigma_{\text{HH}}\sigma_{\text{VV}}) \right] \left[ E^2(\sigma_{\text{HH}})E^2(\sigma_{\text{VV}}) \right]^{-1}. \tag{A2}
\]

Thus, FSD(\(\hat{Z}_{\text{DR}}\)) is written as one expectation taken over the argument given inside the square brackets. This expectation can be computed by integrating this argument over the required Gamma RSD.

Using similar arguments the covariance of \((\hat{Z}, \hat{Z}_{\text{DR}})\) given by (25) can be written as

\[
\text{cov}(\hat{Z}, \hat{Z}_{\text{DR}}) = C* \frac{E(\sigma_{\text{HH}})}{E(\sigma_{\text{VV}})} \left[ E(\sigma_{\text{HH}}) \sigma_{\text{HH}} \right. \\
\left. - E(\sigma_{\text{HH}})E(\sigma_{\text{HH}}\sigma_{\text{VV}}) \right] \left[ E(\sigma_{\text{HH}})E(\sigma_{\text{VV}}) \right]^{-1}. \tag{A3}
\]

After computing \(E(\sigma_{\text{VV}})\) and \(E(\sigma_{\text{HH}})\), \(\text{cov}(\hat{Z}, \hat{Z}_{\text{DR}})\) is computed as the expectation of the variable given inside the square brackets. In order to avoid numerical problems, equations of the type (A1) and (A3) should be used to compute quantities involving \(\hat{Z}_{\text{DR}}\).

**APPENDIX B**

**FSD of \(\hat{Z}_{\text{DR}}\) for Variable Sample Volume**

Let the sampling volume of drops vary as \(V(D) = V_0D^b\). It can easily be shown that if the drop size distribution is described by the gamma form with parameters \(N_0, \Lambda, m\) then the sampled distribution is also gamma with parameters \(N_0, \Lambda, (m + b)\). The \(\hat{Z}_{\text{DR}}\) estimate has to be adjusted for the sample volume in order to get an unbiased estimate. The new estimate of \(\hat{Z}_{\text{DR}}\) becomes

\[
\hat{Z}_{\text{DR}} = \frac{\sum_{i=1}^{N} \sigma_{\text{HH}}(D_i)}{\sum_{i=1}^{N} \sigma_{\text{VV}}(D_i)} = \frac{\sum_{i=1}^{N} \sigma_{\text{HH}}(D_i)D_i^{-b}}{\sum_{i=1}^{N} \sigma_{\text{VV}}(D_i)D_i^{-b}} = \frac{X'}{Y'}. \tag{B1}
\]

This estimator is now defined on a gamma RSD with parameters \(N_0, \Lambda, (m + b)\).

The equivalent of (16) is

\[
E(X') = E(\sigma_{\text{HH}}D^{-b})E(n),
\]

\[
E(Y') = E(\sigma_{\text{VV}}D^{-b})E(n). \tag{B2}
\]

The equivalent equations of (18) are

\[
\text{var}(X') = E(n)[\text{var}(\sigma_{\text{HH}}D^{-b}) + E^2(\sigma_{\text{HH}}D^{-b})]
\]

\[
\text{var}(Y') = E(n)[\text{var}(\sigma_{\text{VV}}D^{-b}) + E^2(\sigma_{\text{VV}}D^{-b})]. \tag{B3}
\]

Substituting (B3), (B2) and covariance term into (20) we get

\[
\frac{\text{var}(\hat{Z}_{\text{DR}})}{\hat{Z}_{\text{DR}}^2} = \frac{1}{E(n)} \left[ \frac{E(\sigma_{\text{HH}}D^{-2b})}{E^2(\sigma_{\text{HH}}D^{-b})} + \frac{E(\sigma_{\text{VV}}D^{-2b})}{E^2(\sigma_{\text{VV}}D^{-b})} - \frac{E(\sigma_{\text{HH}}\sigma_{\text{VV}}D^{-2b})}{E(\sigma_{\text{HH}}D^{-b})E(\sigma_{\text{VV}}D^{-b})} \right]. \tag{B4}
\]

Note that all expectations except \(E(n)\) are to be computed with respect to an RSD with parameters \(N_0, \Lambda, m + b\).

**Example**

Let \(b = 0.5\) (from Gertzman and Atlas 1977). Figure B1 shows the standard deviation of \(\hat{Z}_{\text{DR}}\) in dB normalized with respect to 0.1 ln10\(\sqrt{N_f}\) plotted as a function of \(D_0\) with \(m\) as a parameter.

**APPENDIX C**

**Correlation between \(R_{\text{dis}}\) and \(R_{\text{dis}}(Z)\)**

\(R_{\text{dis}}(Z)\) is the rainfall rate computed from the disdrometer using a Z-R relationship, and \(R_{\text{dis}}\) is the estimate of \(R\) obtained as the 3.67\(^{\text{th}}\) moment of the RSD,

\[
R_{\text{dis}} = C_R \sum_{i=1}^{n} D_i^{3.67}\text{ mm h}^{-1}, \tag{C1}
\]

where \(C_R = \frac{7.12 \times 10^{-3}}{V}\), and

\[
R_{\text{dis}}(Z) = C_i(Z_{\text{dis}})^a. \tag{C2}
\]
where $Z_{\text{dis}} = \frac{1}{V} \sum_{i=1}^{n} D_i^6 \text{ (mm}^6 \text{ m}^{-3})$. For the Marshall-Palmer relationship, $C_1 = 2.08 \times 10^{-4}$ and $a = 0.625$. Expanding $R(Z)$ as a Taylor series expansion about the mean value of reflectivity we get (Mood et al. 1974),

$$R(Z) \approx C_1 E(Z)^a + C_1 a[E(Z)]^{a-1}(Z - E(Z)).$$  

(C3)

Covariance of $R_{\text{dis}}(Z)$ with $R_{\text{dis}}$ can be simplified to (Mood et al. 1974),

$$\text{cov}[R_{\text{dis}}(Z), R_{\text{dis}}] = C_1 a[E(Z)]^{a-1} \text{cov}[Z_{\text{dis}}, R_{\text{dis}}].$$  

(C4)

Similarly, variance of $R_{\text{dis}}(z)$ can be simplified to,

$$\text{var}[R_{\text{dis}}(Z)] = C_1^2 a^2 [E(Z)]^{2(a-1)} \text{var}(Z_{\text{dis}}).$$  

(C5)

The correlation coefficient between $R_{\text{dis}}$ and $R_{\text{dis}}(Z)$ can be written as

$$\rho(R_{\text{dis}}, R_{\text{dis}}(Z)) = \frac{\text{cov}(R_{\text{dis}}, R_{\text{dis}}(Z))}{\sqrt{\text{var}(R_{\text{dis}}) \text{var}(Z_{\text{dis}})}}.$$  

(C6)

Substituting (C3) through (C5) into (C6) and simplifying we get

$$\rho(R_{\text{dis}}, R_{\text{dis}}(Z)) \approx \frac{\text{cov}(Z_{\text{dis}}, R_{\text{dis}})}{\sqrt{\text{var}(R_{\text{dis}}) \text{var}(Z_{\text{dis}})}}.$$  

(C7)

The right-hand side of (C7) can be recognized as the correlation between $R_{\text{dis}}$ and $Z_{\text{dis}}$. Hence,

$$\rho(R_{\text{dis}}, R_{\text{dis}}(Z)) \approx \rho(Z_{\text{dis}}, R_{\text{dis}}).$$  

(C8)

$\rho(Z_{\text{dis}}, R_{\text{dis}})$ has been given in Chandrasekar and Bringi (1987) to be

$$\rho(Z_{\text{dis}}, R_{\text{dis}}) = \frac{\Gamma(m + 10.67)}{[\Gamma(m + 13) \Gamma(m + 8.34)]^{1/2}}.$$  

(C9)

This correlation coefficient is high and typically takes values of the order 0.8.

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