The Transport of Conservative Trace Gases by Planetary Waves

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ABSTRACT

An analytical model for the horizontal and vertical transport of a conservative trace gas by a stationary planetary wave is developed. The wave is associated with a constant background zonal flow. In the absence of dissipation, transports are zero if the wave is trapped, i.e., the background flow is easterly or strong westerly. Propagating waves can carry out appreciable transports. The vertical transport is proportional to the latitudinal gradient of the background mixing ratio and the horizontal transport is proportional to the vertical gradient.

With dissipation in the form of Newtonian cooling, transports are permitted by trapped waves. For strong westerlies, they are very small but for easterlies they can be appreciable. In fact, substantial southward ozone transports in the summertime lower stratosphere are suggested.

The simplicity of the flux expressions suggests that they might be exploited in two-dimensional models of ozone.

1. Introduction

A simple but powerful theorem concerning wave transports follows from the work of Taylor (1915) and Bretherton (1966). It states that the mean horizontal wave transport is zero for some conservative quantity attached to a set of particles all of which deviate in position little from a reference latitude at which that quantity is the same for each particle. The theorem is fairly general in that the conditions for it to be valid are only that the waves are neither growing nor decaying with time, there is no dissipation, and critical levels do not exist where the phase speed matches the background mean wind speed.

The ultralong, quasi-stationary stratospheric waves for which these conditions approximately apply should transport little ozone according to the theorem. Yet we have ample evidence from numerical models of substantial ozone transport especially during winter (Hunt and Manabe, 1968; Clark, 1970; Mahlam, 1975; Cunnold et al., 1975).

A closer examination of the Taylor-Bretherton theorem reveals another implicit assumption in that the displacement of the parcels from their reference latitude is horizontal. Whereas this might be true for Rossby waves for which vertical motions can be neglected, it is not in general a valid assumption for planetary waves.

This paper examines the problem of the transport of a conservative trace species, like ozone in the lower stratosphere, by planetary waves. The approach will be different from the Lagrangian one of Taylor and Bretherton in that it will be Eulerian. Nevertheless, the results are analogous. If their approach were taken, the parcel displacements would have vertical as well as horizontal components.

Dissipation relaxes the constraints of the transport theorem. Transports will be considered with and without Newtonian cooling in the case considered here to determine whether significant transports can occur in situations where none should exist under the adiabatic assumption.

2. Theory

We consider the transport of a conservative trace gas by stationary planetary waves on a midlatitude β-plane. The background atmosphere will have the following properties: 1) it is isothermal with a scale height $H = 7.6$ km; 2) it has a constant zonal flow $U$; and 3) it has constant vertical and north–south gradients of trace constituent mixing ratio, $\Phi$, and $\Phi_\beta$, respectively. The dynamics are quasi-geostrophic and governed in part by the vorticity equation (Clark, 1975)

$$\frac{d}{dt} - V \nabla h + \beta h_z \frac{f \zeta}{g} \left( \frac{w}{H} \right) = 0, \quad (1)$$

where the subscripts denote partial derivatives. The vertical coordinate is $z = H \ln(p_0/p)$, $p_0$ being a reference pressure, $V$ is the horizontal Laplacian, $f_0$ the Coriolis parameter, $\beta = \frac{df}{dy}$, $g = 9.8 \text{ m s}^{-2}$, $h$ is the height of a constant $z$ surface, and $w = dz/dt$.

All fields associated with the disturbance are assumed time-independent and also independent of the north–south coordinate $y$. Thus the linearized substantial
derivative is
\[
\frac{d}{dt} = \frac{\partial}{\partial x}.
\]
(2)

The thermodynamics equation, allowing for dissipation in the form of Newtonian cooling, is
\[
\frac{d}{dt} \left( \frac{\partial h}{\partial x} \right) + \frac{S_\nu}{H^2} \frac{\partial h}{\partial z} = -\alpha \frac{\partial h}{\partial z},
\]
(3)
where \( S = \kappa H (\kappa = 2/7) \) and \( \alpha \) is the Newtonian cooling coefficient.

After \( w \) is eliminated between Eqs. (1) and (3), the potential vorticity equation follows:
\[
\frac{d}{dt} \left( \nabla^2 h + \frac{f_0^2 H^2}{gS} \left( \frac{h_\nu}{H} - \frac{h_z}{H} \right) \right) + \beta h_x = \frac{-c_0 f_0^2 H^2}{gS} \left( \frac{h_\nu}{H} - \frac{h_z}{H} \right).
\]
(4)

All the disturbance fields will vary exponentially in the \( x \) and \( z \) directions, i.e.,
\[
h = h_0 \exp \left[ i(mz + kx) \right],
\]
where \( k \) is related to the integral number \( n \) of wavelengths around a latitude at latitude \( \theta \) by
\[
k = n/\left( \frac{2\pi \cos \theta}{k} \right),
\]
if \( r \) is the earth’s radius. The vertical wavenumber \( m \) is in general complex. Substituting (5) into (4) and linearizing, it can be shown that
\[
mH = -0.5i + \left[ -0.25 \frac{gk^2 S(U-\beta/k^2)^{-1}}{f_0^2 (U-\alpha k)} \right].
\]
(7)

If the adiabatic assumption is made, i.e., \( \alpha = 0 \), \( m \) is imaginary and the planetary wave is trapped if \( U < 0 \) or \( U \geq U_{CR} \), where
\[
U_{CR} = \frac{\beta}{f_0^2} \left( \frac{k^2 + \frac{f_0^2}{4gS}}{-\frac{g}{f_0^2} (U-\alpha k)} \right)^{-1}.
\]
(8)

If \( n = 1 \), \( U_{CR} = 94 \text{ m s}^{-1} \) and if \( n = 2 \), \( U_{CR} = 50 \text{ m s}^{-1} \).

For \( 0 < U < U_{CR} \), \( m \) is complex and the wave can propagate energy vertically.

The trace constituent which is to be transported by these waves is assumed to be conservative. If the background mixing ratio is denoted by \( \Phi \) and the perturbation mixing ratio by \( \phi \), the linearized conservation equation is
\[
\frac{\partial \phi}{\partial x} + v \phi_x + w \phi_z = 0,
\]
where \( v \) is geostrophic, i.e.,
\[
v = \frac{g}{f_0} \frac{\partial h}{\partial x},
\]
(10)

If \( \phi \) has the same exponential dependence on \( x \) as in Eq. (5), it can be expressed in terms of \( h \) from Eqs. (3), (9) and (10), i.e.,
\[
\phi = \frac{H}{\kappa} \left[ \int \left( 1 - \frac{i \alpha k}{f_0 U H} \right) \phi_x + \frac{g \kappa}{f_0 U H} \phi_z \right] h.
\]
(11)

This relation is good if \( U \neq 0 \) since Eq. (9) was solved for \( \phi \) under this assumption.

We define the horizontal and vertical fluxes of the trace species as
\[
\overline{F_H} = \overline{\nu \phi},
\]
(12a)
\[
\overline{F_V} = \overline{w \phi}
\]
(12b)
respectively, where the overbar represents an average over one east–west wavelength. It is understood that the real part of the quantities must be taken before the averaging is done. After Eqs. (3), (10) and (11) are used for \( w, v \) and \( \phi \), respectively, the fluxes can be written
\[
\overline{F_H} = \phi_0 H \left( m_r + \frac{\alpha}{k U} m_i \right) \Phi_x,
\]
(13a)
\[
\overline{F_V} = - \phi_0 H \left( m_r + \frac{\alpha}{k U} m_i \right) \Phi_x,
\]
(13b)
where
\[
\phi_0 = \frac{g |h|^2}{2 f_0 \nu k \cos \theta} e^{-2m_2 z}.
\]
(14)

The symbols \( m_r \) and \( m_i \) represent the real and imaginary parts of \( m \), respectively.

\[\text{Fig. 1. Relation between total flux vector } \mathbf{F} \text{ and gradient vector } \nabla \phi.\]
If a flux vector is defined as
\[ F = \hat{j} F_H + \hat{k} F_V, \]
where \( \hat{j} \) and \( \hat{k} \) are in the \( y \) and \( z \) directions, respectively, then
\[ F = F_{\text{onH}} \left( m_r + \frac{\alpha}{kU} m_s \right) \nabla \Phi \times \hat{1}, \]
\( \hat{1} \) being in the \( x \) direction.

Thus a simple relation between fluxes and gradients results; the horizontal flux is proportional to the vertical gradient and the vertical flux proportional to the horizontal gradient. Clearly, the horizontal flux may be either up or down the horizontal gradient; in fact, it is independent of it. The total flux vector is actually parallel to lines of constant mixing ratio as illustrated in Fig. 1.

The three critical factors determining \( F \) are 1) \( |\hat{k}| \Phi e^{-2mz} \) which is the amplitude squared of the stationary wave doing the transporting (the factor \( e^{-2mz} \) gives the dependence on \( z \)); 2) the background mixing ratio gradients \( \Phi_r \) and \( \Phi_z \); and 3) the \( \epsilon \) factor.

\[ \epsilon = nH \left( m_r + \alpha m_s / kU \right). \]

The factor \( \epsilon \) is shown as a function of \( U \) with and without dissipation in Figs. 2 and 3 for \( n = 1 \) and \( n = 2 \), respectively. Without dissipation a planetary wave can transport a trace species only if it is propagating, i.e., \( 0 < U < U_{\text{CR}} \). Otherwise the velocity fields are 90° out of phase with the mixing ratio field and fluxes are zero. For a westerly background wind the effect of dissipation is small. The change in \( \epsilon \) is never more than 15%. The calculations shown are for a dissipation time scale of 5 days—typical for the lower stratosphere (Blake and Lindzen, 1973). The dissipation does relax the constraints on transport somewhat, in that a slight transport is permitted for \( U > U_{\text{CR}} \).

Dramatic changes in the transport occur with an easterly background flow. Appreciable transports are now possible whereas none are permitted under adiabatic conditions. For a trace species whose mixing ratio increases with height as ozone in the lower stratosphere an appreciable southward transport might be allowed by a stationary, trapped planetary wave. Because of the factor \( \exp(-2mz) \), the wave amplitude squared and thus the transport decays slowly with height. For instance if \( U = -10 \text{ m s}^{-1} \), \( nH = 0.417 \) and the \( \epsilon \) folding depth for amplitude is about 18 km.

For a westerly background flow, a northward transport of O\(_3\) will occur for \( \Phi_z > 0 \). If \( U < U_{\text{CR}} \), \( nH = -0.4 \) for the adiabatic case and the wave amplitude grows with \( z \) as \( \exp(z/2H) \). Thus the northward transport grows with height. The presence of dissipation will decrease the transport somewhat at all heights.

The above prediction of the flux being parallel to lines of constant \( \Phi \) seems to be verified fairly well by the limited observations available. For instance the heat fluxes computed by Reed and German (1965) are reasonably close to being parallel to the mean isentropes. Note that dissipation as included in the above equations will not affect this relationship. However, if the trace species were not conservative, as for example O\(_3\) when photochemistry is important, the flux vector would not be parallel to the mean mixing ratio lines.

3. Conclusion

The above model demonstrates that horizontal planetary wave transports are zero only under very special conditions above and beyond those stated in the introduction. They are either (i) \( \Phi_z \) is zero; (ii) \( w \) is zero, i.e., parcel displacements are horizontal; or (iii) \( w \) is in phase or 180° out of phase with \( v \).

In the particular case of planetary waves on a mid-latitude \( \beta \) plane condition (iii) is satisfied if the wave is trapped, i.e., the background flow is easterly or westerly and above the critical value \( U_{\text{CR}} \) in Eq. (8). Propagating waves can have appreciable horizontal and vertical transports. The transport vector is parallel

![Diagram](Unauthenticated | Downloaded 07/31/22 06:57 PM UTC)
to lines of constant background mixing ratio. Dissipation in the form of Newtonian cooling affects the magnitude of the transport but not its direction.

In the lower stratosphere where O₃ is very close to being a conservative tracer, the wave transport will be directed northward and downward and thus the horizontal transport is countergradient. Wallace (1978) presents a lucid explanation of how the effect of this flux on the mean O₃ distribution is equivalent to the effect of the Stokes drift of parcels passing through the wave pattern. He also shows how a countergradient flux can be maintained through an upward flux of wave energy into the lower stratosphere and the ensuing deformation of constant mixing ratio surfaces.

The planetary-wave transport theory presented here is incomplete in that transports associated with the secondary mean meridional circulation driven by wave heat and momentum fluxes (Kuo, 1956) are ignored. Assuming adiabatic motions and the absence of critical levels where the background wind speed matches the wave speed, Dickinson (1969) found that for quasi-geostrophic motions the mean meridional vertical and north–south velocity components are proportional to the north–south gradients of the heat and momentum transports respectively. In the present model, the momentum transport is zero and the heat transport is independent of north–south distance. Thus, a secondary circulation does not exist. In a more realistic model such a circulation will certainly occur. Andrews and McIntyre (1976) demonstrate, however, that in the absence of critical levels, wave transience and dissipation, transports of a conservative tracer by the mean meridional circulation exactly cancel the wave transports. Even in realistic numerical models (e.g., Hunt and Manabe, 1968; Clark, 1970), where none of these restrictions apply, there is a strong tendency for cancellation between the wave and mean-meridional transports.

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REFERENCES


