A Direct Solution of the Spherical Harmonics Approximation to the Radiative Transfer Equation for an Arbitrary Solar Elevation. Part II: Results

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(Manuscript received 2 January 1975, in revised form 28 February 1975)

ABSTRACT

It is shown that the method based on a direct numerical solution of the spherical harmonics approximation to the radiative transfer equation can be used for obtaining smooth scattered-intensity vs angle-of-observation curves for any arbitrary position of the sun. Results presented for models of moderate (~10) optical thickness and with Rayleigh, haze, as well as non-precipitating cloud phase functions show that intensity values of a given accuracy can be obtained with moderate computing resources.

1. Introduction

In Part I (Dave, 1975), we gave the spherical harmonics approximation to the equation of radiative transfer corresponding to any nth azimuth-dependent component \([I^n(\tau; \mu), n \geq 0]\) of the intensity of radiation \([I(\tau; \mu, \phi)]\) scattered by a plane-parallel atmosphere. [The parameter \(\tau\) represents the optical depth of a level within the atmosphere, \(\mu\) is the cosine of the zenith angle \(\theta\) measured with respect to the negative \(\tau\) axis following Chandrasekhar's (1950) convention, and \(\phi\) is the azimuth angle referred to an arbitrarily chosen, meridian plane.] Expressions were then derived for representing the boundary conditions of no incidence of the scattered radiation on the atmosphere from outside, in a discrete form consistent with the Legendre series representation of \(I^n(\tau; \mu)\). This general spherical harmonics approximation was then reduced by finite differences to a block algebraic system of equations which can be solved by making use of the procedure developed by Canosa and Penafiel (1972).

The numerical solution of the block algebraic system of equations corresponding to an \(L_n\)th spherical-harmonics approximation of the \(n\)th azimuth-dependent component provides values of the moments \(\int^\tau f^n(\tau) d\mu_n\) given by

\[
f^n(\tau) = \int_{-1}^{+1} I^n(\tau; \mu) Y^n_l(\mu) d\mu,
\]

with \(l = n, n+1, n+2, \ldots, L_n\). The functions \(Y^n_l(\mu)\) are the renormalized associated Legendre polynomials (Dave and Armstrong, 1970). The upper limit \(L_n\) is equal to \(L_0 + \delta_n\), where \(L_0\) is the order of the approximation used in the numerical solution of the azimuth-independent term, and the quantity \(\delta_n\) is equal to zero or unity depending upon whether the subscript \(n\) is even or odd, respectively. Values of these moments can then be used to compute values of the \(n\)th component of the source function given by

\[
J^n(\tau; \mu) = \frac{(2-\delta_n)}{4\pi} \sum_{l=-n}^{N} [\Lambda_l(\tau) Y^n_l(\mu) Y^n_l(-\mu)] \frac{L_n}{\sum_{l=-n}^{L_n} \Lambda_l(\tau) f^n_l(\tau) Y^n_l(\mu)}, (2)
\]

when it is assumed that the atmosphere is illuminated by a beam of monochromatic radiation of strength \(\pi F\) per unit area normal to its direction of propagation represented by \((-\mu_0, \phi_0)\). (The quantity \(F\) will be taken to be unity during computations.) The quantity \(\delta_n\) is the Kronecker delta function given by \(\delta_{n, n} = 1\) for \(n = 0\), and otherwise zero, and \(\Lambda_l(\tau)\)'s are the height-dependent Legendre coefficients representing scattering and/or absorbing properties at a level \(\tau\) (e.g., Dave, 1974). The upper limit \(N\) of the first series on the right-hand side of Eq. (2) depends upon the composition of the atmosphere, the wavelength \(\lambda\) of the monochromatic radiation under investigation, and the accuracy with which the normalized scattering phase function has to be reproduced. Values of \(J^n(\tau; \mu)\) are then used to obtain smooth \(I^n(\tau; \mu)\) vs \(\tau\) curves by making use of the following equation with appropriate limits of integration:

\[
I^n(\tau; \pm \mu) = \int_{\mu}^{\tau} J^n(t; \pm \mu) e^{-\tau \tau' \mu} dt. (3)
\]

For values of \(I(\tau; \mu, \phi)\), we then make use of the equation

\[
I(\tau; \mu, \phi) = \sum_{n=0}^{N} I^n(\tau; \mu) \cos(n(\phi - \phi_0)). (4)
\]
It is understood that the second term on the right-hand side of Eq. (2) vanishes for the cases for which \( n > L_n \).

In Section 8 of Part I, we discussed the rationale behind adopting different limits (viz., \( N \) and \( L_n \)) for the two series appearing in Eq. (2). In brief, it permits inclusion of scattering of the direct solar radiation for all \( N \) terms of the Fourier series for intensity. This primary-scattering emission represented by the first term on the right-hand side of (2) can be computed without any difficulty, and with very little effort. Hence, the precise value of the parameter \( N \) is unimportant as long as it is equal to or greater than the number of terms in the Legendre series required for an adequate representation of the scattering phase function used in the atmospheric model [see Eq. (6)]. On the other hand, the second series in (2) is terminated at \( L_n \) which is the order of the spherical harmonics approximation used in the numerical solution of the transfer equation. Evaluation of this second term representing scattering of the atmospheric radiation (viz., contribution due to multiple scattering) requires a good deal of computational effort especially when \( N \), and hence \( L_n \), is very large. The representation of the multiple scattering contribution with the relatively lower order of the spherical harmonics approximation is justified on two counts. First, contributions to the emergent intensity due to primary and multiple scatterings are comparable for models with moderate optical thickness. Second, the distribution of the multiple scattered radiation can be expected to be less anisotropic than that of the primary scattered radiation due to the diffuse nature of its origin. In what follows, we shall therefore represent intensity of the scattered radiation computed with the \( L_n \)th order of the spherical harmonics approximation for the second term, but with all \( N \) terms for the first term in Eq. (2), by \( I_{L_n}(\tau; \mu, \phi) \). Because of the great interest and need of performing calculations for different atmospheric models in a reasonable amount of computer time, another quantity requiring discussion is the ratio \( \rho(\tau; \mu, \phi; L_n/L_{Nm}) \) given by

\[
\rho(\tau; \mu, \phi; L_n/L_{Nm}) = \frac{I_{L_n}(\tau; \mu, \phi)}{I_{L_{Nm}}(\tau; \mu, \phi)}.
\]

Intensities \( I_{L_{Nm}}(0; +\mu, \phi) \) and \( I_{L_{Nm}}(\tau_b; -\mu, \phi) \) of the scattered radiation emerging at the top and bottom, respectively, of a plane-parallel non-absorbing atmosphere were evaluated for models with widely different scattering phase functions by making use of the procedure summarized above. For a given model, these calculations were performed for several values of the parameters \( \tau_b, \theta_b \) and \( L_{nm} \), and intensity values were obtained for a range of parameters \( \mu \) and \( \phi \) determined after studying the scattering phase function. Representative results of analysis of these extensive data are discussed in the following sections. It should be added that values of intensities computed with \( L_{nm} = N \) are known to be accurate to about three significant figures at least (Dave and Armstrong, 1974).

A medium-sized computing facility (IBM System/370 Model 145 VM CMS) accessible to the author was used for performing this task. Computation time for a given model depends upon several parameters directly related to the atmospheric model (e.g., values of the quantities \( \tau, N, L_n \), and incremental optical depth \( \Delta \), as well as upon several coding decisions taken after studying the availability of the peripheral equipment. Because of these, we feel that sketchy information about computer running times will serve no useful purpose.

2. The function \( V^x_l(\mu) \)

An understanding of the behavior of the renormalized associated Legendre polynomials \( V^x_l(\mu) \) is crucial to the development of an accurate and efficient computational procedure. The function \( V^x_l(1.0) \) vanishes for all values of the superscript \( n > 0 \). Hence, the quantities \( j^x_l(\tau) \) [see Eqs. (16), (49), (50), (53), and (56) of Dave (1975)], \( J^x(\tau; \mu) \) and \( J^x(\tau; \mu) \) do not exist for \( n > 0 \) when the sun is overhead \( (\mu_0 = 1.0, \theta_0 = 0) \); this is due to the azimuthal symmetry of the field of the scattered radiation. For any value of \( \mu_0 \), \( J^x(\tau; \mu = \pm 1) \) also vanishes for all values of \( n > 0 \).

In Fig. 1, we have shown variations of the absolute values of \( V^x_l(\mu) \) as a function of the subscript \( l \) for four different values of the parameter \( \theta = \cos^{-1}\mu \), viz., 10°, 20°, 40° and 90°. For \( \theta = 10° \), the absolute values of \( V^x_l(\mu) \) are less than \( 10^{-6} \) for all values of the subscript \( l < 200 \). For \( n > 50 \) and \( \theta < 10° \) for which no results are presented here, absolute values of \( V^x_l(\mu) \) are still smaller. Thus, even for an atmospheric model requiring about
200 terms of the series for an adequate representation of its scattering phase function, contributions to \( I(\tau; \mu, \phi) \) from the 50th and higher-order azimuth-dependent components will be very small for all values of \( \mu \) and \( \phi \) provided \( \theta_0 \leq 10^\circ \) and higher orders of scattering [second term on the right-hand side of Eq. (2)] do not contribute more than the primary scattering. In fact, the upper limit \( N \) of the series given in Eq. (4) is very strongly dependent on values of the parameters \( \mu \) and \( \mu_0 \). All \( N \) terms are required only if \( \mu = \mu_0 = 0 \), a case for which the assumption of a plane-parallel atmosphere breaks down completely.

It should be added that the idea of the dependence of the upper limit of the Fourier series on the directions of incidence and observation has been put forward and used with advantage in the past for obtaining numerical solutions of the transfer equation using the iterative procedure (Dave and Gazdag, 1970), and the doubling technique (Hansen and Pollack, 1970).

3. Rayleigh phase function

For a non-absorbing molecular atmosphere obeying Rayleigh's law of scattering, the Legendre coefficients \([\Lambda_l(\tau)]\) are independent of the parameter \( \tau \) and vanish for all values of the subscript \( l \geq 2 \). The reliability of the numerical results obtained with the present method can be checked to some extent by comparing its results with those obtained with Chandrasekhar's (1950) \( X \)- and \( Y \)-function method. For a strict comparison, values of the intensity of radiation emerging from a non-absorbing Rayleigh atmosphere were computed by using Chandrasekhar’s method, and a scalar version of a vector program described elsewhere (Dave and Warten, 1968). In what follows, we will denote intensity values obtained with Chandrasekhar's method with the suffix CH.

Since the values of \( \mu \) and \( \mu_0 \) for which intensities of the emergent radiation can be calculated without any interpolation depends upon the integration scheme used in computations of the \( X \)- and \( Y \)-functions, and since a Simpson’s rule of integration was used by Dave and Warten (1968), the spherical harmonics program was modified to provide output at equal intervals in \( \mu \).

Values of \( I_{CH}(\tau_0; -\mu, \phi) \) were obtained for the following values of the parameters \( \tau_0, \mu_0, \mu \) and \( \phi = \phi_0 - \phi \): \( \tau_0 = 0.1 \) and 1.0, \( \mu_0 = 0.1, 0.94, 0.80, 0.50 \) and 0.07; \( \mu = 0.02 \) (0.02) 1.00; and \( \phi_0 - \phi = 0^\circ, 90^\circ \) and 180°. The results of this extensive comparison will be given in a very concise form here, as a detailed comparison with tables and graphs for the \( \mu_0 = 1.00 \) case has been given already by Dave and Armstrong (1974).

Fifteen hundred values (combinations of 5 values of \( \mu_0 \), 50 values of \( \mu \), 3 values of \( \phi_0 - \phi \), and at the top as well as at the bottom) of the ratio \( I_{CH}(\tau; \mu, \phi)/I_{CH}(\tau_0; -\mu, \phi) \) were studied. For \( \tau_0 = 0.1 \), values of these ratios were found to lie in the ranges 0.969–0.991, 0.987–1.000 and 0.996–1.003, when values of \( I_{CH}(\tau_0; -\mu_0, \phi_0) \) were obtained with the 9th, 19th and 39th order of the spherical harmonics approximation, respectively. For \( \tau_0 = 1.00 \), the corresponding values of these ranges were 0.973–1.004, 0.989–1.000 and 0.996–1.001, respectively. A value of 0.01 was used for the parameter \( \Delta \tau \) in the spherical harmonics work for this case.

4. Haze phase function

For the case of a moderate anisotropy of the scattering phase function, we have selected a homogeneous, non-absorbing model illuminated by monochromatic radiation of 0.415 \( \mu \)m wavelength, and composed of a spherical polydispersion with its size distribution given by a modified gamma function commonly referred to in the literature as Haze L (Deirmendjian, 1969). In Fig. 2, we have plotted the variations of the normalized Legendre coefficients \( \Lambda_l \) representing the normalized scattering phase function \( p(\cos \Theta) \) [Chandrasekhar (1950)] of such a haze, as a function of the subscript \( l \). It is assumed that the refractive index of the aerosol material is 1.5–0.0i. The function \( p(\cos \Theta) \) can be evaluated for a given value of the scattering angle \( \Theta \) by using the equation

\[
p(\cos \Theta) = \sum_{l=0}^{l=L} \Lambda_l P_l(\cos \Theta),
\]

where \( P_l \)'s are the well-known Legendre polynomials. We found that this series can be terminated at \( L = 69 \) and 89 for obtaining three and four significant figure accuracy, respectively, for values of \( p(\cos \Theta) \) at 181 values of \( \Theta \) given by \( \Theta = 0^\circ \) (1°) 180° (solid curve in Fig. 3). A curve of \( p(\cos \Theta) \) vs \( \Theta \) obtained with only the first 30 terms of the Legendre series \( (L = 29) \) (dotted curve in Fig. 3) exhibits oscillations of about ±15% in the angular range 80°–160°.

For this haze case, the spherical harmonics method was used to obtain values of \( I_{CH}(0; +\mu, \phi) \) and
same type of reliability as the one shown by the cases for which results are presented in this paper.

Several studies on characteristics of the radiation diffusely reflected and transmitted by model atmospheres have been reported by various investigators (e.g., Plass and Kattawar, 1968a, b; Hansen, 1969a, b; Dave and Gazdarg, 1970). Because of this and also because of the fact that the primary purpose of this paper is to show the quality of results obtained with the spherical harmonics method which is considered by some authors to be inappropriate for intensity calculations, we will hold our discussion of $I(\tau; \mu, \phi)$ vs $\theta$ curves to a bare minimum.

The variations of the intensity of the radiation emerging at the top (left half of the diagram) and at the bottom (right half of the diagram) of a Haze L atmosphere are shown as a function of nadir angle or zenith angle $\theta$ in Fig. 4 for three different values of the scattering optical thickness ($\tau_b=0.1, 1.0$ and 10.0). These results which correspond to the sun overhead ($\theta_0=0^\circ$) were obtained with the 19th order approximation for multiple scattering, but with the first 90 terms of the Legendre series for primary scattering. As can be seen, the curves are smooth and the basic features of the scattering phase function (viz., forward peak around $\theta=0^\circ$ in the downward direction, and the glory feature around $\theta=0^\circ$ in the upward direction) are very well reproduced for an optically thin model with $\tau_b=0.1$. These features become less and less distinguishable with increase in the scattering optical thickness of the model.

In order to evaluate the effect of the order of the spherical harmonics approximation on the computed values of intensity, values of the ratio $\rho(0; \tau_b, \mu, \phi; L_\text{n}/L_\text{m})$ and $\rho(\tau_b; -\mu, \phi; L_\text{n}/L_\text{m})$ [see Eq. (5)] are plotted as a function of $\theta_0$ in Fig. 5 for all cases for which the results are presented in Fig. 4. For the radiation

![Diagram](image)

Fig. 3. Normalized scattering phase function of a unit volume of a spherical polydispersion designated Haze L; $m=1.5-0.01$, $\lambda=0.415 \mu m$.

![Diagram](image)

Fig. 4. Intensity of the scattered radiation emerging at the top and bottom of a plane-parallel, homogeneous atmosphere with the spherical polydispersion Haze L for $\theta_0=0^\circ$. These results were obtained with the 19th order of the spherical harmonics approximation for multiple scattering calculations, but with the first 90 terms of the Legendre series for the primary scattering.

![Diagram](image)

Fig. 5. Effect of the order of the spherical harmonics approximation on the computed values of intensity of the scattered radiation emerging at the top and bottom of a plane-parallel, homogeneous atmosphere with the spherical polydispersion Haze L: $\theta_0=0^\circ; N=89$. 

$I_{L_m}(\tau_b; -\mu, \phi)$ for the following values of the various parameters: $N=89; L_m=19, 39, 59$ and 79 for $\tau_b=0.1$ and 1.0; $L_m=19$ and 39 for $\tau_b=10.0; \theta_0=0^\circ (20^\circ) 80^\circ; \theta=0^\circ (1^\circ) 90^\circ; \phi_0-\phi=0^\circ (30^\circ) 180^\circ$; and at $1^\circ$ intervals in $\phi_0-\phi$ in the directions of solar and anti-solar almucantars defined by $\theta_1$.
Emerging at the top of a model with \( \tau_b = 0.1 \) (left side of the bottom section in Fig. 5), the ratio for the 19/79 case (solid curve) deteriorates gradually from 0.998 at \( \theta = 0^\circ \) to 0.994 at \( \theta = 40^\circ \), and rapidly down to a value of 0.981 as the direction of observation approaches close to the horizon. Alternately, we can say that in this case the values of diffusely reflected intensities computed with the 19th order approximation agree within 2% with those obtained with the 79th order approximation. For the radiation emerging at the bottom, values of \( I_{19}(\tau_b; -\mu, 0^\circ) \) and \( I_{79}(\tau_b; -\mu, 0^\circ) \) for \( \tau_b = 0.1 \) agree up to four significant figures in the region of forward peak where the intensity is very high (the ratio varies in the range 0.9998–1.0000 in the angular range 0°–20°), but this excellent agreement deteriorates very rapidly with increase in the slantness due to the relatively smaller contribution from primary scattering. Considering all directions of the emergent radiation (top as well as bottom), we can state that values of intensity for the model with spherical polydispersion Haze L, with optical thickness 0.1, and illuminated by the sun overhead, can be computed with about 4% accuracy by making use of the 19th order of the spherical harmonics approximation. The use of the 39th order of approximation (dotted curves) provide results which agree within 0.4% with those obtained with the 79th order approximation.

Values of \( I_{19}(0; +\mu, \phi) \) and \( I_{39}(\tau_b; -\mu, \phi) \) obtained with the 19th and 39th order of approximation compare more favorably with those obtained with the 79th approximation for the model with \( \tau_b = 1.0 \) (middle section of Fig. 5) than the case with \( \tau_b = 0.1 \). The ratio \( \rho(\tau_b; \mu, \phi) \) now lies in the range 0.985–1.015 for the 19/79 case, and in the 0.997–1.000 range for the 39/79 case. This is probably because the distribution of the multiply scattered radiation becomes less anisotropic with increase in optical thickness. Since values of the intensity of the emergent radiation obtained with the 39th order of the spherical harmonics approximation were judged to be of good quality for most purposes, computations for \( \tau_b = 10.0 \) case were performed with the 19th and 39th orders of approximation, only. The values of the ratio \( \rho(\tau_b; \mu, \phi) \) for an atmospheric model with Haze L (\( \tau_b = 10.0 \) and \( \theta_b = 0^\circ \)) were found to oscillate in the range 0.998–1.006 for the 19/39 case (top section of Fig. 5).

The variations of \( I_{19}(0; +\mu, \phi) \) and \( I_{39}(\tau_b; -\mu, \phi) \) vs \( \theta \) are shown in Figs. 6 and 7, respectively, for Haze L models with \( \tau_b = 0.1, 1.0 \) and 10.0, and illuminated by the sun at 40° from the local zenith. These results are for the meridian planes containing the solar or anti-solar point, and were obtained with the 19th order of the spherical harmonics approximation for multiple scattering calculations. Again, the curves are very smooth, and the basic features of the scattering phase function are clearly visible around the anti-solar and solar points for models with moderate optical thickness. Values of the ratio \( \rho(0; +\mu, \phi; L_m/L_n) \) and \( \rho(\tau_b; -\mu, \phi; L_m/L_n) \) for this \( \theta_b = 40^\circ \) case as a function of angle \( \theta \) are plotted in Figs. 8 and 9, respectively, for the 19/79 and 39/79 cases for models with \( \tau_b = 0.1 \) and 1.0, and for the 39/39 case for the model with \( \tau_b = 10.0 \). The quality of the results obtained with the low order of the spherical harmonics approximation deteriorates with increase of \( \theta_b \). As for example, the range in which values of \( \rho(0; +\mu, \phi; 19/79) \) for models with \( \tau_b = 0.1 \) oscillate, widens from 0.981–0.999 at \( \theta_b = 0^\circ \), to 0.970–1.011 at \( \theta_b = 40^\circ \), and to 0.956–1.114 at \( \theta_b = 80^\circ \) (a case for which no graphical results are presented here). For the 39/79 case, the corresponding values of the ranges are 0.996–1.000, 0.993–1.011 and 0.994–1.035. For the same atmospheric model, the ratio \( \rho(\tau_b; -\mu, \phi; 19/79) \) oscillates in the range 0.958–1.000 at \( \theta_b = 0^\circ \), 0.974–1.050 at \( \theta_b = 40^\circ \), and 0.983–1.232 at \( \theta_b = 80^\circ \), and the ratio

![Figure 6](image-url)  
**Fig. 6.** As in Fig. 4 except for \( \theta_b = 40^\circ \), and for the radiation emerging at the top only.

![Figure 7](image-url)  
**Fig. 7.** As in Fig. 4 except for \( \theta_b = 40^\circ \), and for the radiation emerging at the bottom only.
accuracy for a model with a given phase function, is strongly dependent upon the directions of incidence and observation for which results are required. For an atmospheric model requiring about 69 terms for an adequate (~0.1%) reproduction of its scattering phase function, results obtained with the 19th, 39th and 59th order of the spherical harmonics approximation for multiple-scattering calculations, but with the first 90 terms of the Legendre series for the primary scattering, are accurate to about 4%, 0.5% and 0.1%, respectively, if θ₀ = 0°. For θ₀ = 80°, the corresponding accuracy figures are about 25%, 6% and 1%, respectively. The respectability of these figures increases with decrease in zenith (or nadir) angle of the direction of observation.

From the point of view of efficiency, another quantity of interest is the absolute value of the ratio \( I^H(\tau; \mu) / I^P(\tau; \mu) \), where the subscripts \( H \) and \( P \) stand for the contribution due to the higher order and primary scattering, respectively. (If for a given \( n \), this ratio is of the order of \( 10^{-3} \) or less, it is not necessary to include contributions due to the higher orders for that particular value of \( n \).) Accordingly, \( I^n(\tau; \mu) = I^H(\tau; \mu) + I^P(\tau; \mu) \). The mean of 900 absolute values (90 values of \( \tau \), 5 values of \( \theta_0 \), and for the radiation emerging at the top and bottom) for models with \( \tau_b = 0.1 \) and 1.0, and of 450 absolute values (top only) for the model with \( \tau_b = 10.0 \) of these ratios are plotted as a function of \( n \), the order of the azimuth-dependent component, in Fig. 10. Results for the optically thick model do not include values for the diffuse transmission as they were found to vary
over a wide range \((10^{-4} - 10^{29})\) due to the smallness of \(I_n^p(\tau_s; -\mu)\). Furthermore, they are of very little practical interest as \(I(\tau_s; -\mu, \phi)\) is practically independent of \(\phi\) in such cases (e.g., symmetry around \(\theta = 0^\circ\) line shown by the broken curve in Fig. 7). Results presented in Fig. 10 are based on numerical values obtained with the 79th order (39th order for the model with \(\tau_s = 10.0\)) of the spherical harmonics approximation for computations of \(I_n^p(\tau; \mu)\), but with the first 90 terms of the Legendre series for \(I_n^p(\tau; \mu)\).

Since the curves of \(I_n^p(\tau; \mu)\) vs \(\theta\) show strong oscillations with increase of \(n\) but not those of \(I_n^p(\tau; \mu)\) vs \(\theta\), the mean of the absolute value of \(I_n^p(\tau; \mu)/I_n^p(\tau; \mu)\) depends to a great extent on the number and actual values of \(\theta\) and \(\theta_0\) used in computations of the mean. As for example, this ratio does not exist for \(n > 0\) when the sun is at local zenith. For a model with optical thickness of unity, its magnitude is less than \(10^{-5}\) for \(n > 35\) if \(\theta_0 < 20^\circ\), and for \(n > 55\) if \(\theta_0 < 40^\circ\). Thus, the results presented in Fig. 10 are for the purpose of a general discussion only. With this in mind, we can see that the contribution due to higher orders of scattering is very insignificant for the 30th and higher azimuth-dependent components of intensity when the scattering phase function of the model can be reproduced with \(\pm 0.1\%\) accuracy over the entire angular range of interest with the first 70 terms of the Legendre series.

5. Cloud phase function

For the case of a strong anisotropy of the scattering phase function, we have selected a homogeneous, non-absorbing model illuminated by monochromatic radiation of 0.825 \(\mu\)m wavelength, and composed of a spherical polydispersion designated Cloud C1. The modified gamma function representing the size distribution of this polydispersion can be found in Deirmendjian (1969). This size distribution function has a mode radius of 4 \(\mu\)m, and was used with its lower and upper cutoffs at 2 and 11 \(\mu\)m, respectively. A value of 1.5 — 0.01 was assumed for the refractive index of the material of this polydispersion. Scattering phase functions with still stronger anisotropy do exist in the real atmosphere, e.g., scattering of the visible and ultraviolet radiation by the same polydispersion. However, this particular case was selected as a typical representation of the strong anisotropy because of the desirability of performing the task in a reasonable amount of computer time on a medium-sized computer. Otherwise, we do not foresee any difficulty in obtaining results for models with phase function having anisotropy stronger than the one selected by us.

In Fig. 2 (crosses), we have shown the variations of the normalized Legendre coefficient \(A_1\) representing the normalized scattering phase function of a unit volume of the spherical polydispersion Cloud C1 when illuminated by a wavelength of 0.825 \(\mu\)m wavelength. Three-significant-figure accurate values of \(\rho(\cos\Theta)\) using Eq. (6) could be obtained at all 261 values of \(\Theta\) given by \(\Theta = 0^\circ\) (0.2) 10.0\(^\circ\), 10\(^\circ\) (1\(^\circ\)) 170\(^\circ\), and 170.0\(^\circ\) (0.2\(^\circ\)) 180\(^\circ\) after using a value of 169 for the upper limit \(L\). A plot of \(\rho(\cos\Theta)\) vs \(\Theta\) for this case (see Fig. 11) shows a very narrow forward peak around \(\Theta = 0^\circ\) (insert in the upper left side), a strong but narrow glory feature in the angular range 174\(^\circ\)–180\(^\circ\), and a weak maximum around 162\(^\circ\) (insert in the upper right side). This weak

Fig. 11. Normalized scattering phase function of a unit volume of a spherical polydispersion designated Cloud C1: \(m = 1.5 - 0.01; \lambda = 0.825 \mu\)m.

Fig. 12. Intensity of the scattered radiation emerging at the top and bottom of a plane-parallel, homogeneous atmosphere with the spherical polydispersion Cloud C1 for \(\theta_0 = 0^\circ\). These results were obtained with the 99th (79th for the model with \(\tau_s = 10.0\)) order of the spherical harmonics approximation for multiple scattering calculations, but with the first 170 terms of the Legendre series for the primary scattering.
maximum around 162° is due to the radiation emerging from spheres after undergoing one internal reflection. An attempt to reproduce the curve with the first 130 terms only of the series results in the development of strong oscillations in the angular range 90°–150° (dotted curve in Fig. 11).

The spherical harmonics method was used to obtain values of $I_{L_n}(0; +\mu, \phi)$ and $I_{L_n}(\tau_k; -\mu, \phi)$ for the following values of the various parameters for atmospheric models with the phase function described in Fig. 11: $N=169$; $L_n=59$, 79, 99 for $\tau_k=0.1$ and 1.0; $L_n=59$ and 79 for $\tau_k=10.0$; $\theta=0.0^\circ$ (0.2°) 10.0°, 10° (1°) 89°. The values of $\theta_0$ and $\phi_0=\phi$ as well as the mode of analysis used for this case are the same as the ones used for the Haze L case (see second paragraph of Section 4).

The variations of the intensity of the radiation emerging at the top (left half of the diagram) and at the bottom (right half of the diagram) of a plane-parallel, homogeneous atmosphere consisting of the spherical polydispersion Cloud C1, are shown as a function of angle $\theta$ in Fig. 12 for models with $\tau_k=0.1$, 1.0, 10.0. These results, for the case of the sun at local zenith ($\theta_0=0^\circ$) are shown on a tenfold expanded scale in the angular range 0°–5° for bringing out details of the glory and forward-peak features. They were obtained with the 99th order of the spherical harmonics approximation for multiple scattering contribution (79th order for $\tau_k=10(0)$, but with the first 170 terms of the Legendre series for the primary scattering. As it can be seen, all curves are smooth. Furthermore, the basic features of the glory and the so-called fog-bow (a weak peak around 162° in Fig. 11) are clearly distinguishable in all cases. Even the forward peak around $\theta=0^\circ$ in the downward direction can be recognized in the model with $\tau_k=10.0$. This is not the case for models with the spherical polydispersion Haze L (Fig. 4) for which the scattering phase function is less anisotropic compared to that for the Cloud C1 (see Figs. 3 and 11).

The variations of $I_{L_n}(0; +\mu, \phi)$ and $I_{L_n}(\tau_k; -\mu, \phi)$ vs $\theta$ corresponding to $\theta_0=40^\circ$ are shown in Figs. 13 and 14, respectively, and those for $\theta_0=80^\circ$ in Figs. 15 and 16, respectively. Again, in each case, results are given for models with normal scattering optical thickness of 0.1, 1.0 and 10.0. The 99th order of the spherical harmonics approximation was used for the multiple scattering work (79th order for the model with $\tau_k=10.0$), but the first 170 terms of the Legendre series were used in evaluation of the contribution due to
primary scattering. Except for the angular region of minimum intensity for the model with $\tau_b = 0.1$, $I$ vs $\theta$ curves are generally smooth. This implies that a 99th order of approximation for multiple scattering is not high enough for providing smooth curves of $I$ vs $\theta$ for all combinations of $\tau_b$ and $\theta_0$ when a three-significant figure reproduction of the scattering phase function requires about 170 or more terms of the Legendre series. However, all basic features of the phase function of a unit volume of Cloud C1 polydispersion (Fig. 11), as well as their distortions with $\tau_b$ and $\theta_0$, can be studied without any difficulty from results presented in Figs. 12–16. The $I$ vs $\theta$ curves around the solar and anti-solar points are not shown on an expanded scale for these cases due to a programming oversight.

From our discussion in the preceding paragraph, it is clear that the spherical harmonics approximation of the order greater than 99 must be used for the evaluation of multiple scattering when three-significant-figure accurate values of $I(\tau; \mu, \phi)$ are required for atmospheric models with strongly anisotropic phase function. On the other hand, results obtained with the 99th order are acceptable for most purposes. Furthermore, because of increase in computational requirements with the increase in order of the approximation, an average reader is likely to be more interested in deterioration of the quality of results with decrease of the order. With this in mind, we have plotted the ratio $\rho(\tau; \mu, \phi; L_n / L_m)$ as a function of $\theta$ in Fig. 17 for $\theta_0 = 0^\circ$ ($\tau = 0$ case in the left half of the diagram, and $\tau = \tau_b$ on the right half). For models with $\tau_b = 0.1$ and 1.0, results are given for

Fig. 16. As in Fig. 12 except for $\theta_0 = 80^\circ$, and for the radiation emerging at the bottom only.

Fig. 17. Effect of the order of the spherical harmonics approximation on the computed values of intensity of the scattered radiation emerging at the top and bottom of a plane-parallel, homogeneous atmosphere with the spherical polydispersion Cloud C1: $\theta_0 = 0^\circ$, $N = 169$. 
the 59/99 and 79/99 cases; for the model with $\tau_b = 10.0$, they are for the 59/79 case only. Similar results for $\rho(0; \mu, \phi; L_n/L_m)$ and $\rho(\tau_b; -\mu, \phi; L_n/L_m)$ but for $\theta_0 = 40^\circ$ are presented in Figs. 18 and 19, respectively, and for $\theta_0 = 80^\circ$ case in Figs. 20 and 21, respectively.

From Figs. 17–21, we find that the results obtained with the 59th order of approximation for the multiple scattering part are of the poorest quality for models with $\tau_b = 0.1$ and 1.0 illuminated by the sun at 40° from the local zenith. In particular, the computed values of the intensity of scattered radiation emerging

**Fig. 18.** As in Fig. 17 except for $\theta_0 = 40^\circ$, and for the radiation emerging at the top only.

**Fig. 19.** As in Fig. 17 except for $\theta_0 = 40^\circ$, and for the radiation emerging at the bottom only.

**Fig. 20.** As in Fig. 17 except for $\theta_0 = 80^\circ$, and for the radiation emerging at the top only.

**Fig. 21.** As in Fig. 17 except for $\theta_0 = 80^\circ$, and for the radiation emerging at the bottom only.
at the top are very small, and even acquire a negative sign in the angular range 36°–44° in the meridian plane 180° away from the sun's meridian (see Fig. 18). Another case for which results of such an extremely poor quality are obtained is the computation for near-horizontal directions in the anti-solar meridian of diffuse transmission (bottom-most section of Fig. 19). On the other hand, computed values of \( I_{\phi}(\tau; \mu, \phi) \) for the same cases but in large angular regions surrounding the solar and anti-solar points compare very favorably with those obtained with the 79th or 99th order of the approximation.

In general, values of \( \rho(\tau; \mu, \phi; 79/99) \) are closer to unity than those of \( \rho(\tau; \mu, \phi; 59/99) \). The only exception to this rule is the case of diffuse reflection by the model with \( \tau_b=0.1 \) and \( \theta_b=80° \) (bottom-most section of Fig. 20). This exception is probably due to oscillations within the \( \delta \beta \) interval selected.

Another interesting feature of these diagrams is the relative flatness of \( I_{\phi}(\tau_b; -\mu, \phi)/I_{\phi}(\tau_b; -\mu, \phi) \) vs \( \theta \) curves in diffuse transmission of the model with \( \tau_b=10.0 \), compared with variations of \( I_{\phi}(0; +\mu, \phi)/I_{\phi}(0; +\mu, \phi) \) vs \( \theta \) in diffuse reflection of the same model. We believe that this is primarily due to a very mild anisotropy of the radiation field diffusely transmitted by an optically thick model.

In Fig. 22, we have shown the variations of the quantity \( |P_n(\tau; \mu)/P_n(\tau; \mu)| \) as a function of the order \( n \) of the azimuth-dependent component of \( I(\tau; \mu, \phi) \) for atmospheric models with the spherical polydispersion Cloud C1. This ratio is defined in the penultimate paragraph of Section 4. The mean of 1300 absolute values (130 values of \( \theta_b \), 5 values of \( \theta_b \), and \( \tau=0 \) and \( \tau=10.0 \)) for models with \( \tau_b=0.1 \) and 1.0, and of 650 absolute values (\( \tau=0 \) only) for the model with \( \tau_b=10.0 \) of these ratios are shown in this diagram. A 99th order of the spherical harmonics approximation was used in computations of \( I_{\phi}(\tau; \mu) \) for models with \( \tau_b=0.1 \) and 1.0 (79th order for model with \( \tau_b=10.0 \)), but computations of \( I_{\phi}(\tau; \mu) \) were performed with the first 170 terms of the Legendre series for the phase function. Again, we should remind ourselves of the strong dependence of this ratio on parameters \( \tau_b, \theta_b \) and \( \theta_b \) (refer to discussion in the beginning of the last paragraph of Section 4). As for example, the magnitude of this ratio is less than 10⁻³ for \( n>49 \), \( n>79 \), and for \( n>97 \) for \( \theta_b \leq 20°, \theta_b \leq 40°, \) and \( \theta_b \leq 60° \), respectively, for a Cloud C1 atmosphere with \( \tau_b=1.0 \). With this in mind, we find from Fig. 22 that the contribution to \( I(\tau; \mu, \phi) \) due to \( I_{\phi}(\tau; \mu) \) is greater than that due to \( I_{\phi}(\tau; \mu) \) for the first 80–85 azimuth-dependent components of intensity. This is not the case when the atmospheric models are composed of the spherical polydispersion Haze L (see Fig. 10). Thus, the oscillations in the \( I_{\phi}(\tau_b; -\mu, \phi) \) vs \( \theta \) curves in the angular range 20°–60° on the anti-solar side (e.g., curve for \( \tau_b=0.1 \) in Fig. 16) can be attributed to the omission of higher order scattering contributions in the 100th and higher azimuth-dependent components of intensity.

6. Conclusion

In the preceding sections, we presented representative results of our computations of the intensity of the scattered radiation emerging from a plane-parallel, homogeneous atmosphere as obtained with the spherical harmonics method discussed in Part I of this paper. These results are given for three different atmospheric models, one with Rayleigh phase function requiring only the first three terms of the Legendre series for its complete representation, and the other two for mildly and strongly anisotropic phase functions requiring about 69 and 169 terms, respectively, for their reproduction with three-significant-figure accuracy over the entire scattering range. In each case, results are presented for three different values of normal optical thickness (\( \tau_b=0.1, 1.0, 10.0 \)), and for three different solar zenith angles (\( \theta_b=0°, 40°, 80° \)). Except for the Rayleigh models, the order \( L_a \) of the spherical harmonics approximation used for evaluation of the multiple scattering contribution to the source function is less than \( N \), the total number of terms of the Legendre series required for ±0.1% reproduction of the corresponding phase function. However, the contribution due to primary scattering was fully accounted for, in all cases.

We find that the values of intensity computed with \( L_a \) equal to about \( 1 \) to \( 3 \) of \( N \) are accurate for most purposes even when no restrictions are placed in the angular range for directions of observation and incidence. For a model with a given phase function, results obtained with \( L_a=0.25N \) can be used with some confidence if the angular ranges of interest do not include unfavorable situations. These highly unfavorable situa-
tions occur when the directions of observation and/or of incidence are very close to the horizon, or alternately, the directions of observations are in the angular region with very small contribution due to scattering of the direct sunlight.

In this particular study, our results are for a non-absorbing homogeneous atmosphere with the optical characteristics of a unit volume invariant with height. However, the basic technique has been tested thoroughly for computations of fluxes and the azimuth-independent component of intensity in nonhomogeneous models of the terrestrial atmospheres (Dave and Canosa, 1974; Dave and Armstrong, 1974; Dave and Braslau, 1975). Thus, we may conclude that the method of the direct numerical solution of the spherical harmonics approximation to the transfer equation can be used with great advantage to study all aspects of at least the scalar (without polarization characteristics) transport of solar radiation in homogeneous as well as highly nonhomogeneous models of the terrestrial atmosphere.

Acknowledgments. I would like to take this opportunity to express my sincere thanks to my colleague, Dr. B. H. Armstrong, for a careful reading of the original manuscript and for his very helpful comments.

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