The Instability of a Forced Standing Wave in a Viscous Stratified Fluid:  
A Laboratory Analogue of the Quasi-Biennial Oscillation

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ABSTRACT

An experiment is described in which a standing internal wave is forced at the lower boundary of an annulus of salt-stratified water. At sufficiently large forcing amplitudes, the wave motion generates a strong mean azimuthal circulation which itself exhibits a long-period oscillation. Theoretical calculations, based on the wave-driven theory of the quasi-biennial oscillation of the tropical stratosphere (with suitable modifications), are performed and compared with the experimental results. Agreement is good and the study thus provides substantial confirmation of the fundamental principles of the theory.

1. Introduction

The major features of the quasi-biennial oscillation of zonal wind in the equatorial stratosphere were successfully explained by the theory of Holton and Lindzen (1972) as a consequence of the momentum transfer associated with upward propagating equatorial waves. Plumb (1977, hereafter referred to as P) studied in more detail a simpler model and showed that the phenomenon is not dependent for its existence on the special conditions of the equatorial stratosphere. Rather, the flow arises in the first instance from an instability process which requires only that there be at least two waves whose horizontal phase speeds are of opposite sign, nonzero wave dissipation and sufficiently small mean flow dissipation. This generality renders the phenomenon amenable to laboratory modeling.

The experiment described in this paper concerns the generation and structure of an oscillatory mean flow in an annulus of stratified fluid forced by a standing wave on the lower boundary. The standing wave may of course be regarded as the sum of two equal and opposite travelling waves; this is (neglecting for the moment geometrical differences) precisely the wave configuration discussed in P. In the laboratory, of course, the waves are dissipated by viscous processes rather than by radiative damping.

The real system is inevitably three-dimensional and therefore presents a difficulty in drawing a precise quantitative comparison between theory and experiment. Two-dimensional theory presents, under a WKB approximation, a problem which though nonlinear lends itself to straightforward numerical integration. The nonseparable nature of the three-dimensional problem precludes such a simple solution while full numerical modeling would be impracticable over the parameter range of interest. Therefore, the results for the two dimensional problem are used to assess the agreement between theory and experiment. While geometrical differences will of themselves introduce error factors of order unity, the fundamental nature of the solutions should not be in error. However, two processes which are a consequence of the finite radial scale of the experimental apparatus are important in the parameter range of interest and must be allowed for in the theory. These are radial diffusion of mean momentum and sidewall wave dissipation. The former is represented crudely as a Rayleigh friction acting on the mean flow while the latter effect is included in the determination of the wave structure.

Relevant theoretical results for the mean flow evolution when the waves are dissipated either by viscous diffusion or by Rayleigh friction/Newtonian cooling are presented in Section 2. For a given wavenumber and dissipation mechanism the flow characteristics depend on scaling factors for horizontal velocity, height and momentum flux, which may be derived from linear (no mean flow) theory. In particular, these factors may be used to define two independent dimensionless parameters $\Lambda_1 = T/T_1$ and $\Lambda_2 = T/T_2$, where $T$, $T_1$ and $T_2$ are, respectively, the time scales associated with wave driving, vertical mean momentum diffusion and Rayleigh friction. These two parameters completely determine the (dimensionless) mean flow structure.

The apparatus and experimental procedure are described in Section 3. In Section 4 linear theory is used to determine the structure of the forced waves in the apparatus for the case of no mean flow and hence relevant scaling factors and dimensionless parameters $\lambda_1$, $\lambda_2$ (analogous to $\Lambda_1$, $\Lambda_2$ in the aforementioned theory) are derived. These are used as a basis for the discussion in Section 5 of the experimental results and a comparison with theoretical predictions. The structure and
evolution of the oscillatory mean flow produced in cases of sufficiently small \((\lambda_1, \lambda_2)\) are in excellent qualitative agreement with theoretical results. Quantitative comparisons are drawn between theory and experiment for the stability transition \(i.e.,\) existence or otherwise of a discernable mean flow) and flow profiles. Agreement is reasonably good, given the limitations of the basic theoretical assumptions in the context of this experiment. These limitations are discussed in Section 6.

2. Two-dimensional theory

Plumb (1977) considered the problem of the interaction between the mean flow and waves of the form \(\text{Re}\{\exp[ik_n(\nu-c_n)t]\}\) propagating upward from below through a stratified, radiatively dissipative Boussinesq fluid of infinite depth. The wave-driven contribution to the mean flow acceleration is \(-\sum N\partial \mathcal{F}_n/\partial z\), where \(\mathcal{F}_n = \mathcal{F}_n/z\) are the wave momentum fluxes. Under a WKB approximation \(\mathcal{F}_n\) was derived by P as an integral function of the mean flow \(\bar{v}(\bar{z})\). Here we generalize this to allow for viscous wave dissipation. It is shown in Appendix A that the WKB result for the momentum flux of upward propagating waves is

\[
\mathcal{F}_n(z) = \mathcal{F}_n(0) \times \exp \left\{ - \int_0^z \frac{N}{k_n(\bar{v}-c_n)} \frac{N^3}{k_n(\bar{v}-c_n)} \frac{d\bar{z}}{d\bar{z}} \right\},
\]

(2.1)

where \(\mu\) is the Newtonian cooling rate, \(\nu\) the viscosity and \(N\) the buoyancy frequency. If \(\mu, \nu, N\) are independent of \(z\), \(k_1 = k_2 = k\) and \(c_1 = -c_2 = c\), then (2.1) may be written

\[
\mathcal{F}_n(\bar{z}) = \mathcal{F}_n(0) \times \exp \left\{ - \int_0^\bar{z} \frac{\alpha_1}{(V - C_n)^2} + \frac{\alpha_2}{(V - C_n)^2} \frac{d\bar{z}}{d\bar{z}} \right\},
\]

(2.2)

where \(\bar{z} = z/d\), \(C_n = c_n/c\), \(V = \bar{v}/c\), \(\alpha_1 = N\mu d/k c\), \(\alpha_2 = N\nu d/k c^2\), and \(d\) is a vertical scale which is defined as

\[
d = \left( \frac{N\mu}{k c^2} + \frac{N^3\nu}{k c^4} \right)^{-1},
\]

(2.3)

so that in the linear problem \((V = 0)\), \(\mathcal{F}_n \propto \epsilon^{-1}\).

The mean flow evolution equation is

\[
\frac{\partial \bar{v}}{\partial t} = - \sum \frac{\partial \mathcal{F}_n}{\partial z} + \nu' \frac{\partial \bar{v}}{\partial \bar{z}},
\]

(2.4)

where \(\nu'\) is the Rayleigh friction rate. We restrict attention to the case of equal amplitude waves (standing wave on \(\bar{z} = 0\)), so that \(\bar{F}_1(0) = -\bar{F}_2(0) = \frac{1}{2} \mathcal{F}_n\), \(C_1 = -C_2 = 1\) and define the time scale \(T = cd/F\).

Plumb solved numerically (2.2) and (2.5) for \(\alpha_2 = 0\) (no viscous wave damping) and \(\lambda_2 = 0\). Here we extend the calculations (using the same method) to allow nonzero \(\lambda_2\) and also investigate the case of wave damping by viscosity only \((\alpha_1 = 0, \alpha_2 = 1)\).

The boundary conditions are \(V(0) = 0\) and a free upper surface at \(\bar{z} = d\) where \(\partial V/\partial \bar{z} = 0\). Downward reflection from the upper surface is negligible and the depth of the fluid is thus effectively infinite. This being the case, for given \(\alpha_1\) and \(\alpha_2\) the only parameters determining the evolution of \(V(\bar{z}, \tau)\) are \(\lambda_1\) and \(\lambda_2\).

With no viscous damping, \(\alpha_1 = 1, \alpha_2 = 0\) and \(d = k c^2/N\mu\). P showed that the equilibrium solution of no mean motion is unstable if \(\lambda_1 < 0.112\) when \(\lambda_2 = 0\). Fig. 1 shows the stability curve \((a)\) in \((\lambda_1, \lambda_2)\) space.\(^3\) The curve was determined by integrating (2.2) and (2.5) at spot values of \(\lambda_1\) and \(\lambda_2\). In the region of the

\(^3\) Note the asymmetry with respect to the two dissipation parameters. Vertical diffusion acts through a shear layer of thickness \(O(\lambda_1)\) (Plumb, 1977) and the associated damping rate is \(O(\lambda_1^2)\). For Rayleigh friction, of course, the damping rate is proportional to \(\lambda_2\).
stability transition the resolution is approximately $(\delta \Lambda_1, \delta \Lambda_2) = (0.01, 0.1)$. Below the curve an oscillating mean flow develops; throughout the unstable region the flow structure and evolution are similar to the $\Lambda_2 = 0$ cases presented in Figs. 4b and 4c of P to which the reader is referred for illustration. The oscillation period is almost independent of dissipation, being approximately 14 for small $\Lambda_1$ and $\Lambda_2$ and increasing to about 16 close to the stability transition. The vertical extent of the oscillation decreases slowly with increasing $\Lambda_1$ and $\Lambda_2$.

The transition curve in the case of viscous damping only ($\alpha_1 = 0, \alpha_2 = 1, d = k c / N^2 \nu$) is shown in curve (b) of Fig. 1. The introduction of the stronger nonlinearity in (2.2) leads to a destabilization of the system as compared with curve (a). Indeed, by linearizing (2.2) in $V$ it is straightforward to show that if curve (a) is described by $\chi(\Lambda_1, \Lambda_2) = 0$ then the appropriate expression for curve (b) is $\chi(\frac{1}{2} \Lambda_1, \frac{1}{2} \Lambda_2) = 0$. Presumably for the same reason, the period of the mean flow oscillation found in the unstable region is shorter than in the previous example, varying from about 8 for small damping to about 10 near the transition. The structure of the oscillation (typical cases are shown in Figs. 10b and 11b) is very similar to that in the case of radiative damping, the only marked difference being an exaggerated skewness of the $V$ contours at low levels.

3. Apparatus and experimental procedure

Reference is made to Fig. 2. The experiments were conducted in a transparent cylindrical annulus 0.50 m high whose internal radii were 0.183 and 0.300 m. The annulus, its axis vertical, was mounted on a second shallow annulus of the same radial dimensions and separated from it by a thin rubber membrane. The lower annulus was divided into sixteen equal segments by radial dividers and the rubber membrane was sealed to these dividers to separate each of the segments from its neighbors.

As shown in the figure, alternate segments were connected by tubing to opposite sides of a broad piston driven through an adjustable stroke by a Scotch yoke whose crank was powered by an electronically controlled variable speed motor. The segments and the piston were completely filled with water so that the action of the piston was to force the bottom boundary into a fairly pure standing azimuthal wave of wavenumber $s = 8$, whose amplitude rose smoothly from zero at the inner and outer circumferences to a maximum of up to 10 mm at roughly mid-radius. Forcing frequencies were between $1/15$ and $1/30$ Hz.

The main annular chamber was filled to a depth of about 0.4 m with salt water linearly stratified to a buoyancy frequency $N$ of about $1.5 \, \text{s}^{-1}$. Except for the visualizations presented in Fig. 9, the internal motions were revealed by the movement of neutrally buoyant polystyrene spheres about 2 mm in diameteradrift in the water and illuminated from above by a 10 mm wide circumferential band of light at mid-radius. Cylindrical distortion was avoided by viewing the spheres through a water-filled plane window fixed to the outside of the annulus. Motions were analyzed from 16 mm ciné film records exposed at one frame every 2–4 s.

In operation the forcing began with the fluid in a state of rest. For sufficiently large amplitude forcing zonal mean circulations would appear spontaneously within an hour or two; this flow changed direction with a period of typically one hour.

After a day or so, the stratification within a few centimeters of the upper free surface would tend to disappear, probably as a result of evaporation-induced convection. Nonlinear wave action seemed to be responsible for a weakening of the stratification in the lowest few centimeters, but without much apparent effect on the interior motions or structure.

4. Analysis of data

a. Representation of imposed parameters

The variable imposed parameters of the system are $D$ (the fluid depth), $N$, $\omega$ and $\epsilon$ (the amplitude of the
membrane standing wave). In practice only ω and ε were varied over any significant range (D and N were made as large as practicable in order to minimize reflection from the upper surface; D was always close to 0.40 m, while N varied within the range 1.37–1.57 s−1). In reality, then, ω and ε are the variable parameters of the experiment. In order to facilitate a quantitative comparison with the theory of Section 2, we define two alternative independent parameters λ1 and λ2 which are the dimensionless measures of vertical and radial momentum diffusion relative to wave driving and therefore the experimental equivalents of Λ1 and Λ2.

The structure of linear internal waves in an annulus of viscous fluid is considered in Appendix B. From (B27) (transforming into dimensional quantities) we define wave amplitudes ξn such that the elevation of the forcing membrane is

\[ \xi(r,\psi,l) = \text{Re}\left[e^{i\psi}\sum \xi_n^2 \Pi_n(r/a)e^{-i\omega t}\right], \quad (4.1) \]

where the eigenfunctions \( \Pi_n(x) \) are defined in (B19). The first four of these are plotted in Fig. 3; the corresponding eigenvalues \( \kappa_n \) are listed in Table 1.

Now, assuming that each segment of the membrane is subjected to a uniform pressure difference and is linear in its response, it is shown in Appendix C that

\[ \zeta(r,\psi,l) = \epsilon \chi^B_t(r) \sin \phi \sin \omega t, \quad (4.2) \]

where \( \epsilon \) is the maximum elevation of the membrane. The function \( \chi^B_t(r) \) is shown in Fig. 4.

In terms of the eigenfunctions \( \Pi_n \),

\[ \chi^B_t(r) = \sum_n a_n \Pi_n(r/a), \quad (4.3) \]

where, using (B20),

\[ \alpha_n = \left[ 2/(b^2-a^2) \right] \int_a^b \chi^B(r) \Pi_n(r/a) dr. \quad (4.4) \]

The values of \( \alpha_n \) are listed in Table 1 for the first four modes. Separating (4.2) into its two component traveling modes, \( \zeta = \zeta^+ + \zeta^- \), we then find

\[ \zeta^\pm = \pm \frac{\epsilon}{2} \sum_n a_n \Pi_n(r/a)e^{i(\omega \pm \omega t)}. \quad (4.5) \]

The momentum flux associated with each mode is given in dimensionless form in (B28). In dimensional quantities, this becomes

\[ F_n^M(r, \phi) = \pm \frac{\epsilon \alpha_n^2}{8\kappa_n} \Pi_n(r/a)e^{-i\omega t} \], \quad (4.6) \]

where

\[ d_n^{-1} = \frac{\nu N^2 \kappa_n^3}{\omega^2 a^3} + \left( \frac{2 \nu}{\omega^2 a^3} \right) \frac{\epsilon^2 \theta_n N^3}{\kappa_n (b^2-a^2)} \]. \quad (4.7) \]

Here \( \theta_n \) is a number defined in (B26); its values for \( n \leq 3 \) are listed in Table 1. The two contributions to the attenuation rate in (4.7) are those from internal and sidewall viscous damping, respectively. We now define a momentum flux scale \( F_0 \) such that the flux at mid-channel, \( s=0 \), is

\[ F_n^M(\tfrac{1}{2}(a+b),0) = \pm \frac{\epsilon}{2} F_0, \quad (4.8) \]

![Fig. 4. The membrane shape function \( \chi_t(r) \) for \( b/a=1.64 \).](image)
so that, from (4.7),
\[
F_0 = \frac{1}{2} \varepsilon s N \omega \frac{2a}{\kappa_n} \left( \frac{a + b}{2a} \right) \Pi_s \left( \frac{a + b}{2a} \right).
\] (4.9)

From the values of \( \alpha_n^2 / \kappa_n \) in Table 1 we see that mode \( n = 0 \) carries over 80% of the momentum flux. Therefore, for simplicity other modes can be neglected without incurring serious error. Then with \( b/a = 1.64, \ s = 8, \ F_0 \) takes the values
\[
F_0 = 0.0896 N \omega \varepsilon^2.
\] (4.10)

We adopt this quantity as the momentum flux scaling. The height scale is \( d_0 \) defined in (4.7). The appropriate time scale is then
\[
T_0 = \omega d_0 / k F_0,
\] (4.11)
where \( k = 2s / (a+b) \). Then the experimental parameter equivalent to \( \lambda_1 \) of Section 2 is
\[
\lambda_1 = \frac{\nu T_0}{d_0^2}.
\] (4.12)

As a result of the dependence of \( d_0 \) on \( \omega \) in (4.7) the vertical diffusion rate \( \nu / d_0^2 \) varies markedly within the parameter range of interest. In the radial direction, however, we consider it unlikely that the corresponding rate shows such variation. While the details of the radial mean flow structure are beyond the scope of present theory and the relative importance of internal and boundary layer diffusion is not known, we note from (4.6) that linear theory predicts a forcing function whose radial structure is independent of the variable parameters of the experiment. To a first approximation the radial structure of the response should exhibit a similar lack of sensitivity. On this basis, we assume that radial diffusion of mean momentum may be represented by a Rayleigh friction parameter \( \nu \gamma / (b-a)^2 \) and that \( \gamma \) can reasonably be taken as a constant, to be determined experimentally. The appropriate dimensionless measure of radial dissipation is \( \gamma T_0 \) [cf. (2.6)],
\[
\lambda_2 = \nu \gamma / (b-a)^2.
\] (4.13)

b. Velocity measurements

The positions of selected neutrally buoyant spheres were extracted from the ciné film records (Section 3). The beads were assumed to be located at mid-radius. Because of the finite size of the illuminating beam an error of about \( \pm 5\% \) is introduced into radial position and hence also in the derived velocity measurements. By analyzing \( M \) frames of film (sufficient to cover one complete cycle of the forcing; \( M \) had a value of about 7) \( M-1 \) pairs of frames then gave \( M-1 \) sets of velocity measurements. The tracers were selected to give a uniform data distribution. About six beads were tracked within each 15 mm in the vertical; the velocities of these beads were averaged to give a mean flow velocity appropriate to that height band. The \( M-1 \) values thus obtained for each band were then averaged\(^4\) to give a final value for the mean velocity \( \bar{v} \) and a standard error. These had typical values of 5 mm s\(^{-1}\)

\(^4\)In view of the separation of time scales between wave period and mean flow period (typically 20 s vs 3000 s), the mean flow velocity could be considered constant over the time interval between the \( M \) successive frames.
Fig. 7. Plot of experimental parameters: forcing amplitude $\epsilon$ vs forcing frequency $\omega$. Buoyancy frequency is indicated by symbols: (○) $N=1.37$ s$^{-1}$, (△) $N=1.40$ s$^{-1}$, (△) $N=1.49$ s$^{-1}$, (△) $N=1.57$ s$^{-1}$. Open symbols represent cases with no observable mean flow generated within 4 h of commencement of forcing; closed symbols, mean flow appeared within 4 h.

and 0.5 mm s$^{-1}$, respectively. A representative flow profile is shown in Fig. 5. The process was repeated at time intervals frequent enough to resolve adequately the mean flow evolution.

The same technique was used in an attempt to measure the profile of momentum flux $F=\overline{\epsilon \psi}$ but in fact the errors proved to be so large that little more than the sign of $F$ and an upper limit to its magnitude could be deduced.\footnote{In fact (see below) it proved quite easy to determine the sign of $F$ by eye from the ciné film by noting the characteristic angle of the wave motion.} This was a result of our inability to measure vertical velocity to sufficient accuracy.

5. Results

a. Rayleigh friction coefficient

The coefficient $\gamma$ in (4.13) was determined by running an experiment in which a deep mean flow developed and then switching off the wave forcing (at $t=t_0$). The decay of the flow was then filmed and profiles obtained at $t-t_0=200$, 600, 1000 and 1400 s (the wave motion was judged by eye to have disappeared within less than 200 s). These profiles are shown in Fig. 6.

Neglecting advection by mean meridional circulations the mean flow evolves according to

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial z^2} + \frac{\nu \gamma}{(b-a)^2} \theta$$

on our assumptions. This equation was used (taking finite differences for derivatives) from the data of Fig. 6 to give $\gamma = 11.9 \pm 0.4$ (cf. $\gamma = \pi^2 = 9.87$ for a "half-sine" profile). The value 11.9 is henceforth adopted for $\gamma$; it is assumed that this value is representative throughout the parameter range of the experiments.

b. Regime determination

When the forcing amplitude $\epsilon$ was sufficiently large a mean streaming motion developed to observable magnitude usually within 1–2 h after commencement of the forcing. The existence (or otherwise) of such flow could in practice be determined simply by watching the excursions of the polystyrene spheres. Even quite close (in parameter space) to the transition between stability and instability of the wave motion, the maximum mean flow velocities were of the same order of magnitude and usually somewhat larger than the horizontal velocity amplitude of the wave motion. In some cases this judgement was checked and confirmed by ciné film observation. If after 4 h no mean current was observed it was assumed that none would develop. In practice the cutoff was quite sharp so that the assumption seemed sound. The results of this determination are shown in Fig. 7 as a function of the basic imposed parameters $\omega$, $\epsilon$ and $N$. Mean motion is found if $\epsilon$ exceeds a critical value which apparently decreases slightly as $\omega$ increases from 0.3 to 0.5 s$^{-1}$. The range of $\omega$ was restricted by our desire to maintain the analogy with the quasi-biennial oscillation, i.e., to confine attention to the range of parameters in which the foregoing theory might be expected to apply. For $\omega<0.3$ s$^{-1}$ the wave motions become nonlinear (at a value of $\epsilon$ required to generate mean flows); for $\omega>0.5$ s$^{-1}$ reflection from the upper surface becomes significant.

The data are replotted as a function of the parameters $\lambda_1$ and $\lambda_2$ in Fig. 8. Also shown for comparison are the theoretical transition curves from Section 2, drawn on the assumption that $\lambda_1 = \Lambda_1$, $\lambda_2 = \Lambda_2$. There is some

FIG. 8. Data of Fig. 7 plotted in ($\lambda_1, \lambda_2$) space. See Fig. 7 for definition of symbols. Also shown are the theoretical transition curves (a) and (b) of Fig. 1 on the assumption that $\Lambda_1 = \lambda_1$, $\Lambda_2 = \lambda_2$. Unauthenticated | Downloaded 07/31/22 03:20 AM UTC
Fig. 9. Visualization of the mean azimuthal circulation. Photographs are backlit radial elevations through the viewing window (see Fig. 2). Each frame is numbered in minutes from commencement of the forcing oscillation whose period is 18.5 s. For this experiment $N = 1.23 \, s^{-1}$, $D = 0.39 \, m$, $\varepsilon = 8.5 \, mm$ and $\omega = 0.34 \, s^{-1}$.

The tracers were produced by a vertical tellurium-coated wire cathode suspended at mid-radius and pulsed for 1 s at a fixed phase position in the forcing cycle. Before each photograph the wire was pulsed three times and the picture was taken during the fourth pulse. In some cases (particularly on the right of the frame at 170 min) fossil remnants of previous pulses appear as streaks dissociated from the wire.

Two complete cycles of the long period mean flow oscillation appear in the sequence. Although the vertical profile of mean horizontal motion is smooth, the tracers appear to be modulated in the vertical due to the presence of the small-scale forced wave. Even the details of the small-scale structure are reproduced after a complete cycle of the mean flow oscillation.
overall consistency between the experimental data and the theoretical prediction. At the smaller forcing frequencies \( \omega \) (in Fig. 8, this means small \( \lambda_2/\lambda_1 \)) sidewall wave dissipation becomes relatively small [the first term on the right-hand side of (4.7) is larger than the second by a factor of about 4], while it becomes comparable with internal dissipation at larger \( \omega \) (larger \( \lambda_2/\lambda_1 \)). However, at small \( \omega \) the data show no tendency toward the internal dissipation curve (b). To what extent sidewall dissipation would affect the theoretical transition curve cannot be deduced without recourse to a three-dimensional theory, but curves (a) and (b) illustrate the kind of variation introduced by different wave damping mechanisms. On the whole, in view of geometrical differences between theory and experiment, numerical discrepancies are not unreasonably large. Other shortcomings of the theory in the context of this set of experiments are discussed in the next section.

**c. Flow structure and evolution**

The structure in height and time of the generated mean motion was found to be in excellent qualitative agreement with the evolutionary profiles described in P and arising from the calculations of Section 2 of this paper. Fig. 9 shows a sequence of photographs of dye traces emitted from a vertical wire in mid-channel; this sequence illustrates the evolution of the motion over two complete cycles of the mean flow oscillation. The reproducibility even of small details in the motion from one cycle to the next is easily seen.

Quantitative measurements from the ciné film are shown for two particular examples in Figs. 10 and 11. Included for comparison are results of theoretical calculations with the same parameters (i.e., \( T = T_0, \ d = d_0, \ \lambda_1 = \lambda_1, \ \lambda_2 = \lambda_2 \)) and with wave damping arising from internal viscous dissipation only. In the two experimental cases the internal dissipation term in (4.7) contributes a fraction 0.65 (Fig. 10) and 0.76 (Fig. 11) of the total wave attenuation. The general form of the experimental profiles is clearly in good agreement with theory. In fact the similarity goes beyond the mean flow data. As the lowest positive or negative "jet" is being dissipated (e.g., at times labeled P, Q in Fig. 10), a marked increase in wave activity at higher levels (typically \( z \approx d_0 \)) is seen. The sign of the characteristic angle (\( \arctan \omega/\alpha \)) of this oscillation as seen visually on the film clearly indicated the wave of positive (negative) phase speed. This observation agrees with the theoretical prediction (Holton and Lindzen, 1972)—one that is fundamental to the theory of the quasi-biennial oscillation—that this wave will begin to penetrate to high levels as a result of the disappearance of the barrier to propagation represented by the lowest jet. Strong positive (negative) mean acceleration at the same level was seen to follow this increase in wave activity.
The case shown in Fig. 10 is a typical example of flow at relatively large $\omega$ and moderate $\varepsilon$. The numerical agreement between theory and experiment is good; the experimental oscillation period and vertical height scale are respectively about 1.2 and 0.8 of the theoretical values. The maximum observed value of $|\dot{\theta}|$ is 8.5 mm s$^{-1}$ (cf. 12.9 mm s$^{-1}$ from the theory), but the bottom 20 mm of the fluid could not be observed because of the structure of the viewing window. The theoretical maximum flow speed in the visible region is 10.0 mm s$^{-1}$. The vertical shears are somewhat weaker in the experiment than in the theory.

Agreement between theory and experiment deteriorates with increasing $\varepsilon$ and/or decreasing $\omega$. Fig. 11 illustrates this. As compared with the foregoing example the period and vertical scale of the oscillation and $|\dot{\theta}|_{\max}$ decrease as $\omega$ decreases. Such a trend is predicted by the theory but significant quantitative differences now appear. Theoretical results for the period and velocity amplitude $^6$ (and correspondingly, typical magnitudes of the vertical shear) differ from measured values by factors of about 2. While differences of this magnitude might not be unexpected, given the crudity of the comparison with a two-dimensional theory, it will be suggested below that the theory becomes inappropriate in this region of parameter space for reasons other than those introduced by geometrical differences.

6. Discussion and summary

Derivation of the quantities $\lambda_1$, $\lambda_2$ and the height and time scales $d_0$ and $T_0$ appropriate to the experiment allowed comparison with a purely two-dimensional theory in which the flow characteristics are determined by analogous parameters. Partly because of geometrical differences the basis of the comparison is imperfect. With this proviso, agreement between theory and experiment is good. Where quantitative differences (which may be as great as a factor of 2) do occur, these may arise at least in part from other theoretical inadequacies. This is particularly true at the smaller values of $\omega$, when wave attenuation is dominated by internal viscous diffusion and so, from (4.7), $d_0=\omega^2/\nu N^2 k_0^2$. Now one of the assumptions underlying the WKB analysis is that the Richardson number $Ri=N^2/(\partial \Omega/\partial z)^2$ is large. The appropriate scale for this quantity is $N^3d/\partial^2 \approx \omega^2$. At the smallest values of $\omega$ ($\sim 0.3$ s$^{-1}$), this scale is of order unity or slightly smaller. An additional requirement of the theory is that $d_0 > m_0^{-1}$, where $m_0=Kn/\omega$ is the vertical wavenumber. This inequality also breaks down at the smaller values of $\omega$.

Nonlinearity (other than wave-mean flow interaction), of which a measure is $em_0$, also becomes important for small $\omega$ (i.e., large $m_0$) and large $\varepsilon$. Indeed in this region of parameter space some weak turbulence could be seen on the ciné film (the turbulence could also be in part a consequence of small $Ri$ for such cases). Two consequences of turbulence would be to increase the effective viscosity and to decrease the amplitude of the coherent wave motion, thus increasing the effective values of $\lambda_1$ and $\lambda_2$ above their calculated values, consistent with the discrepancies noted in Section 5.

On the whole, however, these comments do not detract from the main results of the study. The successful generation of oscillatory mean flows in these experi-
ments (and the observed structure of such flows) substantially vindicates the wave-driven theory of the quasi-biennial oscillation (Holton and Lindzen, 1972; Plumb, 1977). As a consequence the work provides indirect confirmation of the basic principles of wave-mean flow interaction on which that theory is based, and which have not hitherto been subjected to extensive experimental investigation.

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**APPENDIX A**

**WKB Theory for Damped Internal Waves in Shear Flow**

The assumptions on which the analysis is based are listed in Plumb (1977). We consider the propagation of a disturbance in \((y,z)\) space through a mean flow \(\bar{v}(z)\), the overbar denoting \(y\) averages and primes deviations from that average. The vorticity and buoyancy equations for the perturbations are

\[
\frac{\partial}{\partial t} \left( \nabla \psi \right) + \frac{1}{\bar{v}} \left( \nabla \bar{v} \right) = \frac{\partial^2 \psi}{\partial y^2} + \bar{v} \left( \nabla \bar{v} \right) \tag{A1}
\]

\[
\frac{\partial^2 \psi}{\partial t^2} - \frac{\bar{v}}{\bar{v}} \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial \psi}{\partial y} \tag{A2}
\]

where \(\psi\) is the streamfunction with \(\psi' = -\partial \psi' / \partial y\), \(w = \partial \psi' / \partial z\); \(N\) is assumed constant. For waves of the form \(\text{Re}[\psi e^{i(kz - \omega t)}]\), Eqs. (A1) and (A2) become

\[
\frac{k^2}{k \bar{v} (\bar{v} - c)} \left[ \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial y^2} \right] + \frac{1}{k \bar{v} (\bar{v} - c)} \left[ \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial y^2} \right] \psi = 0. \tag{A3}
\]

We let \(H\) and \(c\) be the scales for \(z\) and \(\bar{v}\), respectively, and assume that the Richardson number \(RI = N^3 H^2 / c^2\) is large. Further, we write

\[
\begin{align*}
\frac{\mu}{k c} = a_1 \frac{RI^{-1}}{k c} \\
\frac{\nu N^2}{k c^3} = a_2 \frac{RI^{-1}}{k c^3}
\end{align*} \tag{A4}
\]

where \(a_1\) and \(a_2\) are assumed to be of order unity or less. We seek solutions

\[
\ln \psi(z) = \frac{RI}{N} \int (\phi_1 + \phi_2 + \cdots) \, dz. \tag{A5}
\]

Then (A3) yields to leading order

\[
\phi_0 = \pm \frac{i}{\left[ (\bar{v}/c) - 1 \right]} \tag{A6}
\]

where the positive solution represents the upward propagating wave. At next order

\[
-2 \phi_1 = -\left( \ln \phi_0 \right) + \frac{1}{\left[ (\bar{v}/c) - 1 \right]} \left[ a_1 + \frac{a_2}{\left[ (\bar{v}/c) - 1 \right]} \right]. \tag{A7}
\]

To sufficient accuracy, then,

\[
\psi(z) = \frac{C}{\left[ (\bar{v}/c) - 1 \right]} \exp \left( \frac{RI}{2 \left[ (\bar{v}/c) - 1 \right]} \left[ a_1 + a_2 \right] \right) \tag{A8}
\]

where \(C\) is a constant. The momentum flux is

\[
F = \frac{i k}{4} \left( \psi + \psi^* \right) \frac{\partial \psi}{\partial z} \tag{A9}
\]

(1) to leading order. In dimensional terms, this equation yields (2.1).

**APPENDIX B**

**Linear Internal Waves in a Cylindrical Annulus**

We consider the problem of linear internal waves in a cylindrical annulus \((a \leq r \leq b)\) of stratified fluid of buoyancy \(N\) subject both to internal and sidewall viscous dissipation. If radial momentum diffusion is negligible in the fluid interior, then the linearized Boussinesq equations of motion are

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\rho g \sigma}{\rho_0} \frac{\partial \psi}{\partial z} \\
\frac{\partial v}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\rho g \sigma}{\rho_0} \frac{\partial \psi}{\partial z} \\
\frac{\partial w}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\rho g \sigma}{\rho_0} \frac{\partial \psi}{\partial z}
\end{align*} \tag{B1}
\]

where \(\sigma = g \rho \sigma / \rho_0\) and the other symbols have their usual meanings, and the continuity and buoyancy equations are

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r u \right) + \frac{\partial v}{\partial r} &= 0 \\
\frac{\partial v}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r u \right) + \frac{\partial w}{\partial r} &= 0 \\
\frac{\partial w}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r w \right) &= 0
\end{align*} \tag{B2}
\]

\[
\begin{align*}
\frac{\partial \sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \sigma \right) &= 0 \\
\frac{\partial \psi}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \psi \right) &= 0
\end{align*} \tag{B3}
\]

\[\text{We shall be concerned with waves whose vertical scale is much less than } (b-a).\]
Table 2: Scaling parameters for the problem of linear internal waves in a cylindrical annulus.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>a</td>
</tr>
<tr>
<td>z</td>
<td>( \omega a/N )</td>
</tr>
<tr>
<td>t</td>
<td>( \omega^{-1} )</td>
</tr>
<tr>
<td>( u, v )</td>
<td>( \omega a/s )</td>
</tr>
<tr>
<td>w</td>
<td>( \omega a^2/s )</td>
</tr>
<tr>
<td>\sigma</td>
<td>( \omega a N/s )</td>
</tr>
</tbody>
</table>

We shall seek solutions of the form \( e^{i(\sigma-\omega t)} \).

In thin sidewall boundary layers, near \( r = a \),

\[
\begin{align*}
\frac{\partial}{\partial t} (v_i - v) = & -\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial r^2} \\
\frac{\partial}{\partial r} (w_i - w) & + N^2 (w_i - w) = -\frac{\partial^2 w}{\partial r^2 \partial t} 
\end{align*}
\]

(McEwan, 1971), where \( v_i \) and \( w_i \) are the interior solutions as \( r \rightarrow a \). Writing

\[
v = v_i [1 - \eta(r-a)], \\
w = w_i [1 - \xi(r-a)],
\]

then \( \eta(0) = \xi(0) = 1 \) and (B4) has solutions which vanish as \( r \rightarrow a \infty \) of the form

\[
\begin{align*}
\eta(r-a) & = \exp[-(r-a)(-i\omega/v)^{\frac{1}{2}}] \\
\xi(r-a) & = \exp[-(r-a)(-i(N^2-\omega^2)/r)^{\frac{1}{2}}]
\end{align*}
\]

The continuity equation, within the thin boundary layer approximation, integrates to give

\[
u(r-a) \rightarrow \infty = u_i(a) = -\int_a^\infty \left( \frac{isv}{r} + \frac{\partial w}{\partial \xi} \right) dr. \tag{B6}
\]

Now, in the parameter range of interest, the ratio

\[
\epsilon = \omega^2/N^2 \tag{B7}
\]

is small (typically 0.1 in the experiments). In this case the thicker \( v \) layer makes the dominant contribution to (A6) and

\[
u_i(a) = \frac{is}{a} \left( \frac{\nu}{2\omega} \right)^{\frac{1}{2}} v_i(a). \tag{B8}
\]

Similarly, the condition at the outer wall is

\[
u_i(b) = \frac{is}{b} \left( \frac{\nu}{2\omega} \right)^{\frac{1}{2}} v_i(b). \tag{B9}
\]

Eqs. (B1)–(B3) become

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial r} + \frac{\partial^2 w}{\partial \xi^2} & = \frac{\epsilon}{a} \left( \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial \xi^2} \right) \\
\frac{\partial v}{\partial r} + \frac{\partial w}{\partial \xi} & = -\frac{1}{\xi} \left( \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial \xi^2} \right)
\end{align*}
\]

where

\[
\nu N^2/\omega a^2 = \alpha \epsilon. \tag{B11}
\]

The term \( \alpha \) is of order unity in the experiments. The boundary conditions (B8) and (B9) become

\[
\begin{align*}
u(1) & = -\left( 1-i \right) \beta \epsilon v(1) \\
u(-1) & = \left( 1-i \right) \beta \epsilon u(-1)
\end{align*}
\]

where

\[
\beta = \left( \frac{\nu}{\epsilon \omega a^2} \right)^{\frac{1}{2}} = \left( \frac{\alpha}{2} \right). \tag{B13}
\]

Now we seek solutions to Eqs. (B10) in the form

\[
\pi = \text{Re} \left\{ \pi_0(r,s,Z) + \epsilon \left( r,s,Z \cdots \right) e^{i(\sigma-\omega t)} \right\}, \tag{B14}
\]

where \( Z = \epsilon \xi \). To leading order this gives

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \pi_0}{\partial r} \right) - \frac{\sigma^2 \pi_0}{r^2} - \frac{\sigma \partial^2 \pi_0}{\partial \xi^2} = 0 \tag{B15}
\]

and the boundary conditions

\[
\frac{\partial}{\partial r} \left( \pi_0(1) \right) = \frac{\partial}{\partial r} \left( \frac{b}{a} \right) = 0. \tag{B16}
\]

Separation of variables yields

\[
\pi_0 = \rho_n(Z) \Pi_n(r) \exp(\pm i \kappa_\alpha \phi), \tag{B17}
\]

where

\[
\left[ -\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) + \left( \kappa_n^2 - \frac{s^2}{r^2} \right) \right] \Pi_n(r) = 0. \tag{B18}
\]

This equation has the solution

\[
\Pi_n(r) = C_n \left[ J_n(\kappa_n r) + \Gamma_n Y_n(\kappa_n r) \right], \tag{B19}
\]

where \( J_n \) and \( Y_n \) are Bessel functions of the first and second kinds and where \( \Gamma_n \) and \( \kappa_n \) are evaluated by use
of the boundary conditions (B12). $C_n$ is defined by the normalization condition
\[
\int_1^{b/a} \Pi_n^2(r) dr = \frac{1}{2a^2} \left( \frac{b^2}{a^2} - 1 \right).
\]  
(B20)
The first four eigenfunctions for $b/a = 1.64$ are shown in Fig. 3; Table 1 lists the corresponding eigenvalues.
At $O(\varepsilon)$ Eqs. (B10) give
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \pi}{\partial r} \right) - \frac{s^2 \pi}{r^2} \frac{\partial^2 \pi}{\partial z^2} = \sum \iota \kappa_n \left[ \frac{d \rho_n}{dZ} + (i \kappa_n + \kappa_n^2) \rho_n \right] \Pi_n(r) e^{i\kappa_n z},
\]  
(B21)
with the boundary conditions
\[
\begin{aligned}
\frac{\partial \pi}{\partial r}(1) &= - (1 + i) s^2 \rho_n(Z) \Pi_n(1) e^{i\kappa_n Z} \\
\left( \frac{b}{a} \right)^2 &\frac{\partial \pi}{\partial r} \left( \frac{b}{a} \right) = (1 + i) s^2 \left( \frac{b}{a} \right)^2 \rho_n(Z) \Pi_n \left( \frac{b}{a} \right) e^{i\kappa_n Z}
\end{aligned}
\]  
(B22)
Since, using (2.18),
\[
\int_1^{b/a} \Pi_n(r) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \pi}{\partial r} \right) + \frac{(\kappa_n^2 - s^2) \pi}{r^2} \right] rdr = \left[ \frac{\partial \pi}{\partial r} \Pi_n \right]_{r=1}^{b/a},
\]  
(B23)
removal of secularities in (B21) (multiplying by $\Pi_n$ and integrating over $r$ and $z$) gives, using (B22),
\[
\rho_n \propto \exp(-q_n Z),
\]  
(B24)
where
\[
q_n = \frac{1}{2} (i \kappa_n + \kappa_n^3) + \frac{(1 - i) s^2 \theta_n}{\kappa_n \left( (b^2/a^2) - 1 \right)}
\]  
(B25)
\[
\theta_n = \frac{a}{b} \Pi_n \left( \frac{b}{a} \right) + \Pi_n \left( 1 \right).
\]  
(B26)
The solution has now been determined to sufficient accuracy. The complete solution for the vertical particle displacement $\zeta$ is
\[
\zeta = \sum \xi_n \Pi_n(r) \exp(i \kappa_n z - q_n Z) e^{i(\phi - \iota)},
\]  
(B27)
where
\[
\xi_n = \mathcal{N} \exp \left[ -4a^2/(n^2 s^2 - 4) \right] \kappa_n^2(r).
\]  
(C5)
Provided $n s^2 \neq 2$ (which is always satisfied if $s > 1$), Eq. (C4) subject to the boundary conditions (C2) has the solution
\[
\kappa_n^2(r) = \frac{1}{\left| \frac{b}{a} \right|^2 - 1} \left[ \left\{ \left( \frac{b}{a} \right)^{n+2} - 1 \right\} \left( \frac{b}{a} \right)^{n-2} - \left( \frac{b}{a} \right)^{n+2} \right] - 1
\]  
(C6)
The momentum flux associated with the $n$th mode is $F_n = v_n w_n$ which, using (B10) and (B27) is
\[
F_n(r, Z) = \frac{s^2 \kappa_n}{2} \Pi_n^2(r) \exp(-Z/D_n)
\]  
(B28)
where the attenuation rate
\[
D_n = 2 \Re(q_n) = \alpha \kappa_n^3 + \frac{2 s^2 \beta \theta_n}{\kappa_n \left( (b^2/a^2) - 1 \right)}
\]  
(B29)
is the sum of the contributions from interior and side-wall dissipation, respectively. The dimensional forms of (B27)–(B29) are quoted in Section 4.

**APPENDIX C**

**Theoretical Membrane Shape**

If each sector of the membrane is subjected to uniform pressure and is linear in its response, then the membrane displacement $\xi = \text{constant} \times \xi$, where
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \xi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \xi}{\partial \phi^2} = 0,
\]  
(C1)
with the boundary conditions
\[
\xi(a, \phi) = \xi(b, \phi) = \xi(r, 0) = \xi(r, \pi) = 0.
\]  
(C2)
We look for a solution to (C1) in terms of a series
\[
\xi = \sum_n \xi_n(r) \sin(n \pi \phi),
\]  
(C3)
where $n$ is an odd positive integer. Then
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d \xi_n}{dr} \right) - \frac{n^2 \pi^2}{r^2} \xi_n = \frac{4}{n \pi}
\]  
(C4)
In view of the dependence of $\xi_n$ on $n$ in (C5) ($\propto 1/n^3$ for $ns \gg 4$) it is permissible to neglect all terms in (C3) except $n=1$; with $s=8$ the error thereby incurred is of the order of $10^{-8}$. Therefore we may write

$$\zeta = \epsilon \chi_1^i(r) \sin s\phi$$

for the maximum membrane displacement where $\chi_1^i$ is proportional to $\chi_1^i(r)$ (renormalized such that the maximum value of the former is unity) and $\epsilon$ is the maximum membrane elevation. The function $\chi_1^3(r)$ is plotted in Fig. 4.

REFERENCES

