Interactions Between Orographically and Thermally Forced Planetary Waves

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(Manuscript received 2 November 1982, in final form 11 January 1983)

ABSTRACT

A comprehensive analysis has been made of the atmospheric planetary wave response to orographic and thermal forcing in mid-latitudes using a simple model. Vertical heating profiles with maxima at the surface and in the mid-troposphere are considered. The model is quasi-geostrophic on a beta-plane, and has a constant zonal mean basic state wind. With these simplifications it is possible to obtain complete analytic solutions, not only for the wave response with and without Ekman pumping, but also for the secondary effects of the waves on the zonal mean flow. The presence of diabatic heating in the waves results in significant non-zero Eliassen-Palm fluxes and violates conditions for non-acceleration of the zonal mean flow, both for propagating and trapped waves. The potential vorticity transport or, alternatively, the Eliassen-Palm flux divergence is shown to be directly related to the vertical heating profile. However, it is the interaction between orographic and thermally forced waves that is mainly responsible for changes in the zonal mean flow, and the results therefore strongly depend upon the relative phase of the thermal and orographic forcing.

At large heights, remote from the heating, it is shown that it is possible to choose an equivalent mountain that would produce the same response in the planetary waves as the thermal forcing. The equivalent mountain height varies inversely as zonal wavenumber. In addition, the mid-tropospheric heating profile produces a wave with 4-5 times the amplitude of the wave response to the surface heating profile with the same vertically integrated total heating. Consequently it is mainly in zonal waves 1 and 2 where a mid-tropospheric thermally forced wave can dominate or be comparable to the orographic waves.

1. Introduction

It has long been recognized that the stationary component of the atmospheric circulation is linked to the distribution of land and sea, and the large-scale orographic and thermal forcing of the atmosphere. Differences between the circulation in the Northern and Southern Hemispheres make this abundantly clear. However, a continuing debate beginning in the work of Charney and Eliassen (1949) and Smagorinsky (1953) has been the relative roles of orography and diabatic heating in forcing the stationary waves in the Northern Hemisphere.

Simplified models of the atmosphere have been used to consider the effects of mountains and/or heating by Saltzman (1965), Sankar-Rao (1965), Murakami (1967), Sankar-Rao and Saltzman (1969), De Reme and Wiin-Nielsen (1971), Egger (1976a,b), Vernèkar and Chang (1978), Ashe (1979), Grose and Hoskins (1979), Lin (1982) and Opsteegh and Vernèkar (1982). Controlled experiments have also been made with general circulation models whereby runs with and without mountains are compared (Kasahara et al., 1973; Manabe and Terpstra, 1974; Held, 1983). Hayashi and Golder (1977) further analyzed such model output. More complete reviews of the literature are given by Saltzman (1968), Kasahara (1980) and Dickinson (1980).

As indicated above, the results from all these studies have been mixed, most probably because of inadequate specifications of the thermal forcing. The orographic component seems to occur on a smaller scale. However, on balance for the Northern Hemisphere in winter, the weight of evidence from observational (Lau, 1979) and modeling studies is that overall, both thermal and orographic effects are roughly equally important in forcing stationary planetary waves.

As a consequence, a question arises about the degree of reinforcement or cancellation of the orographically and thermally forced waves. Aspects of this have been briefly discussed by Saltzman (1965), Trenberth (1973), Egger (1976a), Bates (1977) and Tung and Lindzen (1979a). In linear theory the net response of waves to multiple forcings is simply the sum of the different forced waves which may reinforce or cancel. In practice, however, the effects of orographic and thermal forcings are not fully separable. Orography acts as a barrier to the flow and thereby plays a major role in air mass formation by limiting mixing. When this effect is combined with orographically induced precipitation and latent heat release, it becomes clear that orography also plays a key role in determining the diabatic heating field. Moreover, as will be shown here, the interaction between two linear waves forced by the different mechanisms is
such that further nonlinear feedback effects occur and act to modify the zonal mean flow, thereby further changing the linear response to the forcings.

For instance, whereas inviscid linear trapped waves forced by either orography or heating alone do not transport heat polewards and the Eliassen–Palm flux is zero, a combination of the forcings will generally result in a net meridional heat flux accompanied by a non-zero potential vorticity transport or Eliassen–Palm flux divergence and modifications to the zonal mean flow. This statement applies if the heating has the same phase at all levels in the vertical. The total result clearly depends upon the relative phase of the heating and the orography, and is linked to the vertical structure of the heating, as will be further shown here. Alternatively, thermal forcing in which surface heating is phase-shifted relative to mid-tropospheric heating would produce the same kind of interaction but between the two trapped thermal waves.

These questions become even more important and less resolved when the interannual variability of the quasi-stationary waves and long-lived atmospheric phenomena, such as blocking, are considered. Clearly, there is considerable capacity for thermal forcing to vary from year to year through, for instance, changes in sea surface temperature (SST) (Namias and Cayan, 1981; Horel and Wallace, 1981). Many attempts have been made to model the effects of SST or heating anomalies on the atmosphere, and recent examples include Roads (1980), Vernekar (1981), Webster (1981, 1982), Hoskins and Karoly (1981), Simmons (1982), Hendon and Hartmann (1982) and Phillips (1982). Although the mountains are fixed, orographic forcing of the atmosphere may also change through fluctuations in the mean flow over the mountains, and this change could result in more than one stable atmospheric response to orographic forcing (Charney and de Vore, 1979). When the nonlinear effects noted above are also introduced, the net response to a change in thermal forcing could potentially be considerable.

In this paper we use a simple model to examine the combined effects of orography and thermal forcing. In principle, the inclusion of orographic forcing in models is reasonably straightforward. In practice, it proves to be much more difficult because of resolution and finite-difference problems, since models usually include only a smooth representation of the actual mountains and air can therefore flow through regions where physical barriers should exist. Instead, mountain forcing is often included through the kinematic lower boundary condition

\[ W = v \cdot \nabla h, \]

where \( W \) is the vertical velocity induced by a horizontal flow \( v \) over topography \( h \). The utility and problems with this approach are discussed by Saltzman and Irsch (1972), Ashe (1979), Dickinson (1980), and Tung (1983).

Our understanding of the diabatic heating field is even less developed, as pointed out by Dickinson (1980). In large part, the difficulty in treatment of diabatic heating in simple models is compounded through the dependence of the heating on the resulting flow field through such things as latent heating and the distribution of clouds and water vapor. Therefore, heating should be parameterized rather than specified. Webster (1981) made one recent attempt to do this. Nonetheless, there is still much that can be learned by arbitrarily specifying the heating field \textit{a priori} (Hoskins and Karoly, 1981; Simmons, 1982), and this is the approach adopted here.

Our analysis of the interaction of thermal and orographically forced waves uses a simplified but continuous and unbounded model of the atmosphere that enables us to obtain analytic solutions. We adopt a quasi-geostrophic model on a \( \beta \)-plane with orographic and thermal forcing. Vertical propagation of forced waves in similar models has been previously considered by Charney and Drazin (1961), Eliassen and Palm (1961), Dickinson (1968a,b, and 1980), Simmons (1974), Tung and Lindzen (1979b) and Held (1983). In this paper, exact solutions are obtained under the highly idealized conditions of a constant zonal mean wind profile. In that case it is possible to calculate the second-order meridional eddy heat fluxes, the induced Eliassen–Palm fluxes, and solve analytically for the induced mean meridional circulation and changes in the zonal mean flow.

In most of the simplified models of the atmosphere it has been necessary to incorporate friction or damping of some kind in order to eliminate the possibility of infinite amplitude resonant waves. Resonance arises when the free modes of the model atmosphere coincide with the forced wave. As Tung and Lindzen (1979b) point out, this can only occur for trapped waves. Consequently, problems with resonance or, in the presence of friction, quasi-resonance, encountered by many studies may have been exacerbated by the artificial nature of the upper boundary condition in most models.

Lindzen et al. (1968) and Bates (1977) note the sensitivity of solutions to the upper boundary condition and the former show the possibility for artificial resonance to ensue. The problem is especially evident in the widely used two-layer model of the atmosphere in which all waves are trapped and therefore potentially resonant. Therefore, the solutions found here will prove useful for validating a two-layer model in related work in progress that considers more general conditions. The continuous solution may also be generalized somewhat using the WKB approximation and related, with respect to the question of resonance,
to the more complete solution of Tung and Lindzen (1979b), but this solution too will be taken up elsewhere.

Models such as the one analyzed in detail here are highly idealized but frequently used as a basis for understanding more complex models. Previous analyses of such models have suffered from several shortcomings, for instance Dickinson (1980), or have not been complete, for example Pedlosky (1979). The idealizations make it inappropriate to compare results with the real atmosphere, but we will attempt to make some more general statements about the differences in planetary waves between the two hemispheres.

In Section 2 the model equations are set up and the zonal mean and perturbation equations are developed. In Section 3 the solution is explored for the case of constant zonal mean wind and solutions for the propagating and trapped waves in the presence of surface friction are developed. These are further examined for specific parameters in Section 4 which also shows that, at large heights, thermal forcing can be replaced by an equivalent mountain forcing. The eddy fluxes and energetics of the forced waves are considered in Section 5 and the forcing of the zonal mean flow is examined in Section 6. Section 7 gives the main conclusions.

2. The model

a. The governing equations

We use a quasi-geostrophic model on a $\beta$-plane. Since we are primarily interested in large-scale stationary waves in which spherical geometry plays an important role, the results we obtain can only be regarded as a qualitative indication of the behavior in the real atmosphere.

We choose as vertical coordinate

$$z = \ln \left( \frac{\rho_0}{p} \right),$$  

(2.1)

where $p$ is pressure, as has also been used by Tung and Lindzen (1979b) and Dickinson (1980).

The development of the mean and perturbation equations for more general sets of equations have been given by Holton (1975), Boyd (1976) and Andrews and McIntyre (1978). Edmon et al. (1980) apply quasi-geostrophic scaling to the Eliassen–Palm theorem. In the following we use standard notation but a full listing of symbols is given in Appendix A.

With the quasi-geostrophic approximation on a $\beta$-plane in this coordinate system, the equations of motion, continuity and the thermodynamic equation may be written as

$$\frac{du_z}{dt} + fu + \frac{\partial \Phi}{\partial y} = \mathcal{F}_x,$$  

(2.2)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \rho_0 w = 0,$$  

(2.3)

$$\frac{d}{dt} \Phi + wS = \kappa Q,$$  

(2.4)

where use has been made of the hydrostatic equation

$$\Phi_z = RT$$  

(2.5)

and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_e \frac{\partial}{\partial x} + \mathbf{v}_e \frac{\partial}{\partial y}.$$  

(2.6)

Here $\mathbf{v}_e = (u_e, v_e) = k \times \nabla \psi$ where $\psi = \Phi/f_0$ is the geostrophic streamfunction. The Coriolis parameter $f$ is regarded as a constant $f_0$ unless differentiated. In addition $w = dz/dt$, $\rho_0(z) = \rho_0 e^{-z}$, and the static stability $S = R(\kappa T + \alpha T)/dz$. For the rhs terms, $\mathcal{F}$ is friction and $Q$ the heating per unit mass.

By elimination of $\Phi$ from (2.2) and (2.3) and with use of (2.4), these equations may be written in more conventional form as the vorticity equation

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi + \beta y) = \frac{f_0}{\rho_0} \frac{\partial}{\partial z} (\rho_0 w),$$  

(2.7)

and thermodynamic equation

$$\frac{\partial \psi}{\partial t} + \frac{J(\psi, \partial \psi)}{\partial z} + \frac{wS}{f_0} = \kappa Q,$$  

(2.8)

with the two unknowns $\psi$ and $w$. Here $J$ is the Jacobian.

The boundary condition at the earth’s surface,

$$w = -\frac{1}{gH} \frac{\partial \Phi}{\partial t} + \frac{W_b}{H},$$  

(2.9)

is applied at $z = 0$; where $H = RT/g$ is the scale height and

$$W_b = v \cdot \nabla h + D_E \nabla^2 \psi.$$  

(2.10)

The surface elevation $h$ is included only through the kinematically induced vertical motion. Surface frictionally induced Ekman pumping is also included with $D_E \approx 100$ m as the characteristic Ekman layer depth (Dickinson, 1980).

b. The mean equations

We now put $\psi = \overline{\psi} + \psi'$ for all the variables, where

$$\overline{\psi} = \frac{1}{2\pi} \int_0^{2\pi} \psi(\lambda) d\lambda$$

is the zonal mean and $\lambda$ is longitude. Further, we recognize from geostrophic theory that $\mathbf{u}$ and $\mathbf{v}$ are
small. The resulting mean flow equations can then be written as
\begin{equation}
\frac{\partial \bar{u}_g}{\partial t} - f\bar{v} = -\frac{\partial}{\partial y} \bar{u}' \bar{v}' + \hat{F}_x,
\end{equation}
\begin{equation}
f\bar{u}_g + \frac{\partial \Phi}{\partial y} = 0,
\end{equation}
\begin{equation}
\frac{\partial \bar{v}}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 \bar{w}) = 0,
\end{equation}
\begin{equation}
\frac{\partial \Phi}{\partial t} + \bar{w}S = -\frac{\partial}{\partial y} \bar{v}' \Phi' + \kappa Q.
\end{equation}
Note that although \( \bar{v} \) and \( \bar{w} \) are small, they must be retained in order to include effects of the mean meridional circulation on the zonal mean flow.

By substituting
\begin{equation}
\bar{v}^* = \bar{v} - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 \frac{\bar{v} \Phi'}{S} \right),
\end{equation}
\begin{equation}
\bar{w}^* = \bar{w} + \frac{\partial}{\partial y} \left( \frac{\bar{v} \Phi'}{S} \right),
\end{equation}
we obtain a set of transformed mean-flow equations
\begin{equation}
\frac{\partial \bar{u}_g}{\partial t} - f\bar{v}^* = \nabla \cdot F + \hat{F}_x,
\end{equation}
\begin{equation}
f\bar{u}_g + \frac{\partial \Phi}{\partial y} = 0,
\end{equation}
\begin{equation}
\frac{\partial \bar{v}^*}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 \bar{w}^*) = 0,
\end{equation}
\begin{equation}
\frac{\partial \Phi}{\partial t} + \bar{w}^* S = \kappa Q,
\end{equation}
where
\begin{equation}
F = \left( -\bar{u}' \bar{v}' \frac{\bar{v} \Phi'}{S}, f \frac{\bar{v} \Phi'}{S} \right)
\end{equation}
is the Eliassen–Palm flux, and its divergence
\begin{equation}
\nabla \cdot F = \frac{\partial F_{\omega}}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left[ \rho_0 F_{\omega, z} \right]
\end{equation}
also corresponds to the potential vorticity transport (Edmon et al., 1980).

As shown by Eliassen and Palm (1961), Boyd (1976) and Andrews and McIntyre (1976, 1978), for steady, conservative, wavelike disturbances of the zonal wind, \( \nabla \cdot F \) is zero giving the Eliassen–Palm theorem and the Charney–Drazin non-interaction theorem. The presence of friction and heating in the waves violates these conditions and thereby permits the waves to interact with the zonal mean flow, as will be considered in more detail later.

c. The perturbation equations

For the perturbations, it is more convenient to work directly with Eqs. (2.8) and (2.9) which becomes upon linearization (and dropping the subscript \( g \))
\begin{equation}
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} \right) + \left( \beta - \bar{u}_y \right) \frac{\partial \psi'}{\partial x} = \frac{f_0}{\rho_0} \frac{\partial}{\partial z} (\rho_0 \omega'),
\end{equation}
\begin{equation}
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \psi'}{\partial x} - \bar{u}_y \frac{\partial \psi'}{\partial x} + \frac{w' S}{f_0} = \kappa Q',
\end{equation}
where use has been made of the thermal wind relation [from (2.6) and (2.18)] in (2.24).

We eliminate \( \omega' \) between (2.23) and (2.24) to obtain the quasi-geostrophic potential vorticity equation
\begin{equation}
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) q' + \beta_e \frac{\partial \psi'}{\partial x} = \frac{f_0}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 \frac{\kappa Q'}{S} \right),
\end{equation}
where
\begin{align}
q' &= \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 \frac{f_0^2}{S} \frac{\partial \psi'}{\partial z} \right), \\
\beta_e &= \beta - \bar{u}_y - \frac{1}{\rho_0} \left( \frac{f_0^2}{S} \bar{u}_z \right),
\end{align}
In order to solve these equations, we choose a beta-plane centered at 45° latitude extending from \( y = 0 \) to \( \pi/\lambda \) over one-half north–south wavelength, with \( \psi' = 0 \) on the boundaries. These boundaries do not necessarily coincide with the equator and pole. We further assume that \( S \) is constant, the zonal mean wind is a function of \( z \) alone, and introduce a new variable \( \Psi \) by putting
\begin{equation}
\psi'(x, y, z, t) = \text{Re} \Psi(z)e^{ik(x-ct)+z/2} \sin y,
\end{equation}
which also represents an expansion in spatial and temporal Fourier components. Here \( \text{Re} \) is the real part. We similarly express the forcing in Fourier components
\begin{equation}
h = \text{Re} e^{ikx} \sin y,
\end{equation}
\begin{equation}
Q' = \text{Re} Q(z)e^{ikx+\phi_H} \sin y,
\end{equation}
where \( \phi_H \) corresponds to the phase of the heating relative to the topography.

Substituting into (2.25)–(2.27) we have
\begin{equation}
\Psi_{zz} + \gamma^2 \Psi = \frac{\kappa e^{i\phi_H-z/2}}{f_0(\bar{u} - c)i k} (Q' - \Psi'),
\end{equation}
where
\[
\gamma^2 = \frac{S}{f_0^2} \left[ \frac{\beta_0}{\bar{u} - c} - K^2 \right] - \frac{1}{4},
\]

(2.32)

\[K = (k^2 + l^2)^{1/2}\] is the total wavenumber,
\[\beta_0 = \beta - \frac{f_0^2}{S} (\bar{u}_{zz} - \bar{u}).\]

Vertical propagation of waves according to an Eq. like (2.31) was first considered by Charney and Drazin (1961). Dickinson (1980) considers a somewhat more general version than that given here in which \(S\) remains a function of \(z\). It is evident that (2.31) will have waveline solutions provided \(\gamma^2 > 0\). Consequently, waves can propagate only in westerlies, if \(\bar{u} > c\), and if \(\bar{u} - c\) is less than a certain critical value \(u_c\), where
\[u_c = \frac{\beta_0}{K^2 + \frac{f_0^2}{4S}}.\]

The lower boundary condition is obtained by applying the thermodynamic Eq. (2.24) at \(z = 0\) and using (2.10). If we now consider only stationary waves, so that \(c = 0\), then using (2.28)–(2.30) the lower boundary condition is
\[\frac{\partial \Psi}{\partial z} + \left[ \frac{1}{2} - \frac{\bar{u}(0)}{\bar{u}(0)} \right] + iE_k \Psi = \frac{S\bar{H}}{f_0 H} + \frac{k\bar{Q}(0)e^{i\phi}}{\bar{u}(0) f_0 k},\]

(2.33)

where
\[E_k = \frac{S D_k (k^2 + l^2)}{f_0 H \bar{u}(0)} k,\]

(2.34)

and this is applied at \(z = 0\). Note that Dickinson (1980) erroneously neglected to include the surface heating term in (2.33) in his analysis.

d. Energetics

The forcing of the zonal mean flow, as shown earlier, and the energetics depend upon the nonlinear eddy fluxes. In these coordinates an element of mass \(dM\) is given by
\[dM = \rho_0 dV,\]

where \(dV = H dz' dA'\), and
\[dA' = 2\pi a \cos \left( \frac{Y_0}{a} \right) \int_0^{2\pi} d\lambda \int_0^{\pi/2} dy,\]

and \(Y_0 = a\pi/4\) is the center of the \(\beta\) plane which extends over one half north-south wavelength. The kinetic energy \(K\) and available potential energy \(A\)
\[K = \int \frac{1}{2} \nabla \Psi \cdot \nabla \Psi \rho_0 dV,\]

(2.35)

\[A = \int \frac{f_0^2}{S} \frac{1}{2} \left( \frac{\partial T}{\partial z} \right)^2 \rho_0 dV,\]

(2.36)

may be divided into zonal \(KZ\) and \(AZ\), and eddy \(KE\) and \(AE\) components. The rates of change of the latter are given by
\[\frac{dKE}{dt} = \langle KZ \cdot KE \rangle + \langle AE \cdot KE \rangle + VKE|_{z=0} - VKE|_{z=\infty},\]

(2.37)

\[\frac{dAE}{dt} = \langle AZ \cdot AE \rangle - \langle AE \cdot KE \rangle + GE,\]

(2.38)

where
\[\langle KZ \cdot KE \rangle = E_0 \int \frac{d\bar{u}}{dy} \bar{u} \nu \bar{\varrho} d\bar{z},\]

(2.39a)
\[\langle AZ \cdot AE \rangle = E_0 \int \frac{R^2}{S} \frac{dT}{dy} \bar{v} \bar{T} \bar{\varrho} d\bar{z},\]

(2.39b)
\[\langle AE \cdot KE \rangle = E_0 \int \frac{Rw\bar{T}}{S} \bar{\varrho} d\bar{z},\]

(2.39c)

\[E_0 = \frac{\pi^2 a^2}{a} \cos \left( \frac{Y_0}{a} \right) H,\]

are the conversions between the different kinds of energy. The vertical flux of geopotential energy through any level \(z\) is
\[VKE = E_0 \rho_0 w \bar{\varrho} \bar{\Phi}.\]

(2.40)

Thus, the flux of geopotential through the lower boundary due to orography and friction is \(VKE|_{z=0}\), and \(VKE|_{z=\infty}\) is the flux of energy radiated to outer space. In addition,
\[GE = E_0 \int \frac{R}{S} \bar{Q} \bar{T} \bar{\varrho} d\bar{z}\]

(2.41)

is the generation of eddy available potential energy by heating.

3. Idealized solution for constant mean wind

In this section we solve these equations exactly for the case where \(\bar{u}\) is not a function of \(z\). In that case \(\beta_0 = \beta\), and the refractive index \(\gamma\) in (2.32) is constant.

We can immediately make some broad statements about the solution from scaling arguments. From the thermodynamic Eq. (2.24), the elimination of the \(\nu \partial T/\partial y\) term requires that, in the presence of diabatic heating, the main balance should be between the heating and the zonal advection term
\[\bar{u} \frac{\partial T}{\partial x} \approx \frac{Q'}{C_p}.\]
However, in the absence of orography and friction, there must also be a positive correlation between $T'$ and $Q'$, and $w'$ and $T'$, from energetics considerations (see Appendix B). Consequently, we anticipate that the temperature perturbation will tend to be locked onto the heating field and nearly 90° out of phase.

a. Specified heating profiles

In order to solve the full equations we need to specify the heating profile $Q(z)$ in (2.30). We will consider two forms, one with surface heating decreasing with height and one with zero surface heating but a maximum heating in the low to mid-troposphere. Suitable forms are

$$Q(z) = Q_0 e^{-\alpha_0 z}, \quad (3.1)$$

$$Q(z) = Q_0 (e^{-\alpha_1 z} - e^{-\alpha_2 z}). \quad (3.2)$$

As $z \to \infty$, $Q \to 0$. To be more specific, we will take $\alpha_0 = 5$ and $\alpha_1 = 2$, $\alpha_2 = 4$ which results in the heating profiles shown in Fig. 1. In the surface heating case, the heating decays to 0.1 times $Q_0$, the surface value at ~630 mb. In the second case maximum heating of $Q_0/4$ occurs at $z = 0.3466$ which is ~700 mb. In practice, neither profile is very realistic although they may easily be combined, perhaps with different phases to obtain more realistic results. It is nevertheless useful to consider these particular forms separately for comparison with other studies. In particular, the internal heating profile will prove useful for comparing with results from a two-layer model.

As the second heating profile involves two terms of the same type as in the first, the solutions involve an additional set of terms with identical format to the particular solution arising from the first profile. We therefore mainly provide the solution for Eq. (3.2), but note that Eq. (3.1) can be simply retrieved by dropping all terms involving $a_2$ and changing $a_1$ to $a_0$.

As a shorthand notation, we put $Q^* = \kappa Q_0/\bar{u}$, where $Q_0$ refers to either $Q_{01}$ or $Q_{02}$.

b. Solution for propagating waves $\gamma^2 > 0$

The solution to the homogeneous equation in (2.31), provided $\gamma^2 > 0$, is

$$\Psi = \Psi_0 e^{i\gamma z} + \Psi_1 e^{-i\gamma z}, \quad (3.3)$$

where

$$\gamma^2 = \frac{S}{f_0^2} \left[ \frac{\beta}{\bar{u}} - K^2 \right] - \frac{1}{4} > 0. \quad (3.4)$$

The upper boundary condition is that, as $z \to \infty$, the upward energy flux should remain finite and the radiation condition requires that any energy flux at infinity should be directed away from the source. From (2.32) the vertical group velocity is found to be

$$w_g = \frac{2\gamma f_0^2 \beta k}{S \left[ \frac{f_0^2}{S} \left( \gamma^2 + \frac{1}{4} + K^2 \right) \right]^2},$$

so that $w_g$ has the same sign as $\gamma$. Therefore $\Psi_1 = 0$ in (3.3). From this point on we deal with stationary waves, where $c = 0$.

The particular solution to (2.31), to be added to (3.3), for the heating profile terms have the form

$$\Psi_j = \frac{iQ^*_0}{f_0 k} \frac{(1 + \alpha_j)}{d_j} e^{i\theta_j - (\alpha_j + 1/3)z},$$

where

$$d_j = \gamma^2 + (\alpha_j + 1/2)^2, \quad j = 0, 1, \text{ or } 2,$$

$$= (\gamma^2 + \frac{1}{4}) + \alpha_j (\alpha_j + 1), \quad (3.5)$$

$$\gamma^2 + \frac{1}{4} = \frac{S}{f_0^2} (K^2 + K^2).$$

We have introduced $K^2 = \beta/\bar{u}$, where $K$ corresponds to the stationary total wavenumber in a barotropic model.

We apply the lower boundary condition (2.33) to find $\Psi_0$ in (3.3) as

$$\Psi_0 = \frac{-1}{f_0^{1/2} + i(\gamma + E_k)} \times \left\{ h^* + \frac{iQ^*_0 e^{i\theta_j}}{k} \left[ \frac{(\gamma^2 + \frac{1}{4}) + iE_k (1 + \alpha)}{d} \right] \right\},$$

and we have defined $h^* = Sh^*/H$, 

![Fig. 1. Vertical heating profiles as a function of z (left) and pressure (right) for thermal forcing. Maximum heating for each profile is 1 K day⁻¹.](image)
\[
\frac{1}{d} = \frac{1}{d_0}, \quad 1 + \alpha = \frac{1 + \alpha_0}{d_0}
\]

for heating profile (3.1) or
\[
\frac{1}{d} = \frac{1}{d_1} - \frac{1}{d_2}, \quad 1 + \alpha = \frac{1 + \alpha_1}{d_1} - \frac{1 + \alpha_2}{d_2}
\]

for heating profile (3.2) so that
\[
\alpha = \frac{\alpha_1 \alpha_2 - (\gamma^2 + \nu)}{\alpha_1 + \alpha_2 + 1}.
\]

In all cases \(d\) and \(\alpha\) are positive.

If we now put
\[
G^2 = \nu + (\gamma + E_k)^2, \quad (3.6a)
\]
\[
\sin \varphi = \frac{\nu}{G}, \quad (3.6b)
\]
\[
\cos \varphi = \frac{\gamma + E_k}{G}, \quad (3.6c)
\]

and then substitute into (2.28), the total solution is
\[
\psi = \psi_h + \psi_Q, \quad (3.7)
\]

where these are respectively the orographically and thermally forced components, given by
\[
\psi_h = -\frac{h^*}{f_0 G} \sin(kx + \gamma z + \varphi)e^{2f_0} \sin y, \quad (3.8)
\]
\[
\psi_Q = -\frac{Q^*}{f_0 k} \sin y \left\{ (\gamma^2 + \nu) \cos(kx + \gamma z + \varphi + \phi_H) \right\}
- E_k(1 + \alpha) \sin(kx + \gamma z + \varphi + \phi_H) \frac{e^{2f_0}}{Gd}
- \left[ \frac{(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} - \frac{(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \right] \sin(kx + \phi_H). \quad (3.9)
\]

For heating profile (3.1) the last term involving \(\alpha_2\) should be set to zero and \(\alpha_1\) set to \(\alpha_0\).

Differentiating with respect to \(z\) we can obtain the corresponding temperature field components \(T = (f_0/R)\varphi/\partial z\) from
\[
\frac{\partial \psi_h}{\partial z} = -\frac{h^*}{f_0 G} (\gamma^2 + \nu)^{1/2}
\times \cos(kx + \gamma z - \theta)e^{2f_0} \sin y, \quad (3.10)
\]
\[
\frac{\partial \psi_Q}{\partial z} = \frac{Q^*}{f_0 k} \sin y \left\{ \frac{(\gamma^2 + \nu)^{1/2}}{Gd} e^{2f_0} \left[ (\gamma^2 + \nu) \times \sin(kx + \gamma z + \phi_H - \theta) + E_k(1 + \alpha) \times \cos(kx + \gamma z + \phi_H - \theta) \right] \right\}
\times \left[ \frac{\alpha_1(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} - \frac{\alpha_2(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \right] \sin(kx + \phi_H), \quad (3.11)
\]

where
\[
\sin \theta = \frac{\sqrt{2} E_k}{G \sqrt{\gamma^2 + \nu^{1/2}}}, \quad \cos \theta = \frac{\gamma \nu + E_k}{G \sqrt{\gamma^2 + \nu^{1/2}}}.
\]

Using (2.24) we can solve for the vertical motion
\[
w = w_h + w_Q,
\]

where
\[
w_h = -\frac{\bar{u} \bar{k} \nu \nu^*}{SG} \times \sin(kx + \gamma z - \theta)e^{2f_0} \sin y, \quad (3.12)
\]
\[
w_Q = -\frac{\bar{u} \bar{Q}^*}{S} \sin y \nu \left\{ \frac{e^{2f_0}}{Gd} \left[ (\gamma^2 + \nu) \times \cos(kx + \gamma z + \phi_H - \theta) \right] \right\}
- E_k(1 + \alpha) \sin(kx + \gamma z + \phi_H - \theta) \left[ \frac{e^{-\alpha_1 z}}{d_1} - \frac{e^{-\alpha_2 z}}{d_2} \right] \cos(kx + \phi_H). \quad (3.13)
\]

Note that in the absence of Ekman pumping, \(\theta = 0\) and these expressions become much simpler. The solutions as a function of certain parameters, the eddy covariances, and the energetics will be considered in detail in later sections.

\(c.\) Solution for trapped waves \(\gamma^2 < 0\)

If \(\gamma^2\) is negative, then the solution is
\[
\Psi = \Psi_0 e^{-\mu z} + \Psi_1 e^{\mu z}, \quad (3.14)
\]

where
\[
\mu^2 = \frac{1}{4} + \frac{S}{f_0^2} \left[ K^2 - \frac{\beta}{\bar{u}} \right] = -\gamma^2. \quad (3.15)
\]

Applying the upper boundary condition requires that \(\Psi_1 = 0\).

In this case the particular solution is more involved owing to a number of special cases that can occur. If there is no surface friction, \(E_k = 0\), then there is a possibility for resonance to occur for \(\mu = \nu/2\). This corresponds to \(K^2 = \beta/\bar{u} = K_0^2\) so that resonance occurs at the same wavenumber as for the barotropic case. It results in an infinite amplitude external...
Rossby wave. It is also an important marker, since there is a marked change in the character of the solution according to whether \( \mu < \frac{1}{2}, (K < K_0) \) or \( \mu > \frac{1}{2}, (K > K_0) \). Even in the presence of friction quasi-resonance occurs at \( \mu = \frac{1}{2} \).

Further special cases occur if \( \mu = \alpha_j + \frac{1}{2}, \ j = 1, 2 \) so that the homogeneous and nonhomogeneous solutions take the same form. Since the \( \alpha_j \) are fairly large for realistic heating profiles, these cases all correspond to fairly short waves and are not of great interest. Moreover, the character of the solution does not change for these special cases which can all be grouped along with other solutions for \( \mu > \frac{1}{2} \).

The general solution for \( \mu \neq \alpha_j + \frac{1}{2} \) is given by

\[
\psi = \psi_h + \psi_Q,
\]

where

\[
\psi_h = -\frac{h^* e^{(1/2-\mu)z} \sin y}{f_0(\frac{1}{2} - \mu)^2 + E_k^2} \times \left[ (\frac{1}{2} - \mu) \cos k x + E_k \sin k x \right], \tag{3.16}
\]

\[
\psi_Q = \frac{Q^*}{f_0 k} \sin y \left\{ \left[ \left( \frac{1}{2} - \mu \right)^2 + E_k^2 \right] d \times \left[ \left( \frac{1}{2} - \mu \right) (\frac{1}{2} - \mu) + E_k (1 + \alpha) \sin (k x + \phi_H) + E_k (\frac{1}{2} - \mu) (\frac{1}{2} - \mu + \alpha) \cos (k x + \phi_H) \right] - \frac{(1 + \alpha_1 e^{-\alpha_1 z})}{d_1} \frac{(1 + \alpha_2 e^{-\alpha_2 z})}{d_2} \right\}, \tag{3.17}
\]

where \( d_j = (\alpha_j + \frac{1}{2})^2 - \mu^2 \), \( j = 0, 1, 2 \) and \( d \) and \( \alpha \) are defined as before, but with \( \gamma^2 = -\mu^2 \). It may be verified that for \( \mu = 0 \) these solutions are identical to (3.8) and (3.9) with \( \gamma = 0 \).

In the event of no Ekman pumping, then provided \( \mu \neq \frac{1}{2} \) these solutions become

\[
\psi_h = -\frac{h^* e^{(1/2-\mu)z} \cos k x \sin y}{f_0(\frac{1}{2} - \mu)} \tag{3.18}
\]

\[
\psi_Q = \frac{Q^*}{f_0 k} \sin (k x + \phi_H) \sin y \\
\times \left[ \frac{(1/2 + \mu)}{d} e^{(1/2-\mu)z} - \frac{(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} + \frac{(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \right]. \tag{3.19}
\]

In all of these cases, the solution is equivalent barotropic and decays with height if \( \mu > \frac{1}{2} \) but increases with height if \( \mu < \frac{1}{2} \). Further, there is a 180° phase shift in \( \psi_h \) in (3.18) across \( \mu = \frac{1}{2} \), as occurs in a barotropic model.

For completeness, we also give the solution for \( \mu = \alpha_1 + \frac{1}{2} \), where \( E_k = 0 \). Then

\[
\psi_Q = -\frac{Q^*}{f_0 k} \sin (k x + \phi_H) \sin y \\
\times \left\{ \left[ \frac{(1/2 - (1 + \alpha_1)z)}{(2\alpha_1 + 1)} + \frac{(1 + \alpha_1)}{d_2} \right] e^{-\alpha_1 z} - \frac{(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \right\}, \tag{3.20}
\]

and, here, \( d_2 = (\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1 + 1) \). For heating profile (3.1) the terms involving \( d_2 \) should be omitted. It is readily verified that as \( \mu \to \alpha_1 + \frac{1}{2} \), (3.19) approximates to (3.20) and there is no change in structure or character of the solution across \( \mu = \alpha_1 + \frac{1}{2} \). In fact all solutions for \( \mu > \frac{1}{2} \) are similar in form.

4. Results

4a. Parameters

In order to consider specific solutions, we adopt typical parameters for a \( \beta \)-plane centered at 45°N, \( f_0 = 1.03 \times 10^{-4} \text{ s}^{-1} \), \( \beta = 1.62 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \), \( T_0 = 250^\circ \), \( H = 7321 \text{ m} \), \( S = 5360 \text{ m}^2 \text{ s}^{-2} \) (corresponding to a Brunt-Väisälä frequency of \( 10^{-2} \text{ s}^{-1} \)), \( \beta \) = 15 m s^{-1}, \( D_E = 100 \text{ m} \), \( h = 100 \text{ m} \), and maximum \( Q/C_p = 1 \text{ K day}^{-1} \). For this value of \( D_E \) the e-folding time for surface friction is \( H/(D_E f_0) = 8.2 \text{ days} \). The stationary wave number \( K_s = (\beta/\alpha)^{1/2} = 1.04 \times 10^{-6} \text{ m}^{-1} \). We also introduce nondimensional wavenumbers in the east–west and north–south directions \( m \), \( n \) such that \( L_m = a^2 \pi \cos(45^\circ) \), \( L_n = a^2 \pi \), where \( a \) is the radius of the earth.

With the above parameters \( E_k \) from (2.34) and \( \gamma^2 \) from (3.4) or \( \mu^2 \) from (3.15) become

\[
\gamma^2 = -\mu^2 = 0.29565 - 0.01245(2m^2 + n^2),
\]

\[
E_k = 5.2595 \times 10^{-3}(2m^2 + n^2)/m.
\]

Fig. 2 presents values of \( \gamma \) or \( \mu \), as appropriate, as a function of \( m \) and \( n \). Also shown are the \( \mu = 0 \) isoline dividing the propagating from the trapped waves and the \( \mu = \frac{1}{2} \) isoline showing the resonance wavenumber corresponding to \( K = K_s \). Fig. 3 shows the corresponding values of \( E_k \) as a function of \( m \) and \( n \).

Note that in the solutions, as given by (3.6), (3.9) or (3.16) and (3.17), \( E_k \) appears along with factors of \( \gamma \) or \( \mu \) and \( \frac{1}{2} \). From Fig. 3 we note that \( E_k \) is small compared with \( \frac{1}{2} \) and is also generally small compared with \( \gamma \). Consequently, the contribution from Ekman pumping to the solution for propagating waves is fairly small and the solution somewhat resembles the inviscid case where \( E_k = 0 \). For trapped waves the same comment applies except for waves near resonance, \( \gamma \), when \( \mu \approx \frac{1}{2} \).
In order to illustrate the solutions given earlier, we choose, in addition to the parameters already given, \( n = 4 \), corresponding to a north–south half-wavelength of 45° latitude. Under these circumstances only wave 1 can propagate and all other waves are trapped. Only results for the mid-tropospheric heating profile (3.2) will be presented in detail.

b. Equivalent mountain to thermal forcing

Before considering the total solutions, it is worthwhile examining the effectiveness of the different kinds of forcing in setting up stationary waves. We consider solutions at large \( z \), i.e., in the regions remote from where the heating is significant. Under these circumstances it is possible to choose an effective mountain forcing that would set up a response identical to the thermally forced wave (Dickinson, 1980).

In the propagating case, (3.9) is similar to (3.8) for large \( z \) provided we choose an equivalent mountain height \( h_E \) to be

\[
h_E = \frac{Q^*H}{Sd k} \left[ (\gamma^2 + \frac{1}{4})^2 + E_k(1 + \alpha)^2 \right]^{1/2}, \tag{4.1}
\]

at

\[
\phi_H = -\frac{\pi}{2} - \tan^{-1}\left[ \frac{E_k(1 + \alpha)}{\gamma^2 + \frac{1}{4}} \right].
\]

For the trapped waves, provided \( \mu < \alpha_1 \), the equivalent mountain height, from (3.17) and (3.18), is

\[
h_E = \frac{Q^*H}{Sd k} \left[ (\frac{1}{4} - \mu^2)^2 + E_k(1 + \alpha)^2 \right]^{1/2}, \tag{4.2}
\]

Notice that the expression in (4.2) is a continuation of (4.1) across \( \mu = \gamma = 0 \). For \( E_k = 0 \), \( h_E \to 0 \) as \( \mu \to \frac{1}{2} \) (resonance); \( \phi_H = -\pi/2 \) for the propagating wave for \( \mu < \frac{1}{2} \) and \( \phi_H = \pi/2 \) for \( \mu > \frac{1}{2} \). This discontinuity in phase is no longer evident with friction included, and \( \phi_H = \pi \) at \( \mu = \frac{1}{2} \). The phase is such that the maximum heating occurs between the effective mountain and the downstream trough for \( \mu < \frac{1}{2} \) but on the upslope for \( \mu > \frac{1}{2} \). In all cases the effective mountain is located such that the vertical motion field generated by the heating results in upward motion on the windward slope of the effective mountain.

For the mid-tropospheric heating profile given by Eq. (3.2) and Fig. 1, with maximum heating of 1° day\(^{-1} \), the values of \( h_E \) for \( E_k = 0 \) are given in Fig. 4. For planetary waves the dominant scale factor enters through the \( k^{-1} \) factor. Roads (1980) previously noted an inverse wavenumber dependence of pressure and temperature to heating.

It is interesting to note that for the same heating profile but with friction included, the equivalent mountain is larger than in Fig. 4 in every case. For instance for \( n = 4 \), and \( m = 1–6 \), rather than the values in Fig. 4, \( h_E \) takes on values of 225, 84, 36, 23, 43 and 68 m; for \( \phi_H = -122, -118, -137, 159, 118, 107^\circ \), respectively. In other words, the presence of surface friction enhances the response to thermal forcing relative to orographic forcing. The reasons for this are discussed in Section 5.

The mid-tropospheric heating profile provides 6.2
\( \times 10^{-3} \text{ J kg}^{-1} \text{ s}^{-1} \) heating averaged over the entire atmosphere which is a reasonable value for planetary-scales (compare with Derome and Wiin-Nielsen, 1971; Ashe, 1979). Typical amplitudes of harmonics of orography are at most 100–200 m (Sankar-Rao, 1965; Tung and Lindzen, 1979a). Consequently, it seems that thermal forcing should be comparable to mountain forcing for wave 2 but may dominate orography for wave 1. However, orography dominates near resonance. These results are consistent with those of Held (1983). The net effect, however, depends on whether cancelation or reinforcement occurs. For the assumed heating rate, the values of \( h_E \) in Fig. 4 are also those values of orographic height for which maximum interaction between heating and orographic forcing occurs.

The expressions for equivalent mountain height in (4.1) and (4.2) also apply for the surface heating profile, but with the appropriate change in \( d \). We may also compare the response to the two different forms of heating profile. Away from the region of heating, for large \( z \), the ratio of the thermally forced streamfunctions is

\[
\frac{\psi_{Q2}}{\psi_{Q1}} = \frac{Q_{02}}{Q_{01}} \left( \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1 + 1}\right) \left[ (\alpha_0 + \gamma^2) + \gamma^2 \right]
\]

Since \( \gamma^2 < \alpha_0, \alpha_1, \text{ or } \alpha_2 \), this is approximately

\[
\frac{\psi_{Q2}}{\psi_{Q1}} \approx \frac{10Q_{02}}{3Q_{01}}
\]

for the \( \alpha_i \) used in Fig. 1. If the maximum heating in each profile is 1° day\(^{-1}\) as in Fig. 1, then this has the value of 13.3. Alternatively, if the integrated vertical column heating is the same then it becomes 4.2. In either case the response to midtropospheric heating is much larger. For the surface heating profile the equivalent mountain heights in Fig. 4 should be reduced by these factors.

c. Solution at the surface

Fig. 5 shows the amplitude and phase of solutions for the thermally and orographically forced waves at the surface evaluated at the center of the \( \beta \)-plane where \( \sin \lambda = 1 \). We adopt the convention that the phase is defined as that of the ridges and increases from zero eastward of the orographic ridge (except in Figs. 7 and 8) and decreases westward. In Fig. 5, therefore, the wave ridge lies to the west of the mountain ridge. At the resonant wavenumber \( (m = 3.74) \) the orographically forced wave is quasi-resonant and has large amplitude whereas the thermally forced wave has zero amplitude at the surface. The latter is true only for this heating profile where there is no heating at the surface. Note that the phase of the thermally forced wave is strongly constrained by the location of the heating. For the orographic wave, the presence of Ekman pumping has blurred the 180° phase shift that would otherwise occur at the resonant wavenumber. The vertical structure of trapped waves is equivalent barotropic, as noted in Section 3c, but is more involved for propagating wave, as follows.

![Fig. 5. Amplitude and phase as a function of wave number m with n = 4 of solutions for the thermally (dashed line) and orographically (solid line) forced waves at z = 0. The phase is that of the first ridge downstream relative to the orography, with the scale given at left for the orography and at right for the heating.](image)
is a 24° shift westward for large $z$. Note also that for large $z$ ($z \gg 2$) the phase changes by $15.5°$ per unit change in $z$ and the amplitude increases as $e^{z/2}$.

The effects of adding both waves together are shown in Fig. 7 for no friction and Fig. 8 for friction. Since the thermally forced wave has larger amplitude (Fig. 6) it tends to dominate the results and in these figures the phase is shown relative to the heating field. They show the amplitude and phase as a function of height versus the phase of the orography relative to the heating $\phi_H$. Also delineated are the regions where the waves slope eastward with height. Note that for large $z$, the slope must be westward at $15.5°$ per unit $z$ and amplitude increasing as $e^{z/2}$. Of note in these figures are the quite large changes in amplitude and structure of the forced wave with either reinforcement or cancelation of the thermally and orographically forced contributions. Largest amplitude at the surface occurs with $\phi_H \approx 60°$ although it corresponds to a weak response in the stratosphere. Largest amplitudes aloft occur with $\phi_H \approx -90°$ (frictionless) or $-120°$ (with friction). Also of note is that the phase of the case with friction is shifted by $\sim 20°$ west, similar to that shown in Fig. 6 for the thermally forced wave.

The dynamics behind these structures can be clarified by considering eddy fluxes and energetics which are discussed in the next section.

5. Eddy fluxes, energetics and Eliassen–Palm fluxes

In Section 2, the energetics and Eliassen–Palm fluxes were shown to depend upon various eddy fluxes. In this problem, $\bar{u}'v' = 0$ so there are no exchanges of eddy kinetic energy with the zonal flow
and only the vertical component of the Eliassen–Palm flux is non-zero.

Expressions for the eddy fluxes may be separated out into the three parts 1) due to orography alone, 2) due to heating alone and 3) due to the interaction between the two. The first two parts may be obtained by setting either the heating or orography to zero. The expressions for the different eddy fluxes are given in Appendix B. For the case of constant $\overline{u}$, the energetics reduces to $GE = \langle AE \cdot KE \rangle$, the divergence of the vertical flux of geopotential [see Eq. (B5) and (B6) in Appendix B]. As shown in Appendix B each of these conversion terms is directly proportional to the heating at all levels in the vertical.

For the propagating wave, these aspects are illustrated in Fig. 9 for no friction and Fig. 10 for the friction case. In the north–south direction all fluxes are proportional to $\sin \gamma$, and only the maximum fluxes at the center of the β plane channel ($\sin \gamma = 1$) are shown.

In the no-friction case, $\overline{vT'}$ and $\overline{w\Psi'}$ are positive everywhere, and the geopotential energy flux due to orography at the surface propagates upwards to infinity. The thermally forced wave dominates in the stratosphere (recall from Fig. 4 that the height of the mountain equivalent to the thermal forcing is 191 m versus the 100 m height used in Figs. 5–10).

Upon adding Ekman pumping, $\overline{w\Psi'}$ is reduced everywhere for the orographic wave but increases in the stratosphere for the thermal wave. This is due approximately to a three-fold increase in $\overline{wT'}$ and, thus, in the generation of $AE$ and conversion to $KE$. It is for this reason that friction can enhance the amplitude of the forced wave. In the lower atmosphere for the thermal wave there is a downward flux of geopotential energy in order to compensate for frictional dissipation.

In the lower panels of Figs. 9 and 10 the mass weighted E–P flux [actually $e^{-\Delta}F_{(s)}$, the $p_{00}$ factor has been omitted] and the E–P flux divergence are presented. For orography alone $p_{00}F_{(s)}$ is constant, and $\nabla \cdot F = 0$. For thermal forcing there is a pronounced change with the inclusion of friction. The interpretation of this is not clear since the Eliassen–Palm and Charney–Drazin theorems are not applicable because of the presence of thermal forcing in the waves. The following section examines this aspect in detail.

When both orography and heating are included together, the resulting maximum eddy fluxes are shown as a function of $\phi_H$ in Fig. 11 without friction and Fig. 12 with friction included. Note the general correspondence between $\psi'$ in Figs. 7 and 8 and the amplitude of $\overline{vT'}$ and $\overline{w\Psi'}$ in the upper atmosphere. Above $z = z_1$, $\overline{vT'} \to 0$ and the E–P theorem holds so that $\overline{vT'}$ and $\overline{w\Psi'}$ are directly related and must be positive. However, both quantities can become negative in the lower part of the atmosphere. This is seen to be linked to strongly positive $\overline{vT'}$ and thus generation of $AE$ and conversion to $KE$ which propagates both upward and downward. As for the case of thermal forcing alone, it is notable that $\overline{wT'}$ increases when friction is included. Thus we may contrast the energetics at $z = 0.5$ for $\phi_H = 0$ versus $\phi_H = 180^\circ$. For $\phi_H = 0$ there is a source of KE from orographic forcing which is destroyed by conversion to $AE$ and loss by diabatic heating $KE \to AE \to GE$. For $\phi_H = 180^\circ$ the reverse applies and orography acts as a sink for $KE$.

The mass weighted E–P flux $e^{-\Delta}F_{(s)}$, which is proportional to $e^{-\Delta}\overline{vT'}$, and its divergence are also shown in Figs. 11 and 12 but are discussed further in the next section.

We have also examined the effects of doubling the size of the mountain to 200 m. This makes it close
to the value giving optimal interaction between the thermally and orographically forced waves. With no friction, the orographic wave is larger than the thermal wave by \(\sim 4\%\), but with friction included the thermal wave is still larger by \(\sim 11\%\). In the no friction case, for large \(z\), the two waves nearly cancel for \(\phi_H \sim 90^\circ\) but reinforce and are a factor of 2 greater than in Fig. 11 for \(\phi_H \sim -90^\circ\) for large \(z\). For instance, from Appendix B, for no friction, it may be shown for large \(z\) and \(\phi_H = \pm 90^\circ\), that

\[
e^{-z\nu T^*} = \frac{\gamma k}{2RF_0G^2} \left\{ \left( \frac{Q^*}{h} \right)^2 \right\}
\]

For the trapped waves, the eddy flux expressions are also given in Appendix B. In the absence of a propagating component, all the eddy fluxes are directly related to the vertical heating. In addition, in the absence of Ekman pumping, or in the case of resonance (\(\mu = 1/2\)), the eddy fluxes are identically zero for either the orographic or thermally forced waves alone. The only non-zero contribution comes from interaction between the two forced waves.

6. Changes in the zonal mean flow

a. Method of solution

Equations for changes in the zonal mean flow were given in Section 2b. Either Eqs. (2.12)–(2.15) or (2.17)–(2.22) may be used. In order to isolate the effects due to thermally and orographically forced waves, the zonal mean friction and heating are set to
zero. In this way, the eddy forcings, set up by the orographic and thermal waves, give rise to a tendency in $\bar{T}$ and $\bar{u}$. Alternatively, for steady state we could define the zonal mean heating and friction required to sustain the steady state.

Since $u'v' = 0$, (2.12) takes on a very simple form, and we have chosen to solve (2.12)–(2.15), but the solutions to the two sets are linked as follows.

We introduce streamfunctions $\chi$ and $\chi^*$ such that (2.14) and (2.19) are identically satisfied.

\[
\begin{align*}
\bar{\theta} = & \frac{1}{\rho_0} \frac{\partial}{\partial z} \rho_0 \chi \\
\bar{w} = & -\frac{\partial \chi}{\partial y} 
\end{align*}
\]

and, similarly for $\bar{\theta}^*$, $\bar{w}^*$ and $\chi^*$. From (2.16)

\[
\chi^* = \chi - \frac{\bar{v}'\bar{\Phi}'}{S}.
\]

Use of (6.1) results in a single equation to be solved

\[
\frac{\partial}{\partial z} \left( \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \rho_0 \chi \right) + S \frac{\partial^2 \chi}{\partial y^2} = \frac{\partial^2}{\partial y^2} v' \bar{\Phi}'.
\]

If we now put $X = e^{-z} \chi$ and $y' = S^{-1/2} f_0 y$, then (6.3) becomes

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y'^2} \right) X = \frac{1}{S} e^{-z} \frac{\partial^2}{\partial y'^2} v' \bar{\Phi}'.
\]

The lower boundary condition at $z = 0$ is described more fully in Appendix C, and from (C2) is

\[
\bar{w} = -\frac{\partial \chi}{\partial y} = \frac{1}{H} v' \cdot \nabla h,
\]

or, from (C5),

\[
\chi = \frac{v' \bar{\Phi}'}{S} + \frac{\kappa Q' v'}{\bar{u} S} - \frac{D_E}{H u} \frac{\psi' \nabla^2 \psi'}{\bar{u}}
\]

at $z = 0$.

Note that for the case of no orography, this be-
FIG. 11. The maximum magnitude of the eddy fluxes as a function of height for $m = 1$ forced by both orography and heating together, as a function of $\phi_H$ for the case of no friction. Shown are $\bar{w}T$ (K m s$^{-1}$), $\bar{w}\bar{W}$ (10$^5$ m$^2$ s$^{-3}$), $\bar{w}T'$ (K s$^{-1}$), $e^{-1}F_{d0}$ (10$^7$ m s$^{-2}$), and $\nabla \cdot \mathbf{F}$ (10$^6$ m s$^{-2}$).
Fig. 12. As in Fig. 11, but with friction included.
comes \( \chi = 0 \) at \( z = 0 \). However, in general, the thermally forced circulation results in large orographically induced vertical motions with apparent mass sources and sinks at the surface. These arise from the Eulerian frame of reference and approximations in using the kinematically induced lower boundary vertical motion (2.11), and, in reality, the mass sources and sinks are canceled by a low level return flow linking the upslope and downslope components (McIntyre, 1980).

Substituting (6.5) [or (C4)] into (2.15) gives the surface zonal mean temperature tendency as

\[
\frac{\partial \Phi_x}{\partial t} = -\frac{\partial}{\partial y} \left[ Q' \frac{\kappa}{\bar{u} S} + \frac{D_E}{H \bar{u}} \nabla^2 \frac{\psi}{\psi} \right]
\]

at \( z = 0 \). Thus for the mid-tropospheric heating profile, where \( Q' = 0 \) at \( z = 0 \), and in absence of friction

\[
\frac{\partial \Phi_x}{\partial t} = 0
\]

at \( z = 0 \). In this case, \( \chi^* = 0 \) at \( z = 0 \) also. This occurs in spite of nonzero tendencies for \( \Phi_x \) in the interior of the fluid.

For the upper boundary condition, we note that as \( z \) becomes large, \( Q \rightarrow 0 \), \( F \) becomes constant and the Charney–Drazin and Eliassen–Palm theorems apply. Therefore \( \nabla \cdot F \rightarrow 0 \), but note from Figs. 9–12 that this only occurs for \( z > 4 \). Consequently, \( \vec{v}^* \), \( \vec{w}^* \rightarrow 0 \) and \( \chi^* \rightarrow \) constant. Also \( \bar{v} \rightarrow 0 \) and \( \rho_0 \chi \) or \( X \rightarrow \) constant. Therefore, a suitable upper boundary condition for solving (6.4) is

\[
\frac{dx}{dz} \rightarrow 0
\]

as \( z \rightarrow \infty \).

For the lateral boundary conditions, two different criteria have been used. For consistency with the \( \beta \) plane channel model with walls at \( y = 0, \pi / l \), it is required that \( \bar{v} = 0 \) and thus \( \chi = 0 \) on the boundaries. However, it is also useful to consider an alternative infinite \( \beta \) plane with cyclic boundary conditions in \( y \). The advantage of the latter is that we may then obtain an analytic solution to (6.4) and (6.5) [see Appendices C and D].

Once the solution for \( X \) has been found, it is straightforward to find \( \vec{v}, \vec{w} \) using (6.1), \( \chi^* \) using (6.2), and \( \vec{v}^* \) and \( \vec{w}^* \). Note that

\[
\frac{\partial \bar{u}}{\partial t} = f \vec{v},
\]  
\[
\frac{\partial \bar{T}}{\partial t} = -\frac{S}{R} \vec{w}^*,
\]

so that \( \vec{v} \) and \( \vec{w}^* \) are proportional to the tendencies of the zonal mean field. If the \( \bar{v} \) field is multiplied by 8.9, we obtain the value of \( \partial \bar{u}/\partial t \) in m s\(^{-1}\) day\(^{-1}\); and from (6.9) by multiplying \( \vec{w}^* \) by \(-1.61 \times 10^6\) we obtain \( \partial \bar{T}/\partial t \) in K day\(^{-1}\). In presenting results, we have therefore included panels showing \( \vec{v} \) and \( \vec{w}^* \) in addition to the total streamlines of the mass flow field. As an alternative to the tendencies in the zonal mean field, we emphasize, again, that we could define zonal mean heating and friction necessary to maintain the steady state.

b. Propagating wave 1

We now specifically consider the solution to these equations for the propagating wave 1 forced by orography and mid-tropospheric heating, as given by (3.8)–(3.13) and the fluxes (B1)–(B4).

Equations of this form may be solved using Green’s functions, as shown by Kuo (1956), but such a solution is complicated by the nature of the lower boundary condition. The appropriate Green’s function for homogeneous boundary conditions are modified Bessel functions of zero order and may be interpreted as influence functions. We have used this method to solve the equations for the case of heating alone, which has \( \chi = 0 \) as the lower boundary condition, and the influence of the lateral boundaries is seen to be mainly confined within 10° latitude from the boundary.

The rhs of (6.4) has a \( \cos 2\theta \) dependence which is compatible with the form of the lower boundary condition on an infinite \( \beta \) plane with cyclic boundary conditions. Consequently, by assuming

\[
X = X(z) \cos 2\theta
\]

in (6.4), the system reduces to a linear differential equation with constant coefficients. The solution is given in Appendix D. It has the advantage that we can easily explore the sensitivity of the solution to changes in parameters.

More generally, solutions to (6.4), (6.5) and (6.7) may be found using finite difference methods and elliptic equation solving software such as subroutine SEPSX4 available through NCAR.

The analytic solution was used as a check on this solution and it was found necessary to apply the upper boundary condition at \( z = 6.0 \) in order for the solution below \( z = 4.0 \) to be unaffected. This is necessary since the rhs of (6.4) is proportional to \( e^{-zF_\theta} \), shown in Figs. 9–12, and the divergence of the E–P flux is still non-zero at \( z = 4.0 \) (see Figs. 9–12).

Results are presented for the \( \beta \) plane with walls at \( y = 0, \pi / l \). Fig. 13 shows the solution for the forcing due only to the orographic wave in the presence of Ekman pumping. Fig. 14 shows the corresponding solution for the thermally forced wave. Shown in each case are \( X = e^{-z}X \) which depicts streamlines of the Eulerian mass flow field, \( e^{-z}X \) showing the corresponding residual circulation, \( \vec{v} \) and \( \vec{w}^* \).

In Fig. 13 the main result is the vertical motion that offsets the convergence of the heat flux. In the case of no friction, there is an exact balance and \( \vec{w}^* = 0 = \vec{v} \). Therefore, the meridional motion evident
Fig. 13. Zonal mean flow solution for forcing by the orographic wave in the presence of Ekman pumping: (a) $\chi = e^{-z} \chi_x$, the Eulerian mass flow streamlines ($10^{-4}$ m s$^{-1}$), (b) $e^{-z} \chi^*$ the residual mass flow streamlines ($10^{-3}$ m s$^{-1}$), (c) $\bar{v}$ ($10^{-5}$ m s$^{-1}$), and (d) $\bar{w}^*$ ($10^{-11}$ s$^{-1}$).

in Fig. 13 is associated with Ekman pumping but is quite small. In this case $\bar{v} = \bar{v}^*$. The residual stream-function shows that the induced Eulerian vertical motion is more than necessary to offset the heat flux convergence and induces a net cooling tendency at high latitudes, a warming at low latitudes, and a consequent increase in the thermal wind although $\bar{u}$ decreases everywhere.

In Fig. 14 the magnitudes of both Eulerian and residual circulations are 1 to 2 orders of magnitude
greater than in Fig. 13. The poleward heat flux by the thermally forced wave drives a direct circulation with maximum $\tilde{u}$, and thus acceleration of $u$, near $z = 2$. The main heat flux convergence at high latitudes occurs at $z > 2$ and results in a net warming tendency.

However, the induced vertical motion therefore results in a cooling trend below $z \approx 2$.

The circulation induced by the presence of both orographically and thermally forced waves are shown in Fig. 15 when Ekman pumping is omitted and in
Fig. 15. Zonal mean flow forced by both orographically and thermally forced waves without surface friction as a function of their relative phase $\phi_H$ and $z$: (a) $\bar{v}$ ($10^{-3}$ m s$^{-1}$) at the center of the $\beta$-plane, (b) $w^*$ ($10^{-2}$ s$^{-1}$) at 57° latitude.

Fig. 16 when friction is included. The first panel shows $\bar{v}$ at the center of the $\beta$-plane, where it is a maximum (e.g., see Figs. 13, 14), as a function of $\phi_H$, the phase difference between the two forcings. The second panel shows $w^*$ at 57° latitude, near where it tends to reach a maximum on the poleward side, as a function of $\phi_H$.

The changes as a function of $\phi_H$ are quite systematic with maximum poleward $\bar{v}$ occurring at $\phi_H \approx 180^\circ$ and maximum equatorward $\bar{v}$ at $\phi_H \approx 0^\circ$. However, there is a marked difference between friction and no friction cases. For $\phi_H \approx 0^\circ$ there is a maximum low level poleward heat flux (Figs. 11, 12) which induces a strong Ferrel-type cell but results in net warming at high latitudes and cooling at low latitudes. For $\phi_H \approx 180^\circ$, the pattern is more like that in Fig. 14 for the thermal wave alone, except that the magnitudes are 2–3 times larger. From Figs. 11 and 12, this corresponds to the case where the low level heat flux is equatorwards, while above $z = 2$ the heat flux is polewards. At low levels there is a resemblance between the heat flux convergence (Figs. 11 and 12) and $-\bar{w}^*$ (or $\partial T/\partial t$) (Figs. 15b and 16b) thereby indicating that the heat flux convergence is only partially offset by the induced circulation and there is a residual warming. This also reflects the dominance of the heat flux forcing over the vertical motion arising from orographic effects and Ekman pumping at the lower boundary.

The magnitudes of the induced changes in the zonal mean flow are not large—at most $5 \times 10^{-2}$ K day$^{-1}$ in $\partial T/\partial t$ and 0.2 m s$^{-1}$ day$^{-1}$ for $\partial \bar{v}/\partial t$. However, since this is induced by heat fluxes in the absence of any zonal mean temperature gradient, the magnitudes should be considerably larger with a more realistic basic state. The points that need to be made here are the very large changes that occur as a function of $\phi_H$ and the substantially enhanced forcing of the basic state when the two forced waves interact compared with the situation when only one forced wave is present.

c. Trapped waves

These points are made even more dramatically when trapped waves are considered. As previously noted and shown in Appendix B, in the absence of friction there is no forcing of the zonal mean flow unless both waves are present.

Eqs. (6.4), (6.5) and (6.7) have also been solved for the trapped waves $m = 2$, 3, and 4 with and without friction. In this case, $v^\Phi_c$ is given in Appendix B by Eq. (B9), and the lower boundary condition is given by (C8) in Appendix C. One case is presented in order to emphasize that trapped waves also can transport heat meridionally and modify the zonal mean flow.

The solution for $m = 2$ with surface friction is shown in Fig. 17 corresponding to the parameters $\mu = 0.06$, $E_k = 0.063$ (see Figs 2 and 3). The solution for $m = 3$ is not very different. Maximum response in the zonal mean flow occurs near $\phi_H = 180^\circ$ and this is the case presented in Fig. 17. The patterns are somewhat more clear for the trapped wave cases because of the absence of the propagating component.
In this case, the equatorward heat flux is partially offset by the induced Eulerian circulation but a residual warming at low latitudes and cooling at high latitudes occurs. Note that the magnitudes are significantly larger than those in Figs. 13-16 for the propagating wave. Changes are as large as 0.3 m s$^{-1}$ day$^{-1}$ in $\partial \bar{u}/\partial t$ and 0.1 K day$^{-1}$ in $\partial \bar{T}/\partial t$.

7. Discussion and conclusions

In this paper we have analyzed in detail a very simple model of the atmospheric planetary wave response to orographic and thermal forcing in mid-latitudes. The assumptions made preclude any attempt to compare the model results with the real atmosphere, although it has revealed a number of points worthy of note. The main virtue of the model is that we can obtain complete analytic solutions not only for the wave response but also for the secondary nonlinear forcing by eddy fluxes on the zonal mean flow. We have therefore found the model to be quite useful for teaching purposes as an introduction to the Eliassen–Palm and Charney–Drazin theorems. The presence of diabatic heating in the waves results in non-zero E–P fluxes and violates the conditions for nonacceleration of the zonal mean basic-state flow, both for propagating and trapped waves. This aspect should not be overlooked in applications of these theorems.

However, as shown here, the presence of heating alone is not the only factor of note. For trapped waves the E–P flux is still zero for the thermally forced wave unless Ekman pumping is also present. Of greater importance and the main result of this paper, is the interaction between orographic and thermally forced waves which results in large increases in the E–P flux and in forcing the zonal mean flow for all cases, with or without surface friction. The net effect is strongly dependent upon $\phi_H$, the relative phase of the orography and the thermal forcing. Results have been presented for the entire range of $\phi_H$ since they are pertinent not only to forcing of the mean stationary planetary waves, but also to the response to anomalous forcing and interannual variability.

Although this model is linear, we have inferred that significant nonlinear feedback effects should be present. This form of nonlinearity should not be confused with that considered by Egger (1976c) and Ashe (1979). Roads (1980) and Phillips (1982) have carried out experiments in which the basic flow was allowed to vary with forcing by thermal anomalies, but neither included orographic forcing as well. Our results may, however, help explain some of Phillips' results with a zonally asymmetric basic flow.

Several other results are also of note:

- For large $z$, remote from the heating, it is always possible to choose an equivalent mountain that would produce the same response in the planetary waves as the heating. This applies to both propagating and trapped waves, with or without friction. However, there is a large change in the phase of the equivalent mountain near the resonant wave number.
- For wave 1, the equivalent mountain height for
the mid-tropospheric heating profile with maximum heating of 1 day\(^{-1}\) is 200–300 m. However, the equivalent mountain height has a \(k^{-1}\) dependence and thermal forcing is therefore less important relative to orography at higher wave numbers and especially near resonance.

* The addition of surface friction decreases the amplitude of the orographic wave but increases the
amplitude of the thermal wave by enhancing the conversion of eddy available to eddy kinetic energy. The equivalent mountain height is therefore higher with friction included.

- The mid-tropospheric heating profile was found to be a factor of 4 to 5 more effective in forcing planetary waves than the surface heating profile with the same integrated column heating. In this regard we note that in order to excite wave 1 with mid-tropospheric heating, it is clear that such heating could not arise as a direct response to latent heating associated with large-scale vertical motion but may well result from the collective effects of latent heating released by midlatitude storms organized into storm tracks, see for example Held (1983).

- The energetics of the waves are fairly straightforward. In the troposphere, for \( \phi_H = 0 \), orographic forcing acts as a source of eddy kinetic energy which is converted to eddy available potential energy and dissipated by diabatic heating. For \( \phi_H = 180^\circ \), orographic forcing and friction act as a sink of eddy kinetic energy while diabatic heating generates eddy available potential energy which is converted to eddy kinetic energy. The latter then propagates up to infinity and down to the ground.

- The solutions here have a basic state of \( \bar{u} \) constant. If \( \bar{T} \) were a function of \( y \), \( \langle AZ \cdot AE \rangle \) immediately becomes nonzero, the temperature perturbation field is no longer locked onto the heating field, and the energy cycle can become enhanced. To a limited extent these aspects can be determined using WKB theory, but this solution will be addressed elsewhere. It seems that the nonlinear feedback effects on the zonal mean flow can be much larger than found here.

An overall goal of this work has been not only to shed some further light on the forcing of planetary waves in the Northern Hemisphere in winter but also to consider the situation in other seasons and in the Southern Hemisphere. Whereas planetary waves transport heat polewards in the Northern Hemisphere in winter, they are equivalent barotropic in the Southern Hemisphere. The contrast is especially great for wave 1 which has large amplitude in both hemispheres (Trenberth, 1980). Two explanations appear to be possible:

1) Owing to the larger meridional scale and thus smaller total wave number, wave 1 can propagate in the Northern Hemisphere whereas it is trapped in the Southern Hemisphere (although this does not explain the differences in scale).

2) Both orographic and thermal forcing are significant in the Northern Hemisphere thereby leading to the interactions discussed in this paper. In contrast, in the Southern Hemisphere, thermal forcing is undoubtedly present and associated with the asymmetry of storm tracks in the Southern Hemisphere (Trenberth 1981, 1982) but orographic effects are small.

Antarctica clearly plays a major role in very high latitudes, not through the kinematic lower boundary condition used in this model, but rather as a major obstacle and dam.

Along with the WKB generalization and comparison with a two-layer model, these aspects will be pursued in future work.

**Acknowledgments.** This research was sponsored by the Climate Dynamics Program, Division of Atmospheric Sciences, National Science Foundation under Grants ATM79-16485 and ATM82-11560. My thanks to Noreene McGhiey who typed the manuscript.

### APPENDIX A

**List of Symbols**

\( a \) \quad \text{radius of earth}

\( a_1, a_4 \) \quad \text{see Appendix B}

\( A \) \quad \text{available potential energy}

\( AZ, AE \) \quad Zonal and Eddy \( A \)

\( b_1, b_4 \) \quad \text{see Appendix B}

\( c_i, c_l \) \quad \text{see Appendices B and C}

\( c \) \quad \text{phase velocity}

\( C_p \) \quad \text{specific heat at constant pressure}

\( D_E \) \quad \text{Ekman layer depth}

\( d, d_0, d_1, d_2 \) \quad \text{see (3.5)}

\( E_0 \) \quad \text{energetics constant, see (2.39)}

\( E_k \) \quad \text{Ekman parameter, see (2.34)}

\( F \) \quad \text{Eliassen-Palm flux, components \([F_{(9)}, F_{(c)}]\)}

\( f \) \quad \text{friction}

\( f_0 \) \quad \text{constant value see (3.6)}

\( G \) \quad \text{generation of} \ AE \text{by heating}

\( g \) \quad \text{gravity}

\( g_1, g_2 \) \quad \text{see Appendix D}

\( H \) \quad \text{scale height}

\( h \) \quad \text{height of topography,} \ \bar{h} \text{ Fourier component}

\( h_E \) \quad \text{equivalent height of topography for thermal forcing} \[ h = S h / H \]

\( i \) \quad \sqrt{-1}

\( J \) \quad \text{Jacobian} \[ J(A, B) = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial B}{\partial x} \frac{\partial A}{\partial y} \]

\( K \) \quad \text{total wavenumber} \[ K^2 = \beta / \bar{u} \]

\( K \) \quad \text{kinetic energy}

\( KZ, KE \) \quad \text{Zonal and Eddy} \( K \)

\( k \) \quad \text{east–west wavenumber}

\( k \) \quad \text{unit vertical vector}

\( l \) \quad \text{north–south wavenumber}

\( l \) \quad \text{[=(S/2)/(foil)]}

\( m \) \quad \text{east–west nondimensional wavenumber}

\( n \) \quad \text{north–south nondimensional wavenumber}

\( p \) \quad \text{pressure,} \ \bar{p}_0 \text{ standard pressure}

\( p_{ij} \) \quad \text{see Appendix D}
The remaining term that arises only from the interaction of both kinds of forcing.

For \( \bar{u} \) constant, we introduce the following shorthand notation:

\[
\begin{align*}
\frac{b_1(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= e^{-\alpha_1 z} - e^{-\alpha_2 z} \\
\frac{b_2(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} - \frac{(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \\
\frac{b_3(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{\alpha_2(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} - \frac{\alpha_2(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \\
\frac{b_4(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1 + \alpha_1}{(1/2 + \alpha_1 + \mu)} e^{-\alpha_1 z} - \frac{1 + \alpha_2}{(1/2 + \alpha_2 + \mu)} e^{-\alpha_2 z}
\end{align*}
\]

\[
\begin{align*}
c_1 &= \left( \frac{\gamma^2 + \frac{1}{4}}{4} \gamma + E_k \right) - \frac{1}{2} E_k (1 + \alpha) \frac{1}{d} \\
c_2 &= -\left( \frac{1}{2} \left( \gamma^2 + \frac{1}{4} \right) + E_k (1 + \alpha) \gamma + E_k \right) \frac{1}{d} \\
c_3 &= E_k \left[ (1 + \alpha) \left( \gamma^2 + \frac{1}{4} + \gamma E_k \right) - \frac{1}{2} \left( \gamma^2 + \frac{1}{4} \right) \right] \frac{1}{d} \\
c_4 &= \left[ \left( \gamma^2 + \frac{1}{4} \right) \left( \gamma^2 + \frac{1}{4} + \gamma E_k \right) + \frac{1}{2} E_k^2 (1 + \alpha) \right] \frac{1}{d} \\
c_5 &= \frac{\gamma}{d^2} \left[ \left( \gamma^2 + \frac{1}{4} \right) E_k^2 (1 + \alpha)^2 \right] \\
c_6 &= \frac{2 \gamma}{d} \left[ \left( \gamma^2 + \frac{1}{4} \right) \sin \phi_H + E_k (1 + \alpha) \cos \phi_H \right] \\
c_7 &= -(\gamma + E_k) \sin \phi_H + \frac{1}{2} \cos \phi_H \\
c_8 &= (\gamma + E_k) \cos \phi_H + \frac{1}{2} \sin \phi_H \\
c_9 &= \frac{1}{2} \gamma E_k \sin \phi_H + (\gamma^2 + \frac{1}{4} + \gamma E_k) \cos \phi_H \\
c_{10} &= \frac{1}{2} E_k \cos \phi_H + (\gamma^2 + \frac{1}{4} + \gamma E_k) \sin \phi_H \\
c_{11} &= \frac{1}{2} \left( \frac{1}{2} - \mu \right) \left( \frac{1}{2} - \mu + \alpha \right) E_k \\
c_{12} &= \frac{1}{2} \left( \frac{1}{2} - \mu \right) \left( \frac{1}{2} - \mu + \alpha \right) E_k \\
\end{align*}
\]

\[
\begin{align*}
\frac{a_1(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= c_1 \cos \gamma z + c_2 \sin \gamma z \\
\frac{a_2(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= c_3 \cos \gamma z + c_4 \sin \gamma z \\
\frac{a_3(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \left( \gamma + E_k \right) \sin \gamma z - \left( \gamma + E_k \right) \cos \gamma z \\
\frac{a_4(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= c_7 \cos \gamma z + c_8 \sin \gamma z \\
\frac{a_5(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1}{2} E_k \sin \gamma z - \phi_H \\
\frac{a_6(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1}{2} E_k \sin \gamma z - \phi_H \\
\frac{a_7(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= c_9 \cos \gamma z + c_{10} \sin \gamma z \\
\frac{a_8(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1}{2} E_k \sin \gamma z - \phi_H \\
\frac{a_9(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1}{2} E_k \sin \gamma z - \phi_H \\
\frac{a_{10}(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= c_9 \cos \gamma z + c_{10} \sin \gamma z \\
\frac{a_{11}(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1}{2} E_k \sin \gamma z - \phi_H \\
\frac{a_{12}(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1}{2} E_k \sin \gamma z - \phi_H \\
\end{align*}
\]

\[
\begin{align*}
\frac{b_1(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= e^{-\alpha_1 z} + e^{-\alpha_2 z} \\
\frac{b_2(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} - \frac{(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \\
\frac{b_3(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{\alpha_2(1 + \alpha_1)}{d_1} e^{-\alpha_1 z} - \frac{\alpha_2(1 + \alpha_2)}{d_2} e^{-\alpha_2 z} \\
\frac{b_4(z)}{\frac{1}{d_1} - \frac{1}{d_2}} &= \frac{1 + \alpha_1}{(1/2 + \alpha_1 + \mu)} e^{-\alpha_1 z} - \frac{1 + \alpha_2}{(1/2 + \alpha_2 + \mu)} e^{-\alpha_2 z}
\end{align*}
\]

\[
\begin{align*}
c_1 &= \left( \frac{\gamma^2 + \frac{1}{4}}{4} \gamma + E_k \right) - \frac{1}{2} E_k (1 + \alpha) \frac{1}{d} \\
c_2 &= -\left( \frac{1}{2} \left( \gamma^2 + \frac{1}{4} \right) + E_k (1 + \alpha) \gamma + E_k \right) \frac{1}{d} \\
c_3 &= E_k \left[ (1 + \alpha) \left( \gamma^2 + \frac{1}{4} + \gamma E_k \right) - \frac{1}{2} \left( \gamma^2 + \frac{1}{4} \right) \right] \frac{1}{d} \\
c_4 &= \left[ \left( \gamma^2 + \frac{1}{4} \right) \left( \gamma^2 + \frac{1}{4} + \gamma E_k \right) + \frac{1}{2} E_k^2 (1 + \alpha) \right] \frac{1}{d} \\
c_5 &= \frac{\gamma}{d^2} \left[ \left( \gamma^2 + \frac{1}{4} \right) E_k^2 (1 + \alpha)^2 \right] \\
c_6 &= \frac{2 \gamma}{d} \left[ \left( \gamma^2 + \frac{1}{4} \right) \sin \phi_H + E_k (1 + \alpha) \cos \phi_H \right] \\
c_7 &= -(\gamma + E_k) \sin \phi_H + \frac{1}{2} \cos \phi_H \\
c_8 &= (\gamma + E_k) \cos \phi_H + \frac{1}{2} \sin \phi_H \\
c_9 &= \frac{1}{2} \gamma E_k \sin \phi_H + (\gamma^2 + \frac{1}{4} + \gamma E_k) \cos \phi_H \\
c_{10} &= \frac{1}{2} E_k \cos \phi_H + (\gamma^2 + \frac{1}{4} + \gamma E_k) \sin \phi_H \\
c_{11} &= \frac{1}{2} \left( \frac{1}{2} - \mu \right) \left( \frac{1}{2} - \mu + \alpha \right) E_k \\
c_{12} &= \frac{1}{2} \left( \frac{1}{2} - \mu \right) \left( \frac{1}{2} - \mu + \alpha \right) E_k \\
\end{align*}
\]
For the propagating wave, the momentum flux \( u'v' = 0 \). The poleward heat flux is given by

\[
\begin{align*}
\overline{v'T_h} &= \frac{kh^2}{RfG^2} \gamma e^{\gamma s(y)} \\
\overline{v'T_Q} &= \frac{Q^*}{Rf_0G^2} \overline{e^{z/2}[c_5e^{z/2} + b_3(z)a_1(z)]} \\
&- \overline{b_2(z)a_2(z)]s(y)} \\
\overline{v'T_I} &= \frac{h^*Q^*}{Rf_0G^2} \overline{e^{z/2}[-c_6e^{z/2} + b_3(z)a_3(z)]} \\
&+ \overline{b_2(z)a_4(z)]s(y)}
\end{align*}
\] (B2)

The vertical flux of geopotential is given by

\[
\begin{align*}
\overline{w\Phi_h} &= \frac{k\overline{uh^2}}{Sg^2} \gamma e^{\gamma s(y)} \\
\overline{w\Phi_Q} &= \frac{\overline{uh^2Q^*}}{Sg} \overline{e^{z/2}[c_5e^{z/2} - b_1(z)a_1(z)]} \\
&\times (\gamma^2 + \mu) - \overline{b_2(z)a_2(z)]s(y)} \\
\overline{w\Phi_I} &= \frac{\overline{uh^2Q^*}}{Sg^2} \overline{e^{z/2}[-c_6e^{z/2} - b_1(z)a_3(z)]} \\
&\times (\gamma^2 + \mu) + \overline{b_2(z)a_4(z)]s(y)}
\end{align*}
\]

The vertical heat flux is given by

\[
\begin{align*}
\overline{w'T_h} &= 0 \\
\overline{w'T_Q} &= \frac{Q^*}{Rsk} \overline{e^{z/2}k\hat{Q}(z)a_2(z)]s(y)} \\
\overline{w'T_I} &= -\frac{h^*}{SRg^2} \overline{e^{z/2}k\hat{Q}(z)a_4(z)]s(y)}
\end{align*}
\] (B3)

and it is readily shown that

\[
\kappa \overline{Q'T} = Sw'T'.
\] (B5)

From the above, the divergence of the vertical flux of geopotential is shown to obey

\[
\frac{1}{\rho_0} \frac{d}{dz} \left( \rho_0 \overline{w\Phi} \right) = Rw'T'.
\] (B6)

In other words, since \( \overline{u} \) is constant \( \langle AZ \cdot AE \rangle = 0 \), \( GE = \langle AE \cdot KE \rangle = \) the flux divergence of geopotential energy, as given in Section 2d.

The divergence of the Eliassen–Palm flux, from (2.21) and (2.22),

\[
\nabla \cdot \mathbf{F} = \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 \frac{f_R}{S} \overline{v'T} \right),
\] (B7)

and has the three components

\[
\nabla \cdot \mathbf{F}_h = 0
\]
\[
\begin{align*}
\nabla \cdot \mathbf{F}_Q &= -\frac{Q^*}{Suk} \overline{e^{z/2}a_1(z)} \frac{1}{\rho_0} \frac{d}{dz} \left[ \rho_0 k\hat{Q}(z)]s(y) \right] \\
\nabla \cdot \mathbf{F}_I &= \frac{h^*}{Suk} \overline{e^{z/2}a_3(z)} \frac{1}{\rho_0} \frac{d}{dz} \left[ \rho_0 k\hat{Q}(z)]s(y) \right]
\end{align*}
\] (B8)

For the trapped waves \( u'v' = 0 \) also. The poleward heat flux is given by

\[
\begin{align*}
\overline{v'T_h} &= 0 \\
\overline{v'T_Q} &= -\frac{Q^*}{Rf_0k} \overline{c_1b_4(z)e^{(1/2-\mu)s(y)}} \\
\overline{v'T_I} &= \frac{h^*Q^*}{Rf_0} \overline{c_2b_4(z)e^{(1/2-\mu)s(y)}}
\end{align*}
\] (B9)

The vertical heat flux is given by

\[
\begin{align*}
\overline{w'T_h} &= 0 \\
\overline{w'T_Q} &= \frac{Q^*}{RSk} \left( \frac{1}{2} - \mu \right) \overline{c_1b_4(z)e^{(1/2-\mu)s(y)}} \\
\overline{w'T_I} &= -\frac{h^*}{RS} \left( \frac{1}{2} - \mu \right) \overline{c_2b_4(z)e^{(1/2-\mu)s(y)}}
\end{align*}
\] (B10)

Eq. (B5) and (B6) also apply for the trapped waves and the divergence of the Eliassen–Palm flux, given by (B7), is

\[
\nabla \cdot \mathbf{F}_h = 0
\]
\[
\begin{align*}
\nabla \cdot \mathbf{F}_Q &= -\frac{Q^*}{Suk} \overline{c_1b_4(z)} \frac{1}{\rho_0} \frac{d}{dz} \left[ \rho_0 k\hat{Q}(z)]e^{(1/2-\mu)s(y)} \right] \\
\nabla \cdot \mathbf{F}_I &= \frac{h^*}{Suk} \overline{c_2b_4(z)} \frac{1}{\rho_0} \frac{d}{dz} \left[ \rho_0 k\hat{Q}(z)]e^{(1/2-\mu)s(y)} \right]
\end{align*}
\] (B11)

Note that in the no friction case \( c_{11} = 0 \) and only the interaction term is non-zero. It should also be noted that for \( \gamma = 0 \) and \( \mu = 0 \) the solutions for the propagating and trapped waves coincide.

**APPENDIX C**

**Lower Boundary Condition**

From (2.10) and (2.11) the steady state lower boundary condition is

\[
w = \frac{1}{H} (v \cdot \nabla h + D E \nabla^2 \psi).
\]

Thus

\[
w' = \frac{1}{H} \left( \overline{u} \frac{\partial h}{\partial x} + D E \nabla^2 \psi \right),
\] (C1)
\[
\hat{\omega} = \frac{1}{H} (\nu' \cdot \nabla h)
= \frac{1}{H} \left\{ -\frac{\partial \nu'}{\partial y} \frac{\partial h}{\partial x} + \frac{\partial \nu'}{\partial x} \frac{\partial h}{\partial y} \right\}.
\]
(C2)

From (2.24)
\[
\tilde{u} \frac{\partial \psi'}{\partial x} + \frac{w'S}{f_0} \frac{\partial \psi'}{\partial y} = \frac{kQ'}{f_0}
\]
and using (C1), we find
\[
\frac{\partial h}{\partial x} = \frac{H \kappa Q'}{H_S} - \frac{H f_0}{S} \frac{\partial \psi'}{\partial x} = \frac{D_E}{\tilde{u}} \nabla^2 \psi',
\]
so that
\[
h = \frac{H}{\tilde{u} S} \int \frac{\kappa Q' \, dx}{S} - \frac{H f_0}{S} \frac{\psi'_x}{\tilde{u}} \int \nabla^2 \psi' \, dx.
\]
(C3)

Substituting into (C2) and making use of
\[
\frac{\partial \nu'}{\partial y} \frac{\partial \psi'}{\partial x} - \frac{\partial \nu'}{\partial x} \frac{\partial \psi'}{\partial y} = \frac{\partial \left( \frac{\partial \nu'}{\partial x} \frac{\partial \psi'}{\partial y} \right)}{\partial y},
\]
\[
\frac{\partial}{\partial x} \left[ \psi' \int p' \, dx \right] = 0 = \frac{\partial \nu'}{\partial x} \int p' \, dx + \psi' p',
\]
where
\[
p' = \frac{kH \kappa Q'}{\tilde{u} S} \frac{\partial \psi'}{\partial y} - \frac{D_E}{\tilde{u}} \nabla^2 \psi'.
\]
Then we can show that
\[
\hat{\omega} = -\frac{\partial \chi}{\partial y}
= -\frac{\partial}{\partial y} \left[ \frac{\nu' \psi'}{S} + \frac{\kappa \nu' \psi'}{\tilde{u} S} - \frac{D_E}{H \tilde{u}} \psi' \nabla^2 \psi' \right].
\]
(C4)

Therefore
\[
\chi = \frac{\nu' \psi'}{S} + \frac{k \nu' \psi'}{S} \frac{\partial \psi'}{\partial y} - \frac{D_E}{H \tilde{u}} \psi' \nabla^2 \psi' + x_0,
\]
(C5)
at \(z = 0\) is the appropriate lower boundary condition. The constant of integration \(x_0\) depends upon the lateral boundary conditions.

If there is no orography \(\hat{\omega} = 0\). Therefore, for the mid-tropospheric heating case, whose solutions are given by (3.8)–(3.17) and (B1)–(B4), at \(z = 0\), since \(Q' = 0\), from (C5)
\[
\frac{\nu' \psi'}{S} = \frac{D_E}{H \tilde{u}} \psi' \nabla^2 \psi', \quad \text{and} \quad x_0 = 0.
\]

For the propagating wave, it is readily shown using either (C2) or (C5) that, at \(z = 0\),
\[
\chi_h = \frac{k h^*}{f_0 S^2} (\gamma + E_k) s(y),
\]
\[
\chi_Q = 0
\]
\[
\chi_I = \frac{h^* Q^*}{f_0 S^2} c_{13} s(y),
\]
for the \(\beta\)-plane channel model, where \(x = 0\) at \(y = 0\), \(\pi / l\). Alternatively, for the infinite \(\beta\)-plane with cyclic continuity, the \(\frac{1}{2} \sin^2 y = s(y)\) in (C7) should be replaced by \(-\frac{1}{4} \cos 2y\), corresponding to a different constant \(x_0\) in (C5). In the latter case we require that \(\chi\) should be cyclic over a wavelength \(\pi / l\) and a mean of zero.

For the trapped waves at \(z = 0\)
\[
\chi_h = \frac{k h^*}{f_0 (\nu^2 - \mu^2 + E_k^2)} s(y),
\]
\[
\chi_Q = 0
\]
\[
\chi_I = \frac{h^* Q^* (\nu^2 - \mu^2 + \nu^2 - \mu + \alpha)}{S f_0 [((\nu^2 - \mu^2)^2 + E_k^2)]}
\times \left[ (\nu^2 - \mu) \cos \phi_H + E_k \sin \phi_H \right] s(y),
\]
and the same change in \(s(y)\) is necessary if a cyclic boundary condition is used.

\section*{APPENDIX D}

\textbf{Mean Meridional Circulation}

From (3.26) we wish to solve
\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y^2} \right) \chi = \frac{1}{S} \frac{\partial^2}{\partial y^2} \nu' \psi_z
\]
(D1)

where \(y' = S^{-1/2} f_0 y\), \(X(y, z) = e^{-z} x(z, y)\); and at \(z = 0\), from (C7)
\[
X(y, 0) = X_0 \cos 2l y',
\]
(D2)

where
\[
X_0 = -\frac{h}{4 f_0 G^2 \tilde{H} [k h^* (\gamma + E_k) + Q^* c_{13}]},
\]
\[
l' = \frac{S^{1/2} f_0}{f_0}
\]

The upper boundary condition is that \(\rho_0 x\), and thus \(X\) should remain finite as \(z \to \infty\). The right hand side of (D1), from (B1) and (B2) may be rewritten in the form
where

\[ \dot{v} \Phi' = e^2 HF(z) \sin^2 l' y', \]  

(D3)

\[ HF(z) = g_1 + e^{-(\alpha_1 + \frac{1}{2})}(g_{21} \cos \gamma z + g_{31} \sin \gamma z) - e^{-(\alpha_2 + \frac{1}{2})}(g_{22} \cos \gamma z + g_{32} \sin \gamma z), \]  

(D4)

with

\[
g_1 = \frac{1}{2 f_0 G^2} \left[ k h^2 y - c_6 h^2 Q^* + \frac{C_5}{k} Q^{*2} \right],
\]

\[
g_{2j} = \frac{Q^*}{2 f_0 G^2 d_j} \left[ \frac{Q^*}{k} (\alpha_j c_2 - c_3) + h^*(\alpha_j c_4 + c_0) \right],
\]

\[
g_{3j} = \frac{Q^*}{2 f_0 G^2 d_j} \left[ \frac{Q^*}{k} (\alpha_j c_2 - c_4) + h^*(\alpha_j c_3 + c_1) \right],
\]

for \( j = 1, 2 \) and where the \( c_j, j = 1 \cdots 10 \) are defined in Appendix B.

Substituting (D3) into (D1) we get

\[
\left[ \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y^2} \right] X = \frac{2 l^2}{S} HF(z) \cos 2l' y'.
\]

(D5)

Owing to the form of (D2) and (D5) we look for solutions of the form

\[ X = X(z) \cos 2l' y', \]

so that (D5) becomes

\[
\left( \frac{d^2}{dz^2} + \frac{d}{dz} - 4 l^2 \right) X = \frac{2 l^2 R}{S} HF(z),
\]

(D6)

with \( X(0) = X_0 \) and \( X \) finite as \( z \to \infty \).

If we now put

\[ \eta^2 = 4 l^2 + \frac{1}{4}, \]

(D7)

the left-hand side may be factored into

\[
\left( \frac{d}{dz} + \frac{1}{2} - \eta \right) \left( \frac{d}{dz} + \frac{1}{2} + \eta \right) X
\]

and solutions are found by standard methods to be

\[
X(z) = \frac{g_1}{2 S} (e^{-(\alpha_1 + \frac{1}{2}) z} - 1)
+ e^{-(\alpha_1 + \frac{1}{2})}(P_{21} \cos \gamma z + P_{31} \sin \gamma z)
- e^{-(\alpha_2 + \frac{1}{2})}(P_{22} \cos \gamma z + P_{32} \sin \gamma z)
+ e^{-(\alpha + \frac{1}{2})}(X_0 - P_{21} + P_{22}),
\]

(D8)

where

\[ P_{2j} = \frac{2 l^2}{S} \left[ p_{1j} g_{2j} + p_{2j} g_{3j} \right] / (p_{4j}^2 + p_{2j}^2), \]

\[ P_{3j} = \frac{2 l^2}{S} \left[ p_{1j} g_{3j} - p_{2j} g_{4j} \right] / (p_{4j}^2 + p_{2j}^2), \]

and

\[ p_{1j} = \alpha_j^2 - \gamma^2 - \eta^2; \]

\[ p_{2j} = 2 \alpha_j \gamma, \quad j = 1, 2. \]

From (D8) it is straightforward to find \( \tilde{v}, \tilde{w} \) using (6.1), \( \chi^* \) using (6.2) and \( \tilde{v^*}, \tilde{w^*} \).

REFERENCES


