A Calculation of the Structure of Stationary Planetary Waves in Winter

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ABSTRACT

A propagation equation valid for determining the vertical and latitudinal structure of stationary planetary waves in winter is derived. This equation reduces to that used by Matsuno (1970) for an isothermal atmosphere where Rayleigh friction and Newtonian cooling are taken to be equal and constant. Quasi-analytical solutions to the propagation equation are obtained for the idealized case of an isothermal atmosphere in constant rotation to illustrate some of its important properties. Solutions are obtained numerically for altitudes between the forcing level at 100 mb and the 100 km level, where a radiation boundary condition is assumed. For various realistic models of the mean zonal wind field, radiative damping rates and mean temperatures profiles, we find that the structure of our computed stationary planetary-scale waves is more sensitive to the former two than to the latter.

The structural behavior of the numerical solutions is interpreted using modal decomposition. Two principal modes compose most of the structure of wavenumber 1, while wavenumber 2 is dominated by a single mode. The trapping of these modes at different heights appears to explain the variability of the wave amplitude and phase with change in the mean wind structure. Overall, the numerical results give reasonable agreement with observations.

We also discuss the associated energy fluxes and conversion terms due to vertically propagating planetary waves.

1. Introduction

Large-scale stationary planetary waves are very prominent features of the winter stratosphere. In the Northern Hemisphere, zonal harmonics 1 and 2 are superimposed upon the mean zonal flow to produce the Aleutian anticyclone which grows in amplitude with height up to 30 km and higher (Muench, 1965; van Loon et al., 1973). The amount of upward energy flux due to these stationary waves is quite large, on the order of 200 ergs cm$^{-2}$ s$^{-1}$ at the tropopause (Newell and Richards, 1969; Miller, 1970), and consequently large amounts of energy are transmitted to the upper atmosphere through these waves. Green (1972) has suggested that stationary planetary waves cause northward heat transport into the high-latitude mesosphere giving rise to the anomalously high temperatures measured near the pole (CIRA, 1972). While observations of planetary wave structure above 30 km are rather limited, evidence from Meteorological Rocket Network, radiometeor and ground-based ionospheric observations suggests that these waves do extend upward to the mesosphere and even the lower thermosphere during winter (Newell and Dickinson, 1967; Brown and Williams, 1971; Cavaliere et al., 1974).

Charney and Drazin (1961) were the first to theoretically investigate the possibility of the propagation of stationary planetary waves from the troposphere to the upper atmosphere. Using a β-plane geometry they showed that these waves could not transmit energy vertically unless the mean zonal wind was westerly and of a lesser magnitude than a cutoff velocity which varies inversely with the meridional and zonal wavenumber. Later, Dickinson (1968a) showed that this cutoff velocity would be increased for planetary waves propagating near the equator and decreased for those propagating close to the pole. He also demonstrated that a spherical geometry should be used to correctly compute this cutoff velocity. In a series of papers following this study, Dickinson (1968b, 1969a,b) traced out the possible propagation paths of planetary waves, studied their interaction with critical levels and examined planetary wave behavior in the presence of radiative dissipation.

Matsuno (1970) solved a numerical boundary value problem in which planetary wave amplitude and
phases at 500 mb were used as a lower boundary condition with a radiation condition at 60 km serving as the upper boundary condition. This model was quasi-geostrophic with a spherical geometry and used observed planetary wave behavior at 500 mb along with a realistic basic state zonal wind field in an attempt to simulate the vertical structure of planetary waves as observed by Muench (1965). He found that the vertical structure of planetary wavenumber 1 (zonal harmonic number 1) could be reasonably reproduced but the computations for wavenumber 2 showed a different vertical structure when compared with the observations of Muench (1965). Matsuno suggested that nonlinear effects might account for this disagreement. However, it has since been pointed out by Tung (1976) that the input data used for Matsuno's simulation were not from the same year as the planetary wave data of Muench (1965). The demonstration of the sensitivity of planetary wave structure to changes in the zonal wind field by Schoeberl and Geller (1976b) together with the observed variability of planetary wave structure (van Loon et al., 1973) leads us to believe that linear planetary wave theory and observations are not inconsistent. Simmons (1974) analytically computed the vertical structure of planetary waves in using simple model atmospheres with vertical shears. His analytical results were in better agreement with Muench's observations.

One of the principal objectives of this study is to examine the structure of planetary waves in response to changes in the polar night jet intensity. Some of these results have been presented previously but with less detail (Schoeberl and Geller, 1976b). The sensitivity of planetary wave structure to the mean temperature state and radiative damping parameters, a presentation of an interpretive framework for our results and a discussion of the energetics are additional subjects dealt with in this paper.

In Section 2 we outline the derivation of a planetary wave propagation equation and use scale analysis to investigate the limits of its validity. Expressions useful for diagnosing the energetics of the planetary wave solutions are also derived in this section. Quasi-analytical solutions to the planetary wave equation are obtained in Section 3 in order to illustrate some of the fundamental properties of the propagation equation. The procedure for numerically integrating this equation as well as the parameters used for our models are presented in Section 4, and the numerical results are presented and discussed in Section 5. These results are interpreted in Section 6 in terms of the analytic solutions obtained in Section 3. A comparison of our results with observed planetary wave amplitude and phase as well as the theoretical models of other workers is given in Section 7. Analysis of the energetics is given in Section 8 and a final summary is given in Section 9.

2. The propagation equation

The planetary wave propagation equation used in this investigation is a slightly generalized version of that which Matsuno (1970) derived from the vorticity and thermodynamic equations for planetary-scale motions. We can derive our equations by this method using the same energy consistency arguments as Matsuno. However, we prefer to obtain our equations from a scaling of the equations of dynamic meteorology, since this procedure has the advantage of providing the time and space scales for which the application of these equations is justified.

The linearized equations for perturbations (primed quantities) upon a mean zonal basic state (overbarrered quantities) as obtained from the primitive equations of dynamic meteorology are

\[
\frac{\partial u'}{\partial t} + \frac{\partial u'}{\partial \lambda} + \frac{\partial w'}{\partial \zeta} = -\frac{1}{a \cos \theta} \frac{\partial \phi'}{\partial \lambda} - F(\zeta), \tag{1}
\]

\[
\frac{\partial \phi'}{\partial t} - \frac{\partial \phi'}{\partial \lambda} + \frac{\partial \phi'}{\partial \zeta} = -\frac{1}{a \cos \theta} \frac{\partial \phi'}{\partial \phi}, \tag{2}
\]

\[
H' = \frac{\partial T'}{\partial t} + \frac{\partial T'}{\partial \lambda} + \frac{\partial T'}{\partial \zeta} + (2\Omega a)^2 S, \tag{3}
\]

\[
\frac{1}{R} \frac{\partial}{\partial \lambda} \left[ (v' \cos \theta - \omega' \cos \theta) + \frac{1}{a \cos \theta} \frac{\partial \phi'}{\partial \phi} \right], \tag{4}
\]

\[
RT' = \frac{\partial \phi'}{\partial \zeta}, \tag{5}
\]

where

\[
Z = 2\Omega \sin \theta - \frac{1}{a} \frac{\partial \phi'}{\partial \phi}, \tag{6}
\]

\[
Z^* = 2\Omega \sin \theta + 2a \tan \theta, \tag{7}
\]

\[
\omega = a/(a \cos \theta), \tag{8}
\]

\[
S = \frac{R}{(2\Omega a)^2} \left( \frac{\partial T'}{\partial \zeta} \right), \tag{9}
\]

In the above \( \phi \) is the geopotential \((\rho h)\), \( u \) and \( v \) are zonal (eastward) and meridional (northward) velocities, \( \theta \) is latitude, \( \lambda \) longitude, \( h \) the height of a pressure surface above the earth's mean surface, \( g \) the acceleration due to gravity, and \( F_\theta \) and \( F_\lambda \) are frictional forces in the meridional and zonal directions, respectively. \( \Omega \) is the rotational frequency of the earth and \( a \) is the earth's radius; \( R \) is the dry air gas constant, \( T \) temperature, and \( z = \log(p_0/p) \), where \( p \) is pressure and \( p_0 \) a reference pressure; \( c_p \) is the specific heat of air at constant pressure; \( H \) is the heating.
rate per unit mass; and \( w \) is \( dz/dt \). We have assumed a geostrophically balanced mean zonal wind field \( U(\theta, \sigma) \) and temperature field \( T(\theta, \sigma) \) such that the basic state equations are

\[
2\Omega \sin \theta \frac{\partial \phi}{\partial \theta} = -\frac{1}{a} \frac{\partial \phi}{\partial \sigma} \\
R \frac{\partial T}{\partial \sigma} = \frac{\partial \phi}{\partial \sigma}.
\]

The meridional and vertical mean zonal winds are taken to be negligible. We use the Rayleigh friction approximations \( F_\phi = \beta_k \nu' \) and \( F_\sigma = \beta_k \nu' \).

As with previous studies (Dickinson, 1968b; Matsuno, 1970; Simmons, 1974), we do not consider the mechanisms through which the waves are forced. Instead, we assume that above some level internal forcing may be neglected so that the wave structure is determined by a propagation equation which can be solved as a boundary value problem along with whatever damping processes exist. We assume the Newtonian cooling approximation for the heating rate, i.e.,

\[
\frac{H'}{c_p} = -a_0 T'.
\]

Since none of the coefficients in the perturbation equations are functions of longitude or time, we assume that perturbation quantities are proportional to \( e^{(m \lambda + \sigma t)} \), where \( \sigma \) is the wave frequency and \( m \) the zonal wavenumber.

Using Eqs. (1) and (2) to derive equations for \( u' \) and \( v' \) and substituting into the continuity Eq. (4) and the thermodynamic Eq. (3) gives

\[
-\frac{iw' a}{\nu_0} \left\{ a \cos \theta - \frac{\partial}{\partial \sigma} \left( \frac{Z^* \cos \theta \partial}{\xi_0 \partial \sigma} + \frac{m \nu_0 \partial}{\partial \sigma} \right) \right\} + \frac{i \partial}{\partial \sigma} \left( \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \theta} \right) + \frac{m \nu_0}{\xi_0} \left( \frac{Z^*}{\cos \theta} \right) - \frac{\partial}{\partial \sigma} \left( \frac{\partial}{\partial \theta} \right) + \frac{m \nu_0}{\xi_0} \left( \frac{Z^*}{\cos \theta} \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \sigma} \right) \right) \right),
\]

and

\[
\left( 2 \Omega a \right)^2 S \left( \frac{\partial \phi}{\partial \sigma} \right) \left( \frac{\partial \phi}{\partial \theta} \right) = -\frac{\partial \phi}{\partial \sigma} - \frac{i}{\nu_0} \left( \frac{\partial}{\partial \sigma} - \frac{m \nu_0 \phi \partial}{\partial \sigma} \right) \right) \right),
\]

where \( \nu_1 = \sigma + \omega m - i a_0 \), \( \nu_0 = \sigma + \omega m - i \beta_k \) and \( \xi_0 = \left( \nu_0 \right)^{-1} \cdot \left( \nu_0 \right)^{-1} \cdot \left( \nu_0 \right)^{-1} \).

The right-hand side of Eq. (6) is a generalized form of Laplace's tidal equation with mean zonal winds included. Eqs. (6) and (7) could be combined to eliminate \( u' \) or \( \phi' \); however, the resulting equation is of such complexity that no advantage is gained.

**Table 1. Basic-state parameters.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Typical</th>
<th>Extreme</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\sigma} )</td>
<td>( 40 )</td>
<td>( 80 )</td>
</tr>
<tr>
<td>( \dot{\sigma} )</td>
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</tr>
<tr>
<td>( \dot{\sigma} )</td>
<td>( 40 )</td>
<td>( 80 )</td>
</tr>
<tr>
<td>( S )</td>
<td>( 0.02 )</td>
<td>( 0.01-0.03 )</td>
</tr>
</tbody>
</table>

Instead, Eqs. (6) and (7) will be scaled to the planetary wave regime individually.

We only sketch out the scaling procedures used to reduce Eqs. (6) and (7) since the analysis is lengthy and we wish here only to illustrate the range of parameters over which the propagation equation is valid. A more detailed analysis, which is generally similar to the analysis made by Burger (1958), is found in Schoeberl and Geller (1976a).

Table 1 lists extreme and typical values of the basic-state parameters for the atmospheric altitude region 10-100 km. These values are taken from model atmospheres given by CIRA (1965) and data presented by Belmont et al. (1975). \( \nu_0 \) and \( \beta_k \) are assumed to be of the same order as \( \dot{\sigma} \). We restrict our analysis to the consideration of quasi-stationary planetary-scale disturbances with length scales on the order of an earth radius. Thus we assume \( m \leq 2 \). \( \lambda / \sigma \leq 3 \) and \( 2 \pi / \sigma > 10 \) days at mid-latitudes. The vertical length scale will be taken to be greater than a scale height as is consistent with observations of planetary waves in the lower stratosphere (van Loon et al., 1973).

With these definitions, we note that

\[ \xi_0 \approx 4k \sin \theta, \quad \dot{Z} \approx \dot{Z} \approx 2k \sin \theta \]

at mid-latitudes indicating that the earth's vorticity generally dominates the total vorticity of the basic state \( \dot{Z} \). Also note that \( \nu_0 \ll \dot{Z} \) or \( \dot{Z} \). With these simplifications Eq. (6) becomes

\[
i(2\Omega a)^2 \cos \theta \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \sigma} \right) = -\frac{i}{\nu_0} \left( \frac{\partial}{\partial \sigma} - \frac{m \nu_0 \phi \partial}{\partial \sigma} \right) \right) \right),
\]

where \( \dot{\xi} = \sigma + \omega m - i a_0 \), \( \nu_0 = \sigma + \omega m - i \beta_k \) and \( \xi_0 = \left( \nu_0 \right)^{-1} \cdot \left( \nu_0 \right)^{-1} \cdot \left( \nu_0 \right)^{-1} \).

The thermodynamic equation [\( \dot{\theta} \)] may be scaled similarly. Note that the term

\[
\dot{Z} \cdot \frac{\partial \phi}{\partial \sigma} \cdot \frac{\partial \phi}{\partial \theta} \cdot \frac{\partial \phi}{\partial \sigma} \cdot \frac{\partial \phi}{\partial \theta}
\]

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may be neglected with respect to \((2 \Omega a)^2 S\), and \(v \partial p'/\partial \theta\) may be neglected with respect to \(m \partial \phi'/\cos \theta\); this gives
\[
\begin{align*}
\lambda'(2 \Omega a)^2 S &= -iv_v + \frac{\partial \phi'}{\partial \pi} + \frac{im \phi'}{\partial \pi} + \frac{1}{a \cos \theta} \frac{\partial \phi'}{\partial z}.
\end{align*}
\tag{9}
\]

The terms which have been deleted from Eqs. (8) and (9) generally give less than a 10% correction to the terms retained at mid-latitudes for the "typical" parameters given in Table 1. At more equatorial latitudes, the approximation \(\bar{Z} = \bar{Z} = 2 \Omega \sin \theta\) breaks down and a different system of equations should be used. As a rule, however, Eqs. (8) and (9) are reasonably valid poleward of 20°N. We now combine Eqs. (8) and (9) to eliminate \(\omega'\), and use the change of variable
\[
\phi' = S \xi \sin \theta \omega'
\]
to cast the system into normal form. The resulting equation is
\[
\begin{align*}
\sin^2 \theta \frac{\partial}{\partial \theta} \left( \frac{\cos \theta \partial \xi'}{\sin^2 \theta} \right) + Q_m \xi' + & \frac{\sin^2 \theta v_v \partial \xi'}{S} + \\
\text{and} \quad & \frac{\sin^2 \theta \partial \xi'}{S v_v} \frac{\partial \xi'}{\partial z} + \frac{\sin^2 \theta \partial \xi'}{S v_v} \frac{\partial \xi_o}{\partial z} = 0,
\end{align*}
\tag{10}
\]

where
\[
Q_m = m \frac{\partial q}{\partial \theta} - m^2 \sin^2 \theta v_v \frac{\sin^2 \theta}{S v_v} \left( \frac{1}{2} \frac{\partial S}{\partial z} \right) - \left( \frac{1}{2} \frac{\partial S}{\partial z} \right) \frac{1}{\cos \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\cos \theta} \right)
\]
\[
\frac{\partial q}{\partial \theta} = 2(\Omega + \bar{\omega}) = -3 \tan \theta - \frac{\partial \phi}{\partial \theta} \sin \theta - \frac{\partial \phi}{\partial \theta} \sin \theta
\]
\[
\frac{\partial q}{\partial \theta} = \frac{3}{4} \frac{\partial S}{\partial z} - \frac{1}{2} \frac{\partial S}{\partial z} - \frac{\partial S}{\partial z} + \frac{4}{2} \frac{\partial S}{\partial z}
\]

This equation is a slightly more generalized form of that used by Matsumo (1970), and his equation may be recovered by assuming \(S\) is constant, and \(a_o = \beta Z\) are constant and equal. \((\partial q/\partial \theta) \cos \theta\) is the meridional derivative of the basic-state potential vorticity.

Eq. (10) can be shown to be energetically consistent within a geostrophic framework by multiplying Eq. (8) by \(\phi'\) and Eq. (9) by \(S^{-2} \phi'/\partial z\) and integrating over the mass of the atmosphere. The resulting energy conversion terms obtained are in agreement with those given by Lorenz (1967) provided \(S\) is taken to be a function of \(z\) only. The energy equations derived from (9) and (10) are
\[
\begin{align*}
\frac{\partial}{\partial t} & K_B = C_K + C_B + E_{FZ} - D_B + E_{FB} \\
\frac{\partial}{\partial t} & A_B = -C_B - G_B + C_A
\end{align*}
\tag{11}
\]

where
\[
K_B = \langle \frac{1}{2} (u'^2 + v'^2) \rangle
\]
\[
A_B = \langle R^2 T'^2 / 2 S (2 \Omega a)^2 \rangle
\]
\[
C_K = \left( \cos \theta u' \frac{\partial \omega_o}{\partial \theta} \right)
\]
\[
C_B = \left( \omega_v \frac{\partial \xi'}{\partial \theta} \right)
\]
\[
C_A = \left( \frac{R^2 T'^2}{S a (2 \Omega a)^2} \right)
\]
\[
E_{FB} = - \int_{q_1}^{q_2} (\phi' \omega_v) \cos \theta \sin \theta \sin \theta d \theta \int_{p_1}^{p_2} \frac{d \phi}{g} \int_{p_1}^{p_2} \int_{p_1}^{p_2} \frac{d \phi}{g}
\]
\[
E_{FB} = - \int_{p_1}^{p_2} \int_{q_1}^{q_2} \frac{d \phi}{g} \int_{p_1}^{p_2} \int_{q_1}^{q_2} \frac{d \phi}{g}
\]
\[
D_B = \langle \beta \theta u'^2 \rangle
\]
\[
G_B = \langle a_o R^2 T'^2 / S (2 \Omega a)^2 \rangle
\]

In the above, \(u'\) and \(v'\) are given by the geostrophic wind equations, except in the definition of \(E_{FB}\), where
\[
v' = \frac{im}{2 \Omega \sin \theta} \left( \frac{\phi'}{\cos \theta} + \bar{\omega}' \right).
\]

As before, the overbar denotes an average over a latitude circle and the angle brackets indicate additional mass integrals over the latitude interval \(\theta_1\) to \(\theta_2\) and pressure interval \(p_1\) to \(p_2\). \(K_B\) and \(A_B\) are the eddy kinetic and available potential energies, respectively. \(C_K\), \(C_B\) and \(C_A\) are energy conversion terms. \(C_K\) and \(C_A\) govern the conversion of the basic-state kinetic and available potential energy to wave energy; \(C_K\) governs the conversion of eddy available potential energy to eddy kinetic energy; \(E_{FZ}\) and \(E_{FB}\) are boundary fluxes of energy in the vertical and horizontal directions. \(D_B\) and \(G_B\) are energy dissipation terms due to Rayleigh friction and Newtonian cooling, respectively. \(w_v\) is \(dp/dt\).

Using an equation similar to Eq. (10) and assuming \(a_o = \beta Z = 0\), Charney and Stern (1962) have shown that a necessary condition for real \(\sigma\) is that \(\partial q/\partial \theta\) does not vanish within the region governed by Eq. (10) and bounded by walls through which the net energy flow vanishes. Thus, we shall take \(\partial q/\partial \theta \neq 0\) within the domain of interest so that \(\sigma\) may be specified as real. In the more general case of nonrigid boundaries and arbitrary damping parameters, \(\sigma\) may be taken to be real provided \(\partial q/\partial \theta\) does not vanish and that the energy flow through the boundaries is completely balanced by energy sources and sinks within the region. We can guarantee that this latter condition is met by not specifying the energy flow at the bounda-
Fig. 1. The eigenvalues $e^{m}_n$, as a function of $u_0$ for an atmosphere in constant rotation for $m=1$ (a) and $m=2$ (b).

ries, but by specifying $\phi'$, its derivative or the direction of the energy flow across the boundaries (radiation condition). The resulting solution of Eq. (10) will yield internal and boundary energy flow patterns which automatically balance the sources and sinks within the domain.

3. Quasi-analytical solution to an isothermal atmosphere in constant rotation

In order to examine some of the fundamental properties of the planetary wave propagation equation [(10)], we shall now consider a solution for a simple model atmosphere with $\omega$ constant and equal to $\omega_0$. This corresponds to an atmosphere in constant rotation with zonal wind velocity $\bar{u}_0$ at the equator. Since this atmosphere has no meridional temperature variation, for convenience we shall take $T$ to be independent of $z$ as well. $a_0$ and $b_0$ are set to be zero. Eq. (10) reduces to

$$\sin^2 \theta \frac{\partial}{\partial \theta} \left( \frac{\partial \psi'}{\sin^2 \theta \partial \theta} \right) + \left( \frac{2(\Omega + \omega_0)m}{\sigma + \omega_0 m} - \frac{m^2}{4S \cos^2 \theta} \right) \psi' + \frac{\sin^2 \theta \partial^2 \psi'}{S \partial^2 \theta} = 0. \quad (12)$$

The solutions to an equation similar to this equation have been briefly discussed by Dickinson (1968a), but it is of interest to reexamine them here. Assume $\psi' = \Theta(\theta)Z(z)$ in Eq. (12) and divide this equation by $\sin \theta$. This leads to two coupled ordinary differential equations:

$$\frac{d^2 Z(z)}{dz^2} + (\epsilon S - \frac{1}{\lambda}) Z(z) = 0, \quad (13)$$

$$\frac{1}{\cos \theta} \frac{d}{d\theta} \left( \frac{\cos \theta}{\sin^2 \theta} \frac{d \Theta(\theta)}{d\theta} \right) + \left[ \frac{(\Omega + \omega_0)}{(\sigma + \omega_0 m) \sin^2 \theta} \right] \Theta(\theta) = \epsilon \Theta(\theta), \quad (14)$$

where $\epsilon$ is the separation constant. Eq. (14) is a form of Laplace's tidal equation and governs perturbations of a fluid constrained to a rotating sphere with rotation rate $\Omega + \omega$ provided that $(\sigma + \omega m)/2\Omega < 1$. This equation is solved as an eigenvalue problem with the boundary conditions $\Theta(\pm \pi/2) = 0$. Figs. 1a and 1b show the variation of the eigenvalue $e^{m}_n$ with respect to $\bar{u}_0$ for $m=1$ and $m=2$ for $\sigma=0$. The number $n$
EIGENFUNCTIONS

\( u = 20 \text{ m s}^{-1} \)

EIGENFUNCTIONS

\( u = 80 \text{ m s}^{-1} \)

Fig. 2. Three of the eigenfunctions of Eq. (14), shown as a function of latitude for \( m = 1 \) (a) and \( m = 2 \) (b) for two different values of \( u_0 \).

denotes the number of zeros the eigenfunction possesses on the open interval \((-\pi/2, \pi/2)\). Values of \( \epsilon_n^m \) for \( \delta \theta < 0 \) are not given. Longuet-Higgins (1968) has shown these values to be large and negative, and this regime is outside our scope of interest. Figs. 2a and 2b show a few of the eigenfunctions \( \Theta \) obtained for \( m = 1 \) and \( m = 2 \) and for two different values of \( u_0 \). The eigenvalues \( \epsilon_1^1 \) and \( \epsilon_2^1 \) as well as their associated eigenfunctions are not plotted since these eigenfunc-
tions are strongly bound to the equatorial region and are of no real importance at mid-latitudes. In what follows we also take the eigenfunctions \( \Theta_n^m \) to be orthonormal, i.e.,

\[
\int_{-\pi/2}^{\pi/2} \Theta_n^m(\theta) \Theta_{n'}^m(\theta) \cos \theta d\theta = \delta_{n, n'},
\]

complete.

The solution of Eq. (13) is straightforward. For boundary conditions we assume the disturbance is specified at \( z = 0 \) by the relation \( \psi(\theta, 0) = G(\theta) \). At \( z = \delta \), we impose a radiation condition or if no vertical energy flow is associated with a particular solution we assume that the kinetic energy density is finite as \( z \to \infty \). Eliassen and Palm (1961) have shown that the radiation condition implies \( \psi' \sim e^\alpha z \), where \( \alpha > 0 \). Finite kinetic energy density as \( z \to \infty \) implies \( |\psi'|^2 \) is constant as \( z \to \infty \). The solution to Eq. (12) is then

\[
\psi'(\theta, z) = \sum_{n=1}^{\infty} A_n^m(\theta) e^{\alpha z},
\]

where

\[
A_n^m = \int_{-\pi/2}^{\pi/2} G(\theta) \Theta_n^m(\theta) \cos \theta d\theta,
\]

\[
\alpha(z) = \begin{cases} \left( \frac{\epsilon_n^m S}{\delta \theta} - \frac{1}{2} \right) \frac{1}{\delta \theta}, & \epsilon_n^m S > \frac{1}{2} \\ \left( \frac{\epsilon_n^m S}{\delta \theta} - \frac{1}{2} \right) \frac{1}{\delta \theta}, & \epsilon_n^m S < \frac{1}{2}. \end{cases}
\]

Note that the solution for \( \epsilon_n^m = 0 \) has \( \omega' = 0 \) by Eq. (9). This case gives the classic Rossby wave solutions for a sphere.

Returning to Fig. 1, the vertical wavelength of the disturbance as determined by Eq. (15) has been plotted on the left-hand side assuming an isothermal atmosphere at a temperature of 240 K. Note that variations of a few meters per second in \( u_0 \) may produce large variations in the vertical wavelength especially for solutions with large \( n \) and small \( u_0 \). We also see that the vertical wavelength for the wavenumber 2 solutions is always larger than those solutions associated with wavenumber 1 for the same values of \( u_0 \) and \( S \). Below the region marked by \( \lambda = \infty \) the solutions are evanescent. It is apparent that the cutoff velocity between evanescent and propagating solutions decreases as \( n \) and \( m \) increase which is consistent with the cutoff velocity formula of Charney and Drazin (1961) for planetary waves propagating in a \( \beta \)-plane channel.

For realistic winter situations in the Northern Hemisphere, lower stratospheric wind velocities are typically 20 m s\(^{-1}\) at mid-latitudes giving a value of \( u_0 \) corresponding to 28 m s\(^{-1}\). From Fig. 1, we see that only the \( n = 1 \) to \( 3 \) solutions for \( m = 1 \) and \( m = 2 \) can propagate vertically. All other solutions are evanescent.
4. The numerical model

The numerical procedure we use to solve Eq. (10) is the method described by Lindzen and Kuo (1969).

For our lower boundary condition on Eq. (10), we specify a wave amplitude and phase based upon observations for each zonal harmonic. Fig. 3 shows the amplitude and phase of zonal harmonic numbers 1 and 2 at 500 mb taken to be representative of a steady state tropospheric forcing for January. These curves were computed from data taken from the MIT General Circulation Library and provided to us by Dr. A. H. Oort.

The vertical grid interval was taken to be 0.2 scale heights or \( \sim 1.4 \) km. The horizontal grid interval was taken to be 5° in latitude. At the upper boundary we impose a radiation condition by assuming the absence of a vertical wind shear, solving Eq. (10) analytically, and choosing only those wavelike solutions which correspond to upward energy flow. For the exponential solutions only evanescent modes are retained. The upper boundary is taken to be located at 100 km, where it is assumed that effects of heat conduction, viscosity and ion drag in the region above would preclude any significant downward wave energy flow. We have tested this assumption using a simple model ionosphere between 100 and 140 km, placing a rigid lid at 140 km. No significant difference between results obtained using the model ionosphere and the radiation condition was detected below 85 km.

In the meridional direction we impose rigid walls at the equator and take the wave amplitude to vanish at the pole as is dictated by the singularity in Eq. (10). The rigid wall at the equator isolates the hemispheres and is consistent with observations which indicate a small amplitude for planetary waves near the equator (van Loon et al., 1973). This is also consistent with the probable absorption of planetary wave energy by critical surfaces (Dickinson, 1969b; Geisler and Dickinson, 1974; Béland, 1976). The critical surface, where \( \theta = 0 \), may lie to the south of the equator, but the zero zonal wind line is shifted to the north of the equator at all heights in our model. To insure complete absorption of wave energy at the critical surface, we have increased the Rayleigh friction parameter from its background value of \( 5 \times 10^{-7} \) s\(^{-1} \) to a Gaussian functional form near the critical surface. The magnitude and width of the Gaussian is chosen on the basis of the strength of the local meridional wind
wind models $\partial q/\partial \theta$ is positive everywhere. Fig. 5 shows three different Newtonian cooling profiles that are based upon different sources. The "fast" profile is taken from Blake and Lindzen (1973) and incorporates the effects of both photochemical and radiative processes, the constant profile is that used by Matsuno (1970), and the "slow" profile is that computed by Dickinson (1973) allowing for radiative cooling processes only.

Fig. 6 shows the non-isothermal static stability profile used in this study. The profile shown is com-

![Fig. 5. Newtonian cooling profiles ($\times 10^{-7}$ s⁻¹).](image)

![Fig. 4. Mean zonal wind profiles representing winter stratosphere/mesosphere conditions for WM (wind model) I (a) and WM II (b).](image)

![Fig. 6. Temperature profile ($T$) and static stability ($S$) as a function of altitude.](image)
computed from the temperature structure given in the CIRA (1972) model atmosphere at 30°N for January. Below 25 km, MIT General Circulation Library data were used. The isothermal profile used assumes an atmosphere at 240 K ($S=0.023$).

5. Results

Figs. 7 and 8 show the amplitude and phase of $\psi'$ for zonal harmonics 1 and 2, respectively. The isothermal temperature state and the slow Newtonian cooling profile were used. Parts a and b correspond to the wind models I and II, respectively, used for each integration of Eq. (10). The phase refers to the longitude of the ridge in degrees west of Greenwich. The amplitude is in decameters.

We immediately note that striking variations in the vertical structure of $\psi'$ are evident as the mean zonal wind is changed. For both wavenumbers, the overall amplitude of the wave in the upper stratosphere at mid-latitudes increases and phase lines become more separated as the strength of polar night jet increases.

![Fig. 7. Amplitude and phase of planetary wavenumber 1 computed using different wind models for WM I (a) and WM II (b).](image1)

![Fig. 8. As in Fig. 7 except for wavenumber 2.](image2)
Note the phase pole in Fig. 7a where the phase lines intersect. Apparently this feature moves northward to a position near the North Pole in Fig. 7b. Structural changes in wave amplitude are also particularly evident south of 45°N for $m=1$, indicating perhaps a change in the efficiency of the equatorial wave guide for planetary waves. In particular, we refer to the extension of a tongue-shaped region of wave amplitude marked by the 20 dam contour at low latitudes. Note how this feature changes between Figs. 7a and 7b.

For $m=2$ a similarly shaped region is also outlined by the 20 dam contour at low latitudes in Fig. 8a. As the polar night jet intensity is increased between Figs. 8a and 8b, this feature loses its distinction from the wave structure near the lower boundary. Equally important is the phase change of the wave between these figures as indicated by the reduced westward tilt of the ridge axis in the stratosphere for WM II.

**Fig. 9.** Amplitude and phase of planetary wave obtained for constant Newtonian cooling profiles for $m=1$ (a) and $m=2$ (b).

**Fig. 10.** As in Fig. 9 except for the fast Newtonian cooling profile.
Figs. 9 and 10 show the different results obtained using the constant Newtonian cooling profile and the fast profile for WM I for wavenumbers 1 and 2, respectively. The temperature state is taken to be isothermal, and parts a and b refer to the different wavenumbers.

Comparing these figures with Figs. 7a and 8a, which show the results obtained for the slow Newtonian cooling profile, we see that the increase in the cooling rate in the upper stratosphere produces an expected reduction in wave amplitude, and this reduction is quite dramatic at lower latitudes. We also note that the amplitude of wavenumber 2 increases slightly at low latitudes in the upper stratosphere from the constant (Fig. 9b) to the slow profile (Fig. 8a). We attribute this contrary result to the larger damping in the lower stratosphere used in the constant profile compared to the slow profile.

Figs. 11a and 11b show the amplitude and phase structure of planetary wavenumbers 1 and 2, respectively, for an atmosphere with a temperature and static stability profile shown in Fig. 6. Since $\psi'$ is defined in terms of $\phi'$ and $S$, the amplitude of $\psi'$ appears somewhat adjusted when Figs. 11a and 11b are compared to Figs. 7a and 7b. Overall, however, there is very little difference between the results obtained for an isothermal and non-isothermal atmosphere. These results suggest that the planetary wave structure is much less sensitive to the observed variations in $S$ than to variations in $\bar{u}$. We have also performed computations where $S$ varies realistically with latitude as well as height with similar results.

6. Fourier-Hough decomposition of wave structure

In Section 3, it was shown that the vertical structure of stationary planetary waves can be partially understood in terms of the vertical structure of individual normalized eigenfunctions excited at the lower boundary by the forcing function. Since $\text{Eq. (10)}$ is nonseparable, the individual eigenfunctions or modes which result from a separable solution become coupled to each other and are not actually eigenfunctions of the system. However, we shall presume in the analysis that follows that mode coupling can be taken into account when comparisons are made between the full solution and the amplitudes of the individual modes which are generated by decomposition. The eigenfunctions we use in this analysis are those obtained from $\text{Eq. (10)}$ by neglecting the vertical shear in the mean zonal wind at the level of interest. We then take the full solution and decompose it in terms of these functions. Since the eigenfunctions are closely related to Hough functions we refer to this process as Fourier-Hough decomposition. An examination of the eigenvalues associated with the eigenfunctions indicates whether the solution has vertically propagating or evanescent characteristics. In regions of strong vertical shear, this procedure may give misleading results but should lend overall insight into the source of the vertical structure variations shown in Figs. 5–9. Note also that with the inclusion of damping, these eigenfunctions are no longer orthogonal. It is still of interest, however, to project our numerical solutions at various levels upon these eigenfunctions to obtain mode coefficients since this procedure allows us to interpret the vertical structure of our solutions in terms of propagating and evanescent modes.

Figs. 12a and 12b show the amplitude of the mode coefficients resulting from the decomposition for wind model I, $m=1$ and $m=2$, respectively. Figs. 13a and 13b show the amplitude of the expansion coefficients for wind model II. The sign of the real part of the eigenvalue associated with the eigenfunction is indicated by points (+) or (−). Points not labeled...
a. Wind model I, wavenumber 1

It is apparent from Figs. 12 and 13 that higher n eigenfunctions are quickly filtered from the middle and upper levels of the numerical model. Attention should be focused upon the $\Theta_1^m$ and $\Theta_2^m$ ($m=1, 2$) eigenfunctions since we find that these are the dominant modes controlling the structure of the total solution at high altitudes. The results for wind model I, $m=1$, indicate that the $\Theta_1^1$ mode is vertically propagating at all heights at low and middle latitudes, while the $\Theta_2^1$ mode is trapped at lower levels. This interpretation is based upon the fact that $\Re(\epsilon_1^1)$ is positive at all levels but $\Re(\epsilon_2^1)$ changes from positive to negative within the lower regions of the model. The trapping of the $\Theta_2^1$ mode produces a reflected wave which interferes with the incident wave-generating nodes (minima) and antinodes (maxima) in wave amplitude below the region where $\Re(\epsilon_1^1)$ changes sign. The amplitude maximum surrounded by the 60 and 80 dam contours at 65°N in Fig. 5a is interpreted as the associated $\Theta_2^1$ antinode. The extended amplitude region surrounded by the 10 dam contour at 50 km at low latitudes suggests the presence of a weak $\Theta_3^1$ antinode. This antinode may be generated by partially reflected waves produced by rapid changes in $\Re(\epsilon_1^1)$ near the jet maximum. The eastward tilt of the disturbance with height poleward of the phase pole in Fig. 7a is consistent with the downward energy flow shown in Fig. 16a.

The effect of mode coupling which is produced by vertical wind shears is difficult to isolate. We note that the $\Theta_3^1$ solution dominates the total solution at 17 km but at 34 km the $\Theta_4^1$ mode has a relatively larger amplitude than the $\Theta_3^1$ mode. We also note that the total amplitude of the wave is larger at 34 km than at 17 km. While some of the increase in the amplitude of the $\Theta_3^1$ mode from 17 to 34 km is probably due to energy conversion processes, we also suspect that mode coupling is at work transferring energy from the $\Theta_2^1$ to the $\Theta_3^1$ mode. In addition, we note that above 70 km the $\Theta_3^1$ mode decays quite rapidly with height, while the $\Theta_4^1$ mode decays very slowly. We suspect that the $\Theta_4^1$ mode is partially maintained in this region by mode coupling.

b. Wind model II, wavenumber 1

The amplitude of the feature surrounded by the 10 dam contour at low latitudes increased from Fig. 7a (WM I) to 7b (WM II) as the polar night jet intensifies. The Fourier-Hough decomposition results shown in Fig. 13a indicate that $\Re(\epsilon_1^1)$ remains positive for wind model II but becomes small enough at 68 km so that $\Re(\epsilon_1^1)S-\frac{1}{2}$ is less than zero; therefore, this solution may be considered evanescent at that level. As a result, a reflected wave forms below 68 km, interfering with the incident wave and giving rise to a well-developed antinode. As the mean zonal wind increases with altitude, the $\Theta_3^1$ mode becomes domi-
nant at middle as well as low latitudes. Thus, the antinode structure appears further northward than in the weak jet case (see Fig. 7a). The $\Theta_2^1$ mode appears to be entirely trapped in the lower regions of the model. The antinode associated with this eigenfunction is surrounded by the 80 dam contour and appears to be bisected by the lower boundary.

c. Wind model I, wavenumber 2

For the $m=2$ solutions, Fig. 12b shows that changes in the sign of the eigenvalues with height are roughly similar to that shown in Fig. 13a for wind model II, $m=1$. The $\Theta_2^1$ mode appears evanescent at all altitudes, while the $\Theta_2^2$ mode appears to be propagating except in the upper regions of the model where the jet maximum occurs (68–85 km). Furthermore, the overall wave structure for $m=1$, wind model II, and $m=2$, wind model I, is quite similar. Both Figs. 7b and 8a contain lobe-shaped amplitude contours at mid-latitudes, extending upward and slightly toward the equatorial upper stratosphere.

The similarities between Figs. 7b and 8a result from the relative importance of two terms in the expression for $Q_m$. For the results obtained for $m=1$, $Q_m$ decreases as the term $\omega^2 \partial \phi / \partial \theta$ decreases due to the fact that $\omega$ grows with altitude. For $m=2$, $Q_m$ is reduced overall by the $m^2 / \cos^2 \theta$ term over $m=1$, particularly in the polar regions. The total effect produces nearly the same variations in altitude for $Q_m$ for $m=2$ as is found for $m=1$ for large values of $\omega$. Thus the solution for the weak zonal jet gives nearly the same wave structure for $m=2$ as the strong jet produces for $m=1$.

The similarities in structure of the solutions for wind model II, $m=1$, and wind model I, $m=2$, also suggest that similar processes are occurring for both solutions. The $\Theta_2^2$ antinode is probably the lobe-shaped region surrounded by 20 dam contour at middle to low latitudes in the stratosphere in Fig. 8a. The appearance of the antinode at a lower altitude and its less definite structure compared with the $\Theta_2^1$ antinode in Fig. 8b indicates that the reflection level for that mode may be a somewhat lower level than the reflection level for the $\Theta_2^1$ mode in Fig. 8b. The $\Theta_2^2$ mode is evanescent at all levels and appears to decay quickly above the lower boundary.

d. Wind model II, wavenumber 2

For $m=2$, wind model II, the Fourier-Hough decomposition shown in Fig. 13b indicates that the $\Theta_2^2$ mode becomes evanescent above the lowest levels. The antinode structure outlined by the 80 dam contour has descended to the lower boundary region as the reflection level descends from Figs. 8a and 8b with increasing jet strength. The amplitude of the solution decays most rapidly with height above 60 km, as both the $\Theta_2^2$ and $\Theta_2^3$ modes are evanescent up to the top of the model. The descent of the $\Theta_2^2$ antinode gives rise to the unusual behavior that the amplitude of $\psi'$ for the planetary wave decreases more slowly with altitude for stronger jet wind models in the lower stratosphere than for weaker jet models. This produces a more rapid increase in $\phi'$ with height in the lower stratosphere for strong polar night jets (Schoeberl and Geller, 1976b).

![Fig. 13. As in Fig. 12 except for WM II.](image-url)
At the lower boundary, the results of the decomposition for $m=2$ indicate that the $\Theta_2^m$ mode is principally excited by the boundary forcing. However, the $\Theta_2^m$ mode is blocked from propagating vertically by the polar night jet westerlies at a fairly low level (30 km). The $\Theta_2^m$ mode penetrates further into the jet and thus dominates the total solution in the upper regions of the model. For strong polar night jets, trapping of the $\Theta_2^m$ mode occurs at high levels and the aninodal, which results from the trapping, obscures the $\Theta_2^m$ solution near the lower boundary. Phase poles appear when a changeover in domination of the total solution from the $\Theta_2^m$ mode to the $\Theta_2^m$ mode occurs. Thus, if the $\Theta_2^m$ mode is never dominant, we do not expect phase poles which is consistent with the results obtained for the wind model II, $m=2$ solution.

**e. Newtonian cooling**

It is apparent from Figs. 9 and 10 that Newtonian cooling plays an important role in producing phase poles. For large values of the Newtonian cooling parameter, the short vertical wavelength $\Theta_3^m$ mode will be more highly damped than the $\Theta_2^m$ mode. Thus the $\Theta_3^m$ should grow less rapidly with height for large values of $a_0$ such as those used in the fast profile, and the $\Theta_2^m$ mode should become the dominant portion of the total solution near the lower boundary. We conclude the phase poles should be present in solutions obtained using fast cooling profiles and should be absent from solutions obtained using the constant profile given the same jet strength. Figs. 9 and 10 indicate this trend especially in the middle regions of the model where the fast cooling rate is much larger than the constant rate. In the lowest scale height where the fast rate is actually smaller than the constant profile we see little difference in the results.

**f. Temperature**

The apparent insensitivity of the vertical structure of planetary waves to the variation of $S$ with altitude, as indicated by Figs. 11a and 11b, may also be understood in terms of the vertical propagation of the individual eigenfunctions. For the separable models, the vertical wavelength is controlled through the term $(\epsilon^2 S - T)$. For the results obtained for wind model I, it was suggested that the vertical structure of the planetary wave could be interpreted as the vertical propagation of the $\Theta_2^m$ mode at low and middle latitudes and the trapping of the $\Theta_2^m$ mode at the higher latitudes. The variation of $S$ with height as occurs with a realistic temperature profile does not
Fig. 15a. As in Fig. 14a except for m=2.

significantly alter the level at which $c_0^s - \gamma$ changes sign. Also, the $\Theta^1_s$ mode does not change from propagating to evanescent over regions large compared to the vertical wavelength of the solution. Essentially, the variation in the height of the reflection level for individual eigenfunction is chiefly controlled through $c_0^s$ because the variations in $c_0^s$ with altitude are relatively greater than the variations in $T$. We conclude that the influence of $S$ and $T$ is secondary to the influence of $\omega$ on vertical propagation of planetary waves and realistic variations in the temperature structure produce relatively minor overall changes in the vertical structure of the waves.

7. Comparison with observations

The large variations in the amplitude and phase structure of $\psi$ in the upper atmosphere with different wind models is translated into even larger variations in geopotential due to the factor $e^{\gamma s}$. While detailed observations of planetary wave structure are still confined to regions below 30 km, some estimates of wave structure in the upper stratosphere and mesosphere have been given by Green (1972). The most complete summary of monthly mean observations in the region between 1000 and 10 mb is given by van Loon et al. (1973). Figs. 14 and 15 show the computed structure of the planetary wave geopotential at 50°N using van Loon's wave amplitude at 100 mb as a lower boundary condition (not to be confused with the wave amplitude shown in Figs. 3a and 3b) for wind models I and II for $m=1$ and $m=2$, respectively. Also plotted are the monthly mean observations made by Muench (1965) and van Loon et al. (1973) and the estimates made by Green (1972) for wavenumber 1. In order to indicate the magnitude of the year-to-year variability in planetary wave structure, the range of the monthly mean wave amplitudes and phase variations below 10 mb over 5 years (1965–69) at 60 and 65°N has also been shown as envelopes at 100 mb and 10 mb.

Fig. 14a shows that the computed amplitude for wind model I seems to be in good agreement with observations reported by van Loon et al. (1973) and by Muench (1965) after adjusting for the different forcing at the lower boundary. A smaller overall amplitude than that suggested by Green (1972) is computed for the stratosphere, and a larger amplitude is computed for the mesosphere. We also note that some of the observed variability in wave amplitude at 10 mb can possibly be accounted for by variation in amplitude of the polar night jet which has also been noted in the model of Holton and Mass (1976). Changes in forcing also produce a large part of this variation as is indicated by the width of the envelope at 100 mb.

Fig. 15b. As in Fig. 14b except for $m=2$. 
The phase structure of wavenumber 1 shows large variations in the upper atmosphere depending on the wind model used; however, the general behavior of the tilt of ridge and trough axis in the lower stratosphere seems to be in agreement with observations. It is also apparent that the computational results show too much westward tilt in the lower stratosphere. Part of this excessive phase tilt appears to be due to accumulative phase errors resulting from the numerical method and this phase error appears as an added phase shift when the observed phase of the wave is compared with the computed phase at the lower boundary. We estimate this error to be not greater than 12° longitude for Fig. 14. The excessive westward tilt of the wave near the lower boundary may also be partly due to small value of $a_0$ used near the lower boundary. Comparing the results obtained for the constant profile in Fig. 9a and the results obtained for the slow profile in Fig. 7a, we note that a larger value of $a_0$ produces a smaller phase shift near the lower boundary. This suggests that better agreement with observations could be obtained using a larger value of $a_0$ near the lower boundary for the slow profile.

In Fig. 15a, for $m=2$, it is apparent that the widely different observational results obtained by Muench (1965) and van Loon et al. (1973) for wavenumber 2 can be simulated by adjusting the strength of the polar jet. We suggest then that Muench's data were gathered during a period when the polar night jet most closely resembled wind model II; van Loon's climatological data seem to be more characteristic of results obtained for the weaker jet case, wind model I.

The computed wave amplitude in the mesosphere is larger than that suggested by Green (1972); however, since mesospheric planetary wave data are still quite sparse, we feel this disagreement is not conclusive. For instance, Brown and Williams' (1971) measurements of height variations in electron density isopleths in the E region suggest that much larger amplitude waves may be present in the mesosphere than suggested by Green.

We also note that the data obtained by Green for the mesosphere and upper stratosphere may be incompatible with the steady-state assumption used in this work. The data used by Green (1972) to compute monthly mean wave amplitude possibly contain major and minor stratospheric warmings, events characterized by time-dependent, nonlinear processes. The reduction in wave amplitude following a warming at upper levels (Matsumoto, 1971) would tend to reduce the observed monthly mean wave amplitude.

The numerical model used in this study contains a variety of adjustable parameters and implicit assumptions. For example, the magnitude of the Rayleigh friction parameter is a critical quantity and the value of $\beta_R$ has only been crudely estimated (Leovy, 1964). Better results might be obtained for $\beta_R$ varying with height and latitude. Also, there is some doubt about the proper treatment of critical levels in numerical models. The question remains open whether they are absorbers of wave energy or reflectors (Béland, 1976). Finally, we point out that the planetary waves are hardly ever in steady-state equilibrium with tropospheric forcing (Hirota and Sato, 1969). In fact, the stratospheric tropospheric system in winter appears to be characterized by con-
The conversion terms $C_K, C_A, D_B, D_E$ and $G_E$ are illustrated in Figs. 16–21 for wind model I. $C_K$ and $C_A$ indicate the conversion of basic state kinetic and available potential energy to eddy kinetic and available potential energy, respectively. $C_E$ governs the conversion of eddy available potential energy to eddy kinetic energy. $D_E$ and $G_E$ are energy dissipation terms due to Rayleigh friction and Newtonian cooling, respectively. $D_E$ removes eddy kinetic energy and $G_E$ removes eddy available potential energy. Part (a) of each figure refers to $m = 1$, part (b) to $m = 2$. An $e^{t/2}$ multiplication scaling factor

Fig. 17. Conversion term $C_K$ for $m = 1$ (a) and $m = 2$ (b).

Fig. 18. Conversion term $C_A$ for $m = 1$ (a) and $m = 2$ (b).

continuously adjusting wave structure. Thus, the comparison of steady-state results with observations depends to a large degree on the time scale for the adjustment process and the goodness of the climatological average.

8. Energetics

Using Eq. (11), the energy conversion and flux terms for stationary planetary waves may be computed. The energy fluxes and the conversion terms
is used to bring out features at the upper levels of the models.

The wave energy sources are the lower boundary energy flux, $C_A$, and $C_E$. These sources may also act as energy sinks; other sinks are $D_E$ and $G_E$ and the energy flux across the upper boundary which is imposed through the radiation condition. Since this is a steady-state model the energy sources and sinks are in complete balance.

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Fig. 19. Conversion term $C_E$ for $m=1$ (a) and $m=2$ (b).

Fig. 20. Dissipation term $D_E$ for $m=1$ (a) and $m=2$ (b).

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Examine Figs. 16-21, we see that for $m=1$ the lower boundary flux provides kinetic energy and $C_A$ provides available potential energy to the wave at mid-latitudes below 40 km. The available potential energy is generated by wave transport of heat across the basic-state vertical shear. Newtonian cooling, as shown by $G_E$, acts as a sink to available potential energy at mid-latitudes but the most important energy sink is the kinetic energy sink at the critical level in
the equatorial region as indicated by the large values of $D_B$. In order to transfer the available potential energy gained through $C_A$ at higher latitudes to $D_B$ at lower latitudes, wave potential energy is converted to wave kinetic energy by $C_K$. The wave kinetic energy is then transferred southward to the critical level as indicated by the energy flow vectors. ($C_K$ is thus large where $C_A$ is large.) The energy source provided by $C_A$ causes a divergence of energy flow vectors at 30 km at mid-latitudes in the meridional direction in Fig. 16a. As wave energy is transferred southward, zonal momentum is transported northward across the meridional shear in $\bar{u}$. This creates the additional energy sink through $C_K$.

Above 40 km the structure of the energy processes alters (at mid-latitudes) as the domination of the total solution changes from the $\Theta_1^2$ to the $\Theta_1^1$ mode in the upper stratosphere. This effect produces poleward energy flow at 60 km and 40°N and also downwind energy poleward of 50°N at the level of the stratopause. As the energy is transferred poleward, zonal momentum is carried by the wave southward across the meridional shear in $\bar{u}$ thus providing wave energy through $C_K$. The $G_B$ available potential energy sink is a result of the large temperature perturbation associated with the phase pole. Energy to this sink appears to be supplied by wave energy fluxes from above and below the region and by $C_K$. In the latter case, wave kinetic energy is converted to available potential energy by $C_B$ as indicated in Fig. 19a. $C_A$ acts as an energy sink above 40 km in the polar region as heat is transported southward across the vertical shear.

Note that actual downward energy flow is seen near 40 km at high latitudes in Fig. 16a. This energy flow distribution is consistent with a convergence of energy flow in regions of energy sinks and divergence in regions of energy sources.

b. Wavenumber 2

For $m=2$, the energy flow patterns shown in Fig. 16b indicate no downward energy flow except very close to the critical level at low latitudes. A much larger vertical energy flow is indicated in the lower levels of the model than has been computed from observations by Miller (1970) and Newell and Richards (1969). This discrepancy can be lessened by using a slightly smaller amplitude wavenumber 2 at the lower boundary. Since the energy terms are proportional to $|\phi'|^2$, differences in boundary forcing values for $\phi'$ tend to be greatly magnified in subsequent energy calculations.

Overall, the energy structure of wavenumber 2 seems quite simple. Wave kinetic energy is provided by lower boundary fluxes and basic-state available potential energy is liberated through $C_A$ as the wave propagates vertically and transports heat northward across the vertical shear in $\bar{u}$. The wave available potential energy is partly removed by the Newtonian cooling sink $G_B$ at mid-latitudes; the rest is converted through $C_B$ to wave kinetic energy which is then transferred southward to sinks provided by the $C_K$ and $D_B$ conversion processes. Note that both $D_B$ and $C_K$ are somewhat larger for $m=2$ than for $m=1$ calculations. Aside from the different amplitude of the forcing, part of this increase is due to the larger associated meridional velocity $\bar{v}$ for wavenumber 2.

Fig. 21. Dissipation term $G_B$ for $m=1$ (a) and $m=2$ (b).
components. This gives a factor of 2 increase in $E_{fb}$, $C_K$ and $D_K$ for $m=2$ over $m=1$ using the geostrophic wind relations.

The sources and sinks for planetary wave energy illustrate the importance of these disturbances in providing energy to the tropical stratosphere, principally through energy fluxes. Energy is also provided to the upper atmosphere. We have computed that the deposition rate of kinetic energy due to both wavenumbers for wind model II is roughly $1 \times 10^{-8}$ ergs cm$^{-3}$ s$^{-1}$ in the 70-100 km region which is about one-fourth of the value suggested by Leovy (1969) required to maintain the neutral temperature in that region. For wind model I, the value is about $3 \times 10^{-9}$ ergs cm$^{-3}$ s$^{-1}$. We have also computed the heating rate due to northward transport of heat at 70 km due to vertically propagating planetary waves. For wind model I the maximum heating rate is nearly 5 K day$^{-1}$ at 70 km at 55$^\circ$N, while for wind model II the heating rate is nearly 8 K day$^{-1}$ at 65$^\circ$N. According to the Charney-Drazin theorem, these large heating rates will be balanced by a zonal mean circulation which counteracts the net effect of planetary wave heating. The magnitude of the mean circulations has not been computed for this study.

9. Summary and conclusions

The objective of this paper has been to examine the steady-state structure of stationary planetary waves using analytical and numerical methods. We have found that the vertical structure of both wavenumbers 1 and 2 is very sensitive to the strength of the polar night jet. The type of Newtonian cooling profile used is also important in determining the vertical structure of the planetary wave. The former result was presented previously by Schoenberi and Geller (1976b). We find that the effects produced by changes in the vertical temperature profile appear to be smaller relative to the effects on wave structure produced by changes in the zonal wind profile and the Newtonian cooling.

We have suggested that the vertical propagation of the planetary waves may be viewed in a manner similar to atmospheric tides (Lindzen and Hong, 1974). The structure of the planetary wave can be understood in terms of the propagation of two eigenfunctions labeled $\Theta_3^m$ and $\Theta_5^m$ which are nearly equivalent to Hough functions which lie on the Rossby branch of the solution to Laplace's tidal equation (Dickinson, 1968a). For $m=1$, and a weak polar night jet, the $\Theta_3^m$ solution dominates at low latitudes and propagates vertically through the equatorial waveguide. The $\Theta_5^m$ solution is trapped by the jet in the lower stratosphere. For stronger jets, the $\Theta_5^m$ solution dominates at mid-latitudes and becomes partially trapped giving rise to antinodes in the wave amplitude below the jet maximum. For $m=2$, the $\Theta_3^m$ mode dominates the structure at mid-latitudes and is partially trapped below weak polar night jets. Stronger jets fully trap the $\Theta_3^m$ mode which causes an antinode in wave amplitude to form in the lower stratosphere giving rise to a rapid wave amplitude increase with height in that region.

In Schoenberi and Geller (1976b) we suggested that the discrepancy between the results obtained by Simmons (1974) and Matsuno (1970) for wavenumber 2 is apparently due to the type of mean zonal wind model used. Matsuno used a weak jet model resembling WM I. We find that for this wind model, the $\Theta_3^m$ mode is only partially trapped. Simmons used a $\beta$-plane model in which the planetary wave is always trapped by the linear vertical shear. This model more closely resembles the strong jet profile (WM II used in this study) in which the $\Theta_3^m$ mode is trapped. We suggest that this effect may be responsible for some of the observed year-to-year variability in planetary wave amplitude in the lower stratosphere possibly associated with changes in the strength of the polar night jet (Belmont et al., 1975; van Loon et al., 1973).

Although generally good agreement with observations and computations is indicated for the lower stratosphere, there is some indication that wave amplitudes predicted for the upper stratosphere and mesosphere are too large when compared with Green's (1972) observations. There is, of course, considerable room for further fine-tuning of the Newtonian cooling parameter and Rayleigh friction coefficient for this model, but until more data become available, we believe that the effort would not be worthwhile. We also note that the steady-state assumption used in this analysis might be questionable for the upper regions of the model as indicated by Matsuno's (1971) calculation.

An examination of the energetics of planetary waves shows that meridional heat transport by the wave across the vertical shear at the base of the jet supplies large amounts of available potential energy to the wave in the lower stratosphere. For both $m=1$ and $m=2$ solutions, this wave energy is converted to kinetic energy which flows southward along with energy supplied by lower boundary fluxes to the equatorial stratosphere where it is deposited at the critical level. Zonal momentum is transported northward by the wave against the meridional shear in the mean zonal wind which provides an additional sink for wave kinetic energy.

In the upper stratosphere and mesosphere for WM I the planetary wave energetics are quite different. Associated with the effects of the superposition of the $\Theta_3$ and $\Theta_5$ modes, southward mean zonal momentum transport by the planetary wave liberates kinetic energy from the meridional shear giving rise to an additional energy source which produces upward and downward energy fluxes in the polar stratosphere.
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