Production of Turbulence in the Vicinity of Critical Levels
for Internal Gravity Waves

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ABSTRACT

Calculations demonstrating that internal gravity waves become unstable in the vicinity of critical levels are reported in this paper. An analytic linear inviscid model is used to look at the geometry of these unstable regions using parameters appropriate to the atmosphere. A nonlinear model is used to establish the usefulness of our simple analytic model. It is argued that turbulence should occur in regions of instability that are tens to hundreds of meters thick in the vertical. It is shown, in a fluid with a sufficiently large kinematic viscosity, that turbulence should no longer occur near critical levels. This condition indicates that turbulence will no longer be produced at critical levels above the altitude of the turbopause, about 100 km. Observational evidence is cited indicating that this production of turbulence near critical levels might be of importance in the planetary boundary layer, in the upper troposphere and stratosphere, in the mesosphere and lower thermosphere, and even in the oceans.

1. Introduction

The level at which an internal gravity wave's horizontal trace velocity becomes equal to the mean flow velocity is called a "critical level." Two features of critical levels are realized to be of particularly great geophysical interest. First, internal gravity waves have essentially zero transmission through critical levels where the wind and density stratification of the mean flow leads to large background Richardson numbers (Bretherton, 1966; Booker and Bretherton, 1967). Second, the horizontal momentum of the gravity wave is strongly coupled to that of the mean flow at a critical level (Lindzen and Holton, 1968; Jones and Houghton, 1971). It is most interesting that the critical level for gravity waves, which at first appears to be a technical detail in the mathematics (a second-order singularity in the linear non-dissipative equations), manifests itself physically as an obvious internal absorbing layer. This has been shown by experiments that have been reported by Bretherton et al. (1967) and Thorpe (1973).

Another aspect of internal wave behavior in the vicinity of a critical level is analyzed in this paper. This is the tendency for large vertical shears in the wave's horizontal velocity as well as large vertical temperature derivatives to develop. These large derivative terms are predicted by the linear inviscid equations, a vastly oversimplified description of the physical problem. It is important to remember, however, that these linear inviscid equations that become obviously invalid very close to the critical level do appear to predict correctly the transmission properties of the gravity wave as well as the coupling of the wave momentum to that of the basic flow. A critical level model by Hazel (1967) which included the effects of dissipation gave the same wave transmission factor as the nondissipative theory of Booker and Bretherton (1967). A numerical nonlinear critical level model by Breeding (1971) showed similar behavior except for some reflection appearing near the critical level.

In the following sections, two mathematical models for internal gravity wave critical levels are analyzed. The first model which is very similar to that of Booker and Bretherton (1967) is a steady-state, linear, inviscid Boussinesq model of a monochromatic internal gravity wave at a critical level. The results of this greatly simplified analysis indicate that for fairly realistic atmospheric gravity wave parameters there exist regions of instability to turbulence with vertical dimensions of tens to hundreds of meters thick. These instability regions are found to lie mostly below the critical level when the wave has its source below the critical level. This simple analytic model is easy to interpret, but to see if the simplifying assumptions inherent in it invalidate the results obtained, a much more realistic numerical model is presented which largely substantiates the more simplified analysis. This numerical model is time-dependent and nonlinear, includes the effects of week dissipation, but is still Boussinesq. The geophysical implications of this analysis are discussed in a later section.

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2. Model

a. Simple analytic model

Fig. 1 shows schematically the model of the problem that we wish to solve. The basic state is given as follows. \( N^2 \), the basic state Brunt-Väisälä frequency squared, is taken to be constant. The basic state horizontal wind \( U(z) \) is taken to be constant between the lower boundary, \( z = -z_B \), and \( z = -L \), as well as between \( z = +L \) and \( z = +\infty \). A linear shear region is taken to exist between \( z = -L \) and \( z = +L \). The wave forcing is a consequence of horizontal flow over the moving sinusoidal rigid boundary which is centered at the location \( z = -z_B \). The lower boundary is taken to be translating in the \( x \) direction with a velocity given by \( c \). The parameters of the problem have been chosen so that a critical level occurs at the level \( z = 0 \) [i.e., \( U(0) = c \)].

Given the assumptions of our idealized problem, i.e., that there exists a steady-state monochromatic forcing that is proportional to \( \exp[i\alpha(x-ct)] \), the equations are linear and inviscid, and the Boussinesq approximation is valid, we get the usual equation for an internal gravity wave propagating in a stably stratified shear flow [Eq. (1.7) of Booker and Bretherton (1967)]:

\[
\frac{N^2}{U_x} \left( \frac{U_x}{U - \alpha^2} \right) w = 0, \tag{1}
\]

where the vertical velocity perturbation is given by \( w(z) \exp[i\alpha(x-ct)] \). In solving Eq. (1), it proves convenient to express \( z \) in terms of a nondimensional parameter \( \xi \) by the relation \( z = \xi L \). Eq. (1) then becomes

\[
w_{\xi\xi} + \left( \frac{N^2}{U_x} \cdot \frac{U_{\xi\xi}}{U - \alpha^2} \right) w = 0. \tag{2}
\]

Referring to Fig. 1, the functional form of \( U(z) \) is as follows:

\[
U(z) = \begin{cases} 
U_1, & \text{for } -b < z < -1 \text{ (region 1)} \\
U_2, & \text{for } -1 < z < +1 \text{ (region 2)} \\
U_3, & \text{for } z > +1 \text{ (region 3)} 
\end{cases} \tag{3}
\]

where \( b = z_B/L \), \( U_1 = -m + c \) and \( U_3 = m + c \). The wind profile given by (3) was chosen to allow for simple analytic solutions in regions 1, 2 and 3 that can be matched with the boundary conditions which are the wave forcing at \( z = -b \) and a Sommerfeld radiation condition at \( z \to \infty \).

In regions 1 and 3, Eq. (2) reduces to

\[
w_{\xi\xi} + \left( \frac{N^2}{m_2} \cdot \frac{U_{\xi\xi}}{U - \alpha^2} \right) w = 0, \tag{4}
\]

which obviously gives the following solutions for \( w(\xi) \) in these regions:

\[
w_1 = A_1 e^{i \xi t} + B_1 e^{-i \xi t} \text{ (region 1)}, \tag{5}
\]

\[
w_3 = A_3 e^{i \xi t} + B_3 e^{-i \xi t} \text{ (region 3)}, \tag{6}
\]

where

\[
k^2 = \frac{N^2}{m_2} - \alpha^2. \tag{7}
\]

In region 2, Eq. (2) becomes

\[
w_{\xi\xi} + \left( \frac{N^2}{m_2^2} \cdot \frac{U_{\xi\xi}}{U - \alpha^2} \right) w = 0. \tag{8}
\]

We solve Eq. (8) by using the method of Frobenius, that is, we insert a solution of form

\[
w_2 = A_2 \xi^r \sum_{n=0}^{\infty} a_n \xi^n \tag{9}
\]

into Eq. (8), which implies the following indicial equation for \( \nu \),

\[
\nu^2 - \nu + \frac{N^2}{m_2} = 0, \tag{10}
\]

and the following recursion relation for the \( a_n \)'s

\[
a_n = \frac{\alpha^2 \nu^2 a_{n-2}}{(\nu + n)(\nu + n - 1)} \left( \frac{U_x}{U - \alpha^2} \right) \tag{11}
\]

where \( a_0 \) is taken to be equal to 1 and all of the \( a_n \)'s with odd values of \( n \) are equal to zero. The solutions to the indicial Eq. (10) are

\[
\nu = \frac{1}{2} \pm i \mu, \tag{12}
\]

where

\[
\mu = (Ri - \xi)^{1/2}. \tag{13}
\]
Here $\tilde{R}_1$ is the Richardson number of the background state in which the Brunt-Väisälä frequency is given by $N$ and the vertical shear of the background wind, $\partial U/\partial z$, is equal to $m/l$ ($\tilde{R}_1$ is assumed to be greater than $\frac{1}{4}$).

Since we now have analytic expressions for $w$ in regions 1, 2 and 3, all that is required to complete the solution is the application of the lower boundary condition at $\xi = -b$, the upper boundary condition at $\xi \to \infty$, and the matching conditions at $\xi = -1$ and $\xi = +1$.

The simplest of these conditions to apply is the upper boundary condition, a radiation condition at $\xi \to \infty$. Looking at expression (6), we see that we can apply this condition by taking $B_2$ to equal zero. This forces the vertical component of the phase velocity to be wholly downward for large $\xi$, and this implies no gravity wave energy propagating downward from infinity.

The lower boundary condition to be imposed is that of the flow induced by horizontal flow moving over the translating sinusoidal rigid boundary that is located at $\xi = -b$ (in the linear sense). The equation for this rigid boundary is

$$h = -b - H \sin \alpha (x - c).$$

Therefore, the linearized vertical motion induced at $\xi = -b$ is

$$w(-b) = -d[U(-b) - c] = -mA H \cos \alpha (x - c) = m d H \cos \alpha (x - c).$$

The amplitude of the forced vertical velocity is then $mdH$. We have our lower boundary condition by requiring the real part of $w(-b)$, as given by expression (5), to be equal to $mdH \cos \alpha (x - c)$. This condition is written mathematically as

$$\begin{aligned} \frac{1}{2} \{ & (A_1^* + B_1) \exp \{ i kl + \alpha (x - c) \} \\ & + (A_1^* + B_1) \exp \{ i kl + \alpha (x - c) \}^* \} \\ & = mdH \cos \alpha (x - c), \end{aligned}$$

(16)

where the asterisk superscript indicates a complex conjugate.

For matching the solutions for $w$ across the interfaces between regions 1 and 2 and regions 2 and 3, we require $w$ to be continuous across $\xi = -1$ and $\xi = +1$, and we require jump conditions on $\partial w/\partial \xi$ at these same interfaces. These jump conditions are a consequence of the discontinuities in $\partial U/\partial \xi$ at $\xi = -1$ and $\xi = +1$, and are obtained by integrating (2) across the region interfaces. The jump condition at $\xi = -1$ is

$$\frac{\partial w_2}{\partial \xi}_{-1} - \frac{\partial w_1}{\partial \xi}_{-1} = w(-1),$$

(17)

while the jump condition at $\xi = +1$ is

$$\frac{\partial w_3}{\partial \xi}_{+1} - \frac{\partial w_2}{\partial \xi}_{+1} = w(+1).$$

(18)

Having formulated the problem, we write the solutions for $w$ as

$$w = \begin{cases} A_1 e^{i \xi + b}, & \text{for } -b < \xi < -1 \\ A_2 \sum_{n \text{ even}} a_n \xi^{n+1} + b_n & \text{for } 1 < \xi < +1 \\ A_1 e^{i \xi + b} & \text{for } \xi > +1 \end{cases}$$

(19)

The solution is then obtained by using the solution (19) containing five unknown complex constants along with five equations: the lower boundary condition at $\xi = -b$ [Eq. (16)]; two equations expressing the fact that $w$ is continuous at the interfaces $\xi = \pm 1$; and the jump conditions [Eqs. (17) and (18)]. We have kept the coefficients $a_n$ and $b_n$ up to $n = 4$ for our calculations.

Having obtained the solution for $w$ in all three regions, the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

(20)

is used to obtain an expression for $u$, the perturbation horizontal velocity. The potential temperature perturbation is obtained by the linearized thermodynamic equation

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial z} = 0,$$

(21)

where $\theta(z)$ is the potential temperature stratification of the basic state.

Thus, we have formulated a very simple linear model of a gravity wave critical level. In a later section, we will use this model to see what region about the critical level might be unstable to turbulence. We will also see how the geometry of this region of instability changes with different choices of the background shear $\partial U/\partial z$, the density stratification as expressed by $\Delta^2$, and the amplitude of the forced wave $W_0 = mdH$. We should realize that the results coming from this simple model are suspect, though, due to our many assumptions. Thus, we look at a more complex numerical model in the next section.

b. Nonlinear numerical model

The simple critical level model outlined in the last section may be viewed as a maximum possible simplification. In this sense, any results obtained with it might establish the plausibility of the critical level instability mechanism, but the physical reality of these results would be quite questionable due to the neglect of nonlinear and dissipative processes. The inclusion of these factors leads to a problem of much greater mathematical complexity making an analytical solution very difficult. Thus, a numerical critical level problem quite similar to the one described in the last section is formulated in
\[ \psi / \theta \text{ are the total values of vorticity, streamfunction and potential temperature, respectively; } D_1 \text{ is the viscous damping term } \left[ = (1/\text{Re}) \nabla^2 \eta \right] (\text{Re is the Reynolds number} ); C_1 \text{ is the heat conduction term } \left[ = 1/\left( \text{RePr} \right) \nabla^2 \theta \right] (\text{Pr is the Prandtl number}); \text{ and } D_2 \text{ is the Rayleigh damping term } \left[ = -K \eta \right] (\text{Rayleigh damping is included for computational purposes}). \nabla^2 \text{ denotes the two-dimensional Laplacian and } (\psi / \theta / \psi / \theta) \text{ is the Jacobian operator. To make the equations nondimensional, we have introduced the characteristic length } \delta, \text{ characteristic horizontal velocity } U_0, \text{ and characteristic temperature difference } \Theta. \text{ This leads to our definition of the layer Richardson number as}

\[ Ri_0 = \frac{g \delta \beta_0}{\Theta_0 U_0^2}, \tag{27} \]

\[ \text{where } g \text{ is the acceleration due to gravity, } \Theta_0 \text{ is the constant reference potential temperature, and } \text{Re} = U_0 \delta_0 / \nu \text{ and } \text{Pr} = \nu / \kappa, \text{ where } \nu \text{ is the kinematic viscosity and } \kappa \text{ the thermal conductivity.}

\text{We separate the equations into horizontally averaged equations and perturbation equations by using the operator}

\[ (\psi / \theta / \psi / \theta) = \frac{1}{L} \int_0^L (\psi / \theta / \psi / \theta) dx, \]

\[ \text{where } L \text{ is the wavelength in the } x \text{ direction. Thus the horizontally averaged equations are}

\[ \frac{\partial \eta}{\partial t} + \frac{\partial (\eta \psi)}{\partial x} = D_1, \tag{28} \]

\[ \frac{\partial \theta}{\partial t} + \frac{\partial (\theta \psi)}{\partial x} = C_1, \tag{29} \]

\[ \frac{\partial^2 \psi}{\partial \zeta^2} = -\bar{\eta}, \tag{30} \]

\[ \text{and the perturbation equations are}

\[ \left( \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial x} \right) \eta - \frac{\partial \eta}{\partial x} \psi = \frac{\partial (\eta \psi)}{\partial x} - \frac{\partial (\eta \psi)}{\partial \zeta} - \frac{\partial (\eta \psi)}{\partial t} = D_1 - D_2 + F(t), \tag{31} \]

\[ \left( \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial x} \right) \theta - \frac{\partial \theta}{\partial x} \psi = \frac{\partial (\theta \psi)}{\partial x} - \frac{\partial (\theta \psi)}{\partial \zeta} + \frac{\partial (\theta \psi)}{\partial t} = C_1, \tag{32} \]

\[ \nabla^2 \psi = -\eta. \tag{33} \]

\text{We have chosen a spectral representation in the horizontal instead of horizontal differencing for these computations. Thus, we expand the dependent variables}
as follows:

\[
\begin{bmatrix}
\eta \\
\psi \\
\theta
\end{bmatrix} = \sum_{n=1}^{N} \begin{bmatrix}
\eta_{e,n} \\
\psi_{e,n} \\
\theta_{e,n}
\end{bmatrix} \cos\alpha x - \sum_{n=1}^{N} \begin{bmatrix}
\eta_{o,n} \\
\psi_{o,n} \\
\theta_{o,n}
\end{bmatrix} \sin\alpha x,
\] (34)

where \( \alpha = 2\pi / L \). All of these Fourier coefficients are functions of both \( \xi \) and \( t \). Expression (34) is then substituted into the perturbation equations (31), (32) and (33) and then the resulting relations are truncated at some value of \( N \) leading to a system of simultaneous equations for the Fourier coefficients. The time integration is accomplished by a combined leap-frog and DuFort-Frankel method.

The upper and lower boundary conditions are specified at the outside boundaries of the sponge layers. All of the Fourier coefficients of vorticity, streamfunction, and potential temperature are taken to vanish at these boundaries. Furthermore, the horizontally averaged values of vorticity and potential temperature are constrained to remain constant with time at the boundaries. Physically, the zero streamfunction means that the vertical velocity is zero at the boundaries; the zero vorticity implies a stress-free condition when the vertical velocity equals zero; and of course the potential temperature cannot change at the boundaries.

Sponge layers are used to avoid spurious reflections. The coefficients of Rayleigh drag are taken to be

\[
K_+ = K_0 \exp(d/s_1) - \exp[(1 + d - \xi)/s_1]/\exp(d/s_1) - 1
\] (35)

in the top-side sponge layer and

\[
K_- = K_0 \exp(d/s_2) - \exp[(d + \xi)/s_2]/\exp(d/s_2) - 1
\] (36)

in the bottom-side sponge layer. We have used \( d \) to denote the thickness of these sponge layers while \( K_0, s_1 \) and \( s_2 \) are constants.

The wave forcing is taken to be monochromatic and is specified by a vertical velocity component on one horizontal grid line only for simplicity. This type of forcing is implicitly related to \( F(t) \) in Eq. (31). Forcing is introduced by the relations

\[
w_f = W_0 \sin \alpha(x - ct) \cdot f(t),
\] (37)

where

\[
f(t) = \begin{cases} 
0, & t \leq 0 \\
[t/l_0, 0 < t < l_0] \\
1, & t \geq l_0
\end{cases}
\] (38)

Thus, the forcing is introduced gradually.

Vertical differencing is carried out as follows. The entire computational domain is separated into \( M_t \) layers. The physical domain, \( 0 \leq z \leq \zeta_0 \), contains \( M_y \) of the total number. Thus, the spacing unit is given by

\[
\Delta z = \zeta_0 / M_y.
\] (39)

3. Results

We have defined a local instantaneous Richardson number as

\[
R_i = \frac{\left(\frac{g}{\theta + \delta} \frac{\partial \delta}{\partial z} \frac{\partial \theta}{\partial z} \right)}{\left(\frac{\partial U}{\partial z} \frac{\partial U}{\partial z}\right)^{1/2}}
\] (40)

in the notation of our analytic model described in Section 2a, and as

\[
R_i = Ri \frac{\partial \delta}{\partial z}/\left(\frac{\partial U}{\partial z}\right)^{1/2}
\] (41)

in the notation of our numerical model described in Section 2b. In order to see the physical significance of this local instantaneous Richardson number \( R_i \), let us recall that there are three important values of the Richardson number. \( R_i < 0 \) implies a situation in which the stratification is unstable, i.e., a situation where buoyancy will accelerate a fluid parcel away from its equilibrium state. In the atmosphere, \( R_i < 0 \) is simply another way of indicating the existence of a super-adiabatic lapse rate. \( R_i < 1/4 \) implies that sufficient energy may be extracted from a shear flow to overcome a stable stratification, perhaps initiating turbulence (Miles, 1961; Howard, 1961). \( R_i < 1 \) appears to be the condition for turbulence, once established in a fluid flow, to persist (Woods, 1969). Our conceptual model is that in a region about a critical level where \( R_i < 1/4 \), any small perturbation will tend to produce turbulence which will then spread, filling the region where \( R_i < 1 \). Thus, we inquire in what region about a critical level \( R_i \) is less than 0, \( 1/4 \), or 1.

Fig. 3 shows some results of the simple linear model that was described in Section 2a. The parameters used in this calculation were as follows:

\[ S_0 = \partial U / \partial z = \text{basic state shear} = 0.024028 \text{ s}^{-1} \]
The Fourier expansion of the dependent variables [expression (34)] was truncated at $N=3$ for this calculation. The vertical variation of the Richardson number of the initial state is shown in Fig. 2. Note that initially the minimum value of $\text{Ri}$ is $\sim 2$ at the location of the critical level ($z=700$ m) and remains quite constant in the vicinity of the critical level. The parameters of this nonlinear model are seen to agree very closely with those

$R_i = \frac{\nu}{\gamma}$

$N^2 = \text{basic state Brunt-Väisälä frequency squared}$

$W_0 = \text{magnitude of vertical velocity forcing}$

$\alpha = \text{horizontal wavenumber}$

$T = \text{wave period}$

$Re = \text{Reynolds number}$

$Pr = \text{Prandtl number}$

$\Delta z = \text{grid size}$

$t_o = \text{rise time for forcing}$

$d = \text{thickness of sponge layers}$

$K_0 = \text{constant multiplicative factor in Rayleigh friction}$

$s_1 = s_2 = d$

The very fact that the action of the wave is dominating the basic state in determining the value of the local instantaneous Richardson number in the vicinity of the critical level should prompt us to examine the importance of the nonlinear effects. Fig. 5 shows some results obtained with the nonlinear numerical model that was described in Section 2b. These results were obtained using the following parameters:

$U_o = \text{velocity difference across the layer}$

$z_0 = \text{layer thickness}$

$\theta_i = \text{potential temperature difference across the layer}$

$R_i = \text{layer Richardson number}$

$W_0 = \text{magnitude of vertical velocity forcing}$

$\alpha = \text{horizontal wavenumber}$

$T = \text{wave period}$

$Re = \text{Reynolds number}$

$Pr = \text{Prandtl number}$

$\Delta z = \text{grid size}$

$t_o = \text{rise time for forcing}$

$d = \text{thickness of sponge layers}$

$K_0 = \text{constant multiplicative factor in Rayleigh friction}$

$s_1 = s_2 = d$
of the linear model with \( W_0 = 3 \text{ cm s}^{-1} \) that is shown in Fig. 4, with the exception of the forcing period.

The values of \( \text{Ri} \) as given by (41) are shown in the top section of Fig. 5 for an elapsed time of \( 0.7 \times 10^4 \text{ s} \) after the initiation of the forcing (about \( 3 \frac{1}{2} \) wave periods). The middle section is for an elapsed time of \( 0.8 \times 10^4 \text{ s} \) (about \( 4 \) wave periods), and the bottom section is for an elapsed time of \( 1.4 \times 10^4 \text{ s} \) (about \( 11 \) wave periods). The darkly shaded areas in Fig. (5) show regions where \( \text{Ri} < \frac{1}{4} \), and the dashed enclosed regions where \( \text{Ri} < 1 \). Note that due to the nature of the line source there exists a region of instability in the vicinity of the source region. Noting that the computational grid has outputs at every \( \Delta z = 10/3 \text{ m} \), we see that the maximum vertical thickness of the shaded region is \( 11 \Delta z \) (about \( 37 \text{ m} \)) at \( 0.7 \times 10^4 \text{ s} \), \( 12 \Delta z \) (about \( 40 \text{ m} \)) at \( 0.8 \times 10^4 \text{ s} \), and \( 16 \Delta z \) (about \( 53 \text{ m} \)) at \( 1.4 \times 10^4 \text{ s} \). In interpreting these numbers we note that due to the nonlinear exchange of momentum between the wave and the mean flow at the critical level, the critical level itself is moving downward as is shown in Fig. 6. Thus, after \( 0.7 \times 10^4 \text{ s} \) the critical level has moved downward by \( 3 \text{ m} \) to \( Cr' \), and by \( 1.4 \times 10^4 \text{ s} \) it has moved downward by \( 22 \text{ m} \) from its original position to \( Cr'' \). We interpret the thickness of the \( \text{Ri} < \frac{1}{4} \) region in Fig. (5) to be due to two effects: the thickness of the region as given by the simple linear calculation (in the case being discussed about \( 33 \text{ m} \)), and the downward movement of the critical level itself (for instance, \( 3 \text{ m} \) at \( t = 0.7 \times 10^4 \text{ s} \) and \( 22 \text{ m} \) at \( t = 1.4 \times 10^4 \text{ s} \)). The sum of these, \( 36 \text{ m} \) and \( 55 \text{ m} \), agrees very closely with the vertical dimensions of the \( \text{Ri} < \frac{1}{4} \) region in the nonlinear calculation which are about \( 37 \text{ m} \) and \( 53 \text{ m} \) at the corresponding times.

**Fig. 6.** Mean velocity profiles near the critical level at various times as given by the numerical nonlinear model. Numbers in parentheses indicate values in MKS units. Other numbers indicate nondimensional values.

**Fig. 7.** Configuration of \( \text{Ri} < \frac{1}{4} \) envelopes as given by the analytical linear model for various values of the basic state Richardson number.

It should be noted that the nonlinear critical level model of Breeding (1971) leads to conclusions about the stability of flow near a critical level that are quite different from ours if one looks rather superficially at both results. Breeding found that the Richardson number never fell below \( \frac{1}{4} \) in his simulations. We attribute this to his using a rather large value of eddy viscosity in his computations, \( 2 \text{ m}^2 \text{ s}^{-1} \). Breeding found it necessary to use this large value of eddy viscosity for computational purposes. We believe then that Breeding's eddy viscosity process simply smoothed out the shears that give rise to instability.

Calculations using the simple analytic model were also performed using different basic state shears and density stratifications leading to different basic state Richardson numbers. Some of these results are shown in Fig. 7 where results are displayed for basic state Richardson numbers of 2.0819, 20.819 and 62.46, with a vertical velocity forcing of \( 10 \text{ cm s}^{-1} \). In what at first appears to be a surprising result, the \( \text{Ri} < \frac{1}{4} \) envelope has a greater vertical thickness for the larger basic state Richardson numbers. For \( \text{Ri} = 20.819 \), it is \( 121 \text{ m} \); and for \( \text{Ri} = 62.46 \), it is \( 173 \text{ m} \). This variation in the thickness of the instability region with varying \( \text{Ri} \) is less surprising when one realizes that the wave shears dominate the basic state shear close to the critical level. One sees from the critical level solutions (see the Appendix) that a more stable basic state leads to more wave shearing in the vicinity of the critical level. Fig. 7 illustrates this fact in that there is a more obvious layering for the larger basic state Richardson numbers with each individual layer decreasing in thickness with increasing \( \text{Ri} \), reflecting the stabilizing effect of the basic state. The principal results of these simple model calculations are shown in Table 1 where the maximum thickness of the unstable region above and below the critical level, \( Th^+ \) and \( Th^- \), respectively, are shown for many different values of \( \text{Ri} \) that are obtained with different values of the basic shear \( \partial U/\partial z \) (denoted by \( S \)) and the Brunt-
Table 1. Dependence of $\text{Th}_+^{\text{a}}$ and $\text{Th}_-^{\text{b}}$ on the basic state shear and stratification. $\text{Ri} = 2.0819$, $S_0 = 0.024028 \text{ s}^{-1}$, $W_s = 10 \text{ cm s}^{-1}$, and $N_0^2 = 1.202 \times 10^{-3} \text{ m}^{-1}$.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$N_0^2$</th>
<th>$\text{Ri}$</th>
<th>$\text{Th}_+^{\text{a}}$ (m)</th>
<th>$\text{Th}_-^{\text{b}}$ (m)</th>
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</tr>
<tr>
<td>$S_0/2$</td>
<td>$N_0^2$</td>
<td>4Ri$_0$</td>
<td>0.25</td>
<td>110</td>
</tr>
<tr>
<td>$S_0$</td>
<td>$10N_0^2$</td>
<td>10Ri$_0$</td>
<td>0.009</td>
<td>121</td>
</tr>
<tr>
<td>$S_0$</td>
<td>$30N_0^2$</td>
<td>30Ri$_0$</td>
<td>$=0$</td>
<td>173</td>
</tr>
<tr>
<td>$S_0$</td>
<td>$50N_0^2$</td>
<td>50Ri$_0$</td>
<td>$=0$</td>
<td>$\approx 200$</td>
</tr>
</tbody>
</table>

Väisälä frequency squared, $N^2$. One sees that $\text{Th}_+^{\text{a}}$ decreases consistently with increasing $\text{Ri}$, while there is a general but weak trend toward increasing $\text{Th}_-^{\text{b}}$ for increasing $\text{Ri}$. This is shown to be quite consistent with the analysis that is done in the Appendix which indicates that $\text{Th}_-^{\text{b}}$ should vary approximately as $\text{Ri}^3$.

Finally, Table 2 summarizes some calculations for differing horizontal wavenumbers. We see that waves with smaller wavenumbers (larger wavelengths) produce larger regions of instability. This also follows from the analysis that is presented in the Appendix which indicates that $\text{Th}_-^{\text{b}} + \text{Th}_+^{\text{a}}$ varies as $\alpha^3$.

4. Discussion

In the previous section we saw that a simple linear analysis of the critical level problem was capable of explaining the region of instability that was present in the vicinity of the critical level. The main nonlinear influence in the problem, at least in determining the dimensions of the unstable region, appeared to be the downward motion of the critical level and even this can be treated in a quasi-linear sense through the Reynolds’ stress as was done by Lindzen and Holton (1968) and Jones and Houghton (1971). Thus, the formation of these unstable regions can be explained by linear theory although a nonlinear treatment is needed if one wants quantitatively correct values of wind shear and temperature derivatives very close to the critical level.

Neither the linear nor the nonlinear model treated in this paper is capable of describing how this instability develops. The linear model does not have this capability since it is steady-state, and the nonlinear model cannot give the correct unstable modes because it does not possess sufficient degrees of freedom, i.e., the horizontal length scales that might be expected to go unstable, reasoning from Kelvin-Helmholtz instability theory (Tanaka, 1975a), are not resolved. More sophisticated models of gravity wave critical level interaction are being planned by us to model the unstable modes that should develop in the vicinity of the critical level.

The generation of thin turbulent layers in the vicinity of critical levels might be geophysically important in a number of contexts. The numerical model that was presented in Section 2b was developed for application to the planetary boundary layer (PBL) by Tanaka (1975b). Thin wavy echo layers have been observed by such workers as Gossard et al. (1971) and Metcalf (1974) at a height of several hundred meters above the ground using an FM-CW radar. Such layers have also been noted by Fukushima et al. (1974) using an acoustic sounder. The existence of many internal gravity waves in the stable PBL has been noted by Beran et al. (1973). Tanaka (1975b), using his numerical model of a gravity wave critical level, has found unstable regions that compare quite favorably with such observed regions in the PBL when parameters appropriate to a stable PBL were used.

The structure of Clear Air Turbulence (CAT) has been reviewed in Pao and Goldberg (1969) as well as in other sources. Although CAT is a complicated phenomenon, at least one manifestation of it is seen in rather thin patches which occur in a rather sporadic fashion in strongly stable air. Hardy et al. (1969) have observed very thin echo layers, which they interpret as turbulent layers, at an altitude near 12 km. Their observed region of turbulence appears to be quite consistent with that implied by the critical level mechanism that is discussed in this paper. It is well-known that CAT occurs more frequently over mountains and continents than over flat terrain and oceans (Reiter, 1969). In Lilly (1971) a situation is seen in which a critical level for mountain lee waves, a level of zero wind, appears to be a region of light turbulence (see Lilly’s Figs. 1 and 2).

Turbulence in the lower thermosphere has been detected by Blamont and de Jager (1961), for example, by observing vapor trails released by rockets. A most striking feature of many of these observations is the existence of a level below which the flow appears to be turbulent and above which the flow appears to be laminar. This level is referred to as the turbopause and occurs at an altitude of ~100 km. It is interesting to note that turbulence production by the critical level mechanism should also cease at about this altitude. This may be seen by referring to some results of Hazel (1967) and looking at Table 3. Hazel (1967) defined a characteristic vertical scale for the viscous critical level prob-

Table 2. Dependence of $\text{Th}_+^{\text{a}}$ and $\text{Th}_-^{\text{b}}$ on the horizontal wavenumber $\alpha$. $W_s = 10 \text{ cm s}^{-1}$, $\text{Ri} = 2.0819$, and $N_0^2 = 10^{-3} \text{ cm}^{-1}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\text{Th}_+^{\text{a}}$ (m)</th>
<th>$\text{Th}_-^{\text{b}}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4\alpha_0$</td>
<td>1.1</td>
<td>23</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>3.4</td>
<td>71</td>
</tr>
<tr>
<td>$\alpha_0/4$</td>
<td>8.8</td>
<td>176</td>
</tr>
</tbody>
</table>
lem by the relation

\[ Z_r = \left( \frac{\nu}{\frac{\partial U}{\partial z}} \right)^{\frac{1}{\alpha}}. \] (42)

Hazel noted that whereas the gravity wave exchanges momentum with the mean flow only at the critical level in the inviscid problem, the interaction takes place over several units of \( Z_r \) in the viscous problem. We have calculated values of \( Z_r \) using values of \( \nu \) taken from the winter 45\(^\circ\)N values of U. S. Standard Atmosphere Supplements, 1966, and representative values for \( \alpha \) (= 2\( \pi / \phi \)) and \( \partial U / \partial z \) from Rosenberg (1968). We see that \( Z_r \), for example, will quickly start to exceed reasonable values for (Th+ + Th−) at an altitude near 100 km implying that the action of viscosity will eliminate the large shears that form the basis for the turbulence generation mechanism. Thus, we have a mechanism for the generation of turbulence that should cease sharply at the turbopause.\(^4\) It should also be mentioned that there is evidence for thin layers of turbulence in the lower thermosphere in the observations reported by Rees et al. (1972) and Zimmerman et al. (1973).

Thin mixed layers have been observed in the ocean in the vicinity of the thermocline by Woods and Wiley (1972). These layers are observed to be only 1–2 m thick. The common presence of internal gravity waves in this part of the ocean has been argued by Phillips (1966). Orlanski and Bryan (1969) have argued that mixing on an order of magnitude larger vertical scale might be induced by the breaking of large-amplitude internal gravity wave modes. The thinner layers are probably due to another mechanism. Woods and Wiley (1972) have suggested Kelvin-Helmholtz instability for this purpose, but the critical level mechanism discussed here could be important.

5. Conclusions

We have seen that internal gravity waves in the atmosphere become unstable in the vicinity of critical levels and that this should lead to regions of turbulence of the order of tens to hundreds of meters thick and perhaps tens of kilometers long. We have seen that turbulent regions having this type of geometry are commonly seen in various regions of the atmosphere as well as in the ocean. Viscosity is capable of stabilizing this process and it is predicted that this should occur at about the level of the turbopause, 100 km.

It should be made clear that we are not suggesting that this critical level mechanism is the only or even the principal turbulence production mechanism in the free atmosphere. Certainly such mechanisms as Kelvin-

\(^4\) It appears that this is also the reason for the absence of turbulence in the critical level experiments that were reported by Thorpe (1973). Our calculations imply that \( Z_r \) exceeds Th+ + Th− in his experiments.

<table>
<thead>
<tr>
<th>( z ) (km)</th>
<th>( \nu ) (m(^2) s(^{-1}))</th>
<th>( Z_r ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.317\times10^{-6}</td>
<td>2.188</td>
</tr>
<tr>
<td>10</td>
<td>3.539\times10^{-6}</td>
<td>3.046</td>
</tr>
<tr>
<td>20</td>
<td>1.642\times10^{-4}</td>
<td>5.074</td>
</tr>
<tr>
<td>30</td>
<td>8.187\times10^{-4}</td>
<td>8.669</td>
</tr>
<tr>
<td>40</td>
<td>4.494\times10^{-4}</td>
<td>15.29</td>
</tr>
<tr>
<td>50</td>
<td>1.972\times10^{-4}</td>
<td>25.03</td>
</tr>
<tr>
<td>60</td>
<td>6.586\times10^{-4}</td>
<td>37.42</td>
</tr>
<tr>
<td>70</td>
<td>2.349\times10^{-4}</td>
<td>57.17</td>
</tr>
<tr>
<td>80</td>
<td>9.411\times10^{-4}</td>
<td>71.58</td>
</tr>
<tr>
<td>90</td>
<td>4.760</td>
<td>155.9</td>
</tr>
<tr>
<td>100</td>
<td>3.004\times10^{1}</td>
<td>288.1</td>
</tr>
<tr>
<td>110</td>
<td>1.700\times10^{2}</td>
<td>513.4</td>
</tr>
</tbody>
</table>

Helmholtz instability and the breaking of internal gravity waves (Hodges, 1967) must be considered. It is only suggested that some of the thin turbulent layers in the atmosphere and in the ocean, perhaps, might be due to this critical level mechanism.

Many other authors have suggested that turbulence might occur near critical levels. Among these are Booker and Bretherton (1967), Hines (1968), Maslowe (1972), and Bekofski and Liu (1972). While Booker and Bretherton (1967) and Hines (1968) have suggested the identical mechanism discussed in this paper, the ideas of Maslowe (1972) and Bekofski and Liu (1972) are quite different. Maslowe (1972) suggests that there exist unstable thin diffusive layers bounding the critical layer region and Bekofski and Liu (1972) suggest that the interaction between the gravity wave and the mean flow produces a ledge in the mean flow which is unstable. Much of the analysis by Bekofski and Liu (1972) has been severely criticized by Holton and Lindzen (1973).

It is very important to see how much this idea of gravity wave instability in the vicinity of critical levels changes our basic ideas on critical level interaction. For instance, if the wave becomes unstable some of its energy and momentum must go into the unstable modes and, perhaps, to turbulence leading to momentum exchange with the mean flow well below the critical level. Some of these instability modes might be gravity waves with different trace velocities and hence might pass through the original critical level. We are looking into these problems.

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ments of Prof. J. R. Holton in reviewing this paper were also very helpful. We especially want to thank Ms. C. Martin for typing this manuscript.

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APPENDIX

Analysis of Critical Region Thickness Dependence on \( \text{Ri} \), \( W_0 \), and \( \alpha \)

In order to estimate the dependence of \( \text{Th} + \text{Th} - \) on \( \text{Ri} \), \( W_0 \) and \( \alpha \), we start with Eq. (1):

\[
\frac{N^2}{(U-c)} - \frac{U_{zz}}{U-c} - \frac{\alpha^2}{2} \frac{w_{zz}}{w} = 0. \tag{A1}
\]

Expanding \( U(z) \) in a Taylor series about \( z = 0 \) and keeping only the term in the brakets that is largest near the critical level gives

\[
w_{zz} + \frac{\alpha}{z^2} w_{zz} = 0, \tag{A2}
\]

where \( \text{Ri} \) is the Richardson number of the basic state at \( z = 0 \). Using a nondimensional height variable defined by \( z = \xi \) gives

\[
w_{zz} + \frac{\alpha}{\xi} w_{zz} = 0. \tag{A3}
\]

The leading term of the Frobenius solution to (A3) is

\[
w = W_0 \xi^{1+i\mu}, \text{ where } \mu = (\text{Ri} - \frac{1}{2})^4 \tag{A4}
\]

(see Booker and Bretherton, 1967). For simplicity, we will keep only

\[
w = W_0 \xi^{1+i\mu} \tag{A5}
\]

as the solution.

Using the continuity equation \([\text{Eq. (20)}]\) along with Eq. (A5) gives

\[
u = -\frac{i}{i\alpha} \frac{1}{W_0} \frac{\alpha}{\xi} \xi^{1+i\mu}, \tag{A6}
\]

which is differentiated to give

\[
\frac{\partial u}{\partial z} = -\frac{i}{\alpha^2} \frac{\alpha}{\xi} \xi^{1+i\mu}. \tag{A7}
\]

Using Eq. (21) along with Eq. (A5) gives

\[
\theta = -\frac{i}{\alpha(U-c)} \frac{\alpha}{\xi} \xi^{1+i\mu} \tag{A8}
\]

which can be written near the critical level as

\[
\theta = \frac{i}{\alpha(U-c)\xi} \xi^{1+i\mu} \tag{A9}
\]

Differentiating Eq. (A9) gives

\[
\frac{\partial \theta}{\partial z} = -\frac{i}{\alpha}(\frac{\partial \theta}{\partial z} + \frac{\alpha}{\xi} \xi^{1+i\mu}) \tag{A10}
\]

We wish to put \( \partial u/\partial z \) and \( \partial \theta/\partial z \) into Eq. (40). To do this, we should take the real parts of Eqs. (A7) and (A10). This gives

\[
\text{We first consider the thickness of the Ri} \leq 0 \text{ envelope. For Ri} < 0, \text{ we must have}
\]

\[
\frac{\xi^{1+i\mu} W_0}{\alpha(P\partial U/\partial z)} \{\mu \cos[\alpha(x-c\xi) + \mu \xi] - \frac{1}{2} \sin[\alpha(x-c\xi) + \mu \xi] \}
\]

\[
\left[ 1 + \frac{W_0}{\alpha(P\partial U/\partial z)} \right]^{-1} \left[ \frac{W_0}{\alpha(P\partial U/\partial z)} \right]^{-1} \sin[\alpha(x-c\xi) - \mu \xi]^2 \right] \tag{A11}
\]

where \( \xi = \ln \xi \). In writing Eq. (A11), we have ignored a term \( \theta/\theta \) compared to 1. This appears to be a reasonable approximation at the boundaries of the Ri < 0 and Ri < \( \frac{1}{2} \) instability regions.

We may identify the regions where \( \partial \theta/\partial z \) dominates \( \partial \theta/\partial z \) and where \( (\partial u/\partial z)^2 \) dominates \( (\partial U/\partial z)^2 \) separately. Physically, when \( \partial \theta/\partial z \) dominates \( \partial \theta/\partial z \), Ri becomes negative and convective instability should result. When \( (\partial u/\partial z)^2 \) dominates \( (\partial U/\partial z)^2 \), \( |\text{Ri}| \) becomes very small and shearing instability should result.

\[
\text{For moderately large } \text{Ri}, \text{ the term with coefficient } \mu \text{ will dominate the term with coefficient } \frac{1}{2}. \text{ Also, } \mu \approx \text{Ri}^4.
\]
Therefore, the maximum vertical extent of the $\text{Ri} < 0$ envelope denoted by $\xi_{\text{mug}}$ will be determined by the relation

$$\frac{W_0 \text{Ri}^{1/2}}{a^{2/3} (\partial U/\partial z)^{2/3}} \sim \frac{1}{\bar{\text{Ri}}^{1/2} W_0^{2/3}}$$

implying that the thickness of the $\text{Ri} < 0$ envelope should vary as

$$Z_{\text{mug}} \sim W_0^{1/6} \bar{\text{Ri}}^{1/3} \alpha^{-2/3}.$$  \hspace{1cm} (A14)

The region of shear instability, where $\text{Ri} > \frac{1}{4}$, should occur when the term of highest order in the denominator of (A11) becomes sufficiently large, i.e., when

$$\frac{1}{\bar{\text{Ri}}^{1/6} W_0^{2/3}} \sim \frac{1}{a^{2/3} (\partial U/\partial z)^{2/3}} < -1.$$ \hspace{1cm} (A15)

The maximum extent of the region of shear instability, denoted by $Z_{1/4}$, should vary as

$$Z_{1/4} \sim W_0^{1/6} \bar{\text{Ri}}^{1/3} \alpha^{-2/3}.$$ \hspace{1cm} (A16)

Thus, both the thickness of the $\text{Ri} > \frac{1}{4}$ envelope and that of the $\text{Ri} < 0$ envelope show the same dependence on $W_0$, $\text{Ri}$ and $\alpha$.

REFERENCES


