A New Look at Equatorial Quasi-Biennial Oscillation Models

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ABSTRACT

Simplified quasi-biennial oscillation (QBO) models are investigated in light of bifurcation theory. If the two components of the wave forcing are symmetric (i.e., it is a standing wave), the model has a trivial steady solution of no mean zonal flow. The steady solution becomes unstable with respect to an oscillatory eigenmode when the amplitude of the wave forcing exceeds a critical value. Periodic solutions branch off at that point from the steady solution as a result of a Hopf bifurcation. The periodic solutions are well known QBO-type solutions.

If the two components are not symmetric as in the case of a Kelvin wave and a Rossby–gravity wave, the model has a nontrivial steady solution with nonzero mean zonal flow. As in the symmetric case a Hopf bifurcation takes place; periodic solutions appear that are not symmetric with respect to time.

A two-level, prototype model is developed to analyze the QBO mechanism analytically and to get dynamical insights easily. Both vertical diffusion and the shielding effect, which is an integrated effect of wave-momentum flux absorption below a given level, are necessary to obtain periodic solutions (Hopf bifurcations).

1. Introduction

The quasi-biennial oscillation (QBO) of the mean zonal wind in the equatorial stratosphere is one of the clearest examples of wave–mean flow interactions in geophysical fluid dynamics. A theory of the QBO was proposed by Lindzen and Holton (1968) based on the interactions of vertically propagating gravity waves with the mean zonal flow. Absorption of gravity waves at critical levels was the key mechanism in their theory; zonal momentum flux of the waves is deposited in the mean zonal flow to change the vertical profile. An updated theory was developed by Holton and Lindzen (1972); the critical-layer interaction was replaced by attenuation of waves due to infrared cooling. Two components of the equatorial waves, a Kelvin wave and a Rossby–gravity wave, were enough to produce a plausible oscillation of the mean zonal wind. (See recollections by Lindzen 1987 for an account of the development of these theories.)

Fundamental aspects of the QBO theory were investigated in detail by Plumb (1977) who replaced equatorial wave driving by the simpler analog of internal gravity waves. He clearly described how the oscillations take place. Furthermore, Plumb and McEwan (1978) demonstrated oscillations due to the wave–mean flow interactions in a laboratory experiment; a standing internal gravity wave which was forced at the bottom boundary of an annulus generated a strong mean azimuthal circulation, which exhibited a long-period oscillation. The laboratory experiment gave us an assurance that the wave–mean flow interaction can produce an oscillation of the mean flow in a real fluid.

During this period, on the other hand, bifurcation theories were developed in applied mathematics. The theories were applied to some problems in dynamic meteorology: e.g., a simplified model of topographically forced Rossby waves (Yoden 1985) and a stratospheric vacillation model (Yoden 1987a). It has been demonstrated that the bifurcation theories are useful for understanding the nonlinear structure of each system in parameter space. In this paper the QBO models, particularly Plumb’s simple version (1977), are investigated in the light of bifurcation theories. We here stress the mathematical structure inherent in the simple QBO models. Descriptions of the model and the analysis procedure are given in sections 2 and 3, and results are in section 4. In section 5 a two-level analog of the model is developed to provide mathematical and dynamical insights into the QBO models more easily. Discussions are given in section 6 and conclusions in section 7.

2. Model description

Holton and Lindzen (1972) developed a one-dimensional prototype model of the QBO with parameterized effects of attenuating equatorial waves. In this paper we mainly use Plumb’s version (1977) of that...
model in which the wave forcing is due to internal gravity waves. He derived a nondimensionalized equation for the mean zonal flow \( u(z, t) \):

\[
\frac{\partial u}{\partial t} = - \sum_{n=1}^{N} \frac{\partial F_n}{\partial z} + \Lambda \frac{\partial^2 u}{\partial z^2}, \tag{2.1}
\]

where independent variables are height, \( z \), and time, \( t \). The first term in the rhs of Eq. (2.1) is a summation of the convergence of wave momentum flux \( F_n \) over all the wave components \( N \). The vertical momentum flux of gravity waves is parameterized by using a WKB approximation:

\[
F_n(z, t) = F_n(0) \exp \left[ - \int_0^z g_n(z', t) \, dz' \right], \tag{2.2}
\]

where

\[
g_n(z, t) = \frac{\alpha}{k_n(u(z, t) - c_n)^2}. \tag{2.3}
\]

Constants \( k_n \) and \( c_n \) are dimensionless wavenumber and phase velocity of \( n \)th wave, respectively. The dissipation rate of waves is denoted by \( \alpha \). Dimensionless \( \Lambda \) is a coefficient of vertical diffusion. Equations (2.1) and (2.2) can be written as

\[
\frac{\partial u(z, t)}{\partial t} = \sum_{n=1}^{N} F_n(0) g_n(z, t) \times \exp \left[ - \int_0^z g_n(z', t) \, dz' \right] + \Lambda \frac{\partial^2 u(z, t)}{\partial z^2}. \tag{2.4}
\]

Top and bottom boundary conditions are given by

\[
\frac{\partial u}{\partial z} = 0 \text{ at } z = z_T, \tag{2.5}
\]

\[
u = 0 \text{ at } z = 0. \tag{2.6}
\]

We use the same vertical difference as that in Plumb (1977) with \( \Delta z = 0.05 \). If we set \( U_j(t) = u(j \Delta z, t); j = 0 \sim J \), Eqs. (2.4) and (2.3) become as follows:

\[
\frac{dU_j}{dt} = \sum_{n=1}^{N} F_n(0) G_{nj} \times \exp \left[ - \left( \frac{1}{2} (G_{n0} + G_{nj}) + \sum_{k=1}^{j-1} G_{nk} \right) \Delta z \right] + \Lambda \frac{U_{j+1} - 2U_j + U_{j-1}}{(\Delta z)^2}, \tag{2.7}
\]

where

\[
G_{nj} = \frac{\alpha}{k_n(U_j - c_n)^2}. \tag{2.8}
\]

The boundary conditions (2.5) and (2.6) become

\[
U_j = U_{j-1}. \tag{2.9}
\]

\[
U_0 = 0. \tag{2.10}
\]

In (2.7) the trapezoidal rule was used for the integration. Equations (2.7) and (2.8) can be regarded as an autonomous dynamical system for \( U = (U_1, \ldots, U_{J+1}) \):

\[
\frac{dU}{dt} = f(U; \Lambda, \alpha, F_n(0), k_n, c_n). \tag{2.11}
\]

In this paper we consider a simple case of two waves, \( N = 2 \). We set \( k_1 = k_2 = 1 \) and \( c_1 = -c_2 = 1 \); the two waves have the same horizontal wavelength and frequency but propagate in opposite directions. Furthermore, we use constant values of \( \Lambda = 0.02 \) and \( \alpha = 1 \) unless otherwise mentioned. Therefore, the controllable parameters in (2.11) are wave momentum fluxes at the bottom boundary, \( F_1(0) \) and \( F_2(0) \); hereafter, we use \( A_n \) instead of \( F_n(0) \). Because the top boundary condition is applied at \( z = 4 \) (\( J = 80 \)), the system (2.11) consists of 79 components of \( U_j \).

3. Analysis procedure

a. Steady solutions

A steady solution \( \hat{U} \) satisfies the simultaneous nonlinear algebraic equation, \( f(\hat{U}) = 0 \). If the wave momentum flux at the bottom is symmetric, i.e., \( A_1 = -A_2 \), Eq. (2.11) has a trivial steady solution of \( \hat{U} = 0 \) for any \( A_1 \). As for nontrivial steady solutions, we search for these using Powell's hybrid method in the IMSL package.

b. Linear stability of steady solutions

If we assume a small perturbation \( U' \) from the steady solution \( \hat{U} \), Eq. (2.11) can be linearized for \( U' \):

\[
\frac{dU'}{dt} = f'(\hat{U})U', \tag{3.1}
\]

where \( f'(\hat{U}) \) is a coefficient matrix, whose elements are \( \partial f_i/\partial U_j \), evaluated at \( U = \hat{U} \). Linear stability of the steady solution is investigated by obtaining eigenvalues and eigenvectors of the matrix \( f' \). Elements of \( f' \) for the symmetric case \( A_1 = -A_2 \) are shown in the Appendix.

It may be useful to show the linearized equation of the continuous form (2.4) in considering the dynamics of instability. If we write a steady solution \( \bar{u} \) and a small perturbation from it \( u' \), the corresponding linearized equation becomes as follows:

\[
\frac{\partial u'}{\partial t} = \sum_{n=1}^{N} A_n \frac{\alpha}{k_n(\bar{u} - c_n)^2} \times \exp \left[ - \int_0^z \frac{\alpha}{k_n(\bar{u}(z') - c_n)^2} \, dz' \right] \times \left[ - \frac{2u'}{\bar{u} - c_n} + \int_0^z \frac{2\alpha u'(z', t)}{k_n(\bar{u}(z') - c_n)^3} \, dz' \right] + \Lambda \frac{\partial^2 u'}{\partial z^2}. \tag{3.2}
\]
Here we used Taylor's expansions for fractional expressions and exponential functions. If we consider the symmetric case of \( \bar{u} = 0 \), Eq. (3.2) becomes Eq. (3.1) in Plumb (1977). Following him, we call the two terms within the braces of (3.2) the "amplification term" and the "shielding term." The last term in (3.2) is a diffusion. The amplification term comes from the perturbation of \( g_0(z, t) \) in (2.4); a perturbation \( \bar{u}' \) changes the attenuation rate of each wave to produce an acceleration of \( u' \). While the shielding term comes from the exponential function in (2.4); it is an integrated effect of wave-momentum flux absorption below a given level, \( z \).

c. Time-dependent solutions

We adopt a leapfrog scheme with periodic smooth restarts for time-integrations of (2.11). The initial condition is a steady solution \( \bar{U} \) with small perturbations. Integrations are performed until the solution converges to a periodic solution. No chaotic solution was obtained in this study.

4. Results

We analyzed the solutions for the two-dimensional parameter space of \( A_1 \) and \( A_2 \) along three intersections of \( a-a', b-b', \) and \( c-c' \) (Fig. 1). The intersection \( a-a' \) is the symmetric case with \( A_1 = -A_2 \), while the others are nonsymmetric cases. The system (2.11) has an antisymmetry with respect to \( A_1 \) and \( A_2 \); if \( U_0 \) is a solution for \( A_1 = A_{10} \) and \( A_2 = A_{20} \), then \( -U_0 \) is a solution for \( A_1 = -A_{10} \) and \( A_2 = -A_{20} \), too. Therefore, the analyses for the nonsymmetric cases are the same except for the sign along broken lines in Fig. 1.

a. Symmetric case

When the wave forcing is symmetric, i.e., it is a standing wave forcing as in the laboratory experiment by Plumb and McEwan (1978), Eq. (2.11) has a trivial steady solution of \( \bar{U} = 0 \). We could not obtain any other steady solution by Powell's method with a lot of combinations of initial guesses.

Linear stability of the steady solution \( \bar{U} = 0 \) is shown in Fig. 2 where the intensity of the wave-momentum flux at the bottom, \( A_1 \), is the parameter. Below a critical value of \( A_1 = 0.0877 \), the steady solution is stable. Above that value, on the other hand, it is unstable for an oscillatory eigenmode of complex eigenvalue. This is a Hopf bifurcation point with pure imaginary eigenvalue and it is expected that periodic solutions bifurcate from this point (cf. Yoden 1987). The steady solution

![Image](https://via.placeholder.com/150)

**Fig. 1.** A two-dimensional parameter space of \( A_1 \) and \( A_2 \). We make the analysis along three intersections: \( a-a' \), \( b-b' \), and \( c-c' \).

**Fig. 2.** Linear stability of the trivial steady solution \( \bar{U} = 0 \). (a) Growth rate \( (\sigma) \). Real eigenvalue is denoted by a dot while complex conjugates are by \( x \). Hopf bifurcation points are indicated by arrows. (b) Period \( (2\pi/|\sigma|) \). Stable (negative \( \sigma \)) eigenvalue is denoted by \( x \) and unstable (positive \( \sigma \)) one by \( * \). A star indicates the period of periodic solution in the nonlinear system (cf. Fig. 4).
becomes unstable with respect to another oscillatory eigenmode at $A_1 = 0.289$. These critical values of $A_1$ depend on the parameters fixed in Eq. (2.11), $\Lambda$, $\alpha$, $k_n$ and $c_n$.

Neutral eigenmodes of $U'(z, t)$ at the Hopf bifurcation points are shown in Fig. 3. Both modes show periodic descent of the perturbed mean flow. The first mode (a) has a node around $z = 1$, while the second mode (b) has a smaller vertical scale of $U'$ with three nodes.

Time-dependent solutions were obtained to confirm the Hopf bifurcation. Near the first bifurcation point (Fig. 4a) there exists a periodic solution which is similar to the neutral eigenmode of Fig. 3a. The period of the oscillation is also close to that given by the linear theory, $2\pi / |\sigma|$ (see Fig. 2b). This is a case of supercritical Hopf bifurcation. If the wave forcing $A_1$ increases, the amplitude of the oscillation becomes larger and the period becomes shorter (Fig. 4b–f). For large $A_1$ the nodal structure disappears (d–f): amplitude is largest below $z = 1$ and decreases monotonically with height. The variation is symmetric with respect to time; $U(t + T/2) = -U(t)$, where $T$ is a period.

We could not obtain any periodic solution similar to the second eigenmode around $A_1 = 0.289$, however. Even if we take a very small amplitude of the eigenmode as an initial condition, the time-dependent solution is finally attracted into a periodic solution closely resembling that of the first linear mode. Thus, we conclude that this is a case of subcritical Hopf bifurcation.

Bifurcation properties of the symmetric case are summarized in Fig. 5. The trivial steady solution $\bar{U} = 0$ becomes unstable at $A_1 = 0.0877$ with respect to an oscillatory eigenmode. There appears a periodic solution as a result of supercritical Hopf bifurcation. As the amplitude of wave forcing increases, the amplitude of the mean wind oscillation becomes "saturated" around 0.6 at $z = 1$. If an oscillation is assumed to consist of a single frequency (like the eigenmode of Fig. 3), the amplitude is theoretically expected to be $\sqrt{\frac{A_1}{a} + b}$ as shown by the dotted line in Fig. 5 (e.g., see Marsden and McCracken 1976). However, $A_1$ sufficiently greater than its value at the bifurcation point, the amplitude of oscillation departs from the theoretical curve due to higher harmonics of the fundamental frequency which are produced by the nonlinearity of the system. The time-variation is no longer sinusoidal in Fig. 4e and f particularly near the bottom boundary. The saturation of the amplitude is consistent with the physical constraint on the upper bound of the amplitude, $|U_0| < |c_n| = 1.0$ for the present case.

Figure 6 shows each term in Eq. (3.2) for the neutral eigenmode at the bifurcation point. It is clear from (3.2) that the amplification term is in phase with $u'$. Its maximum variation appears around $z = 0.4$; above there it decreases with height due to the exponential factor. The shielding term has its maximum variation around $z = 1$. This term is nearly out of phase with $u'$ particularly in the lower layers. The diffusion term has its primary maximum variation around $z = 0.3$ and a secondary one around $z = 1.3$. This is also nearly out of phase with $u'$ below $z = 1$. The acceleration is determined as the residual of these three large terms; near the bottom boundary, the amplification and the diffusion terms are large and almost cancel each other. While the shielding term becomes important in the middle layers ($0.5 \leq z \leq 2.5$).

Generally, the linearized equation is not very useful for analysis of the periodic solutions except very close to the Hopf bifurcation point; from Fig. 5 the linearization would be justified for $A_1 < 0.14$.

Figure 7 shows each term in Eq. (2.4) for a nonlinear period solution of $A_1 = 0.3$ (Fig. 4e), because the linear analysis in Fig. 6 is no longer appropriate for this parameter value. The wave with the same sign of phase speed as the mean zonal flow attenuates near the bottom boundary ($z < 0.2$); its flux divergence is largely canceled by the diffusion. The other wave penetrates into the upper layers to change the mean zonal flow. The level of maximum value of the wave drive descends as the zero-wind line descends. The diffusion term becomes important below $z = 1$ during the period when switching of mean zonal flow takes place, as clearly described by Plumb (1977).

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**Fig. 3.** Time–height sections of neutral eigenmode of $U'(z, t)$ at the Hopf bifurcation points: $A_1 = 0.0877$ (a) and 0.289 (b). Contour interval is arbitrary. Negative values are shaded. Vertical line denotes one period.
b. Nonsymmetric cases

If $A_1$ is different from $-A_2$, Eq. (2.11) no longer has any trivial solution. First we show a result for the nonsymmetric case in which $A_2$ is small ($= -0.05$; $b$--$b'$ in Fig. 1). As shown in Fig. 8, only one steady solution is obtained for each value of the parameter $A_1$. Linear stability analysis shows that all of these steady solutions are stable.

Vertical profiles of the stable steady solutions are shown in Fig. 9 for several values of $A_1$. When $A_1$ is close to zero ($=0.001$), the mean zonal flow has a negative shear below $z \approx 0.5$ and is nearly constant above that level. As $A_1$ increases, the mean zonal flow in a steady state increases at all the levels. The increment is large in the upper layers because the wave with positive phase speed does not attenuate very much in the lower layers due to large negative flow. As $A_1$ approaches $-A_2$, the mean zonal flow tends to approach the symmetric case of $\bar{U} = 0$. When $A_1$ exceeds $-A_2$, the mean zonal flow has a mirror image of that for $A_1$ just below $-A_2$. (Compare the profile of $A_1 = 0.04$ in Fig. 9a with that of $A_1 = 0.06$ in Fig. 9b.) For further increased $A_1$, a strong positive flow appears near the bottom boundary with strong negative flow above it.

Figure 10 shows the balance among the three terms on the rhs of Eq. (2.4) for six steady solutions. When $A_1$ is very small (a), the wave $\mathcal{N} = 2$ with negative phase speed attenuates below $z = 0.8$ and convergence of its momentum flux is balanced by vertical (down-
ward) diffusion. If $A_1$ increases (b), wave 2 penetrates into upper layers due to a smaller (negative) mean flow. Wave 1 is dominant above $z \sim 1.2$ and tends to produce a negative curvature of the mean zonal flow. The symmetric forcing of $A_1 = -A_2$ (c) is a special case with $D = 0$ due to $\overline{U} = 0$; the two waves cancel out. Panel (d) is nearly a mirror-image of (b). For large $A_1$, (e) and (f), wave 1 attenuates rapidly near the bottom boundary and mostly balances the vertical diffusion. Wave 2 also attenuates around $z = 1$ to produce a positive curvature. As a result the mean zonal flow is nearly constant above $z = 1.5$ in (f).

If the fixed wave forcing of $A_2$ has a negatively large value ($-0.15$), the bifurcation properties are qualitatively different from the small value case. Figure 11 shows a bifurcation diagram for $A_2 = -0.15$. As in the small value case we obtained only one steady solution for a given value of $A_1$. However, just as in the symmetric case, the steady solution becomes unstable at $A_1 = 0.087$ with respect to an oscillatory eigenmode. As a result of the Hopf bifurcation there exist periodic solutions above this point. The critical value is nearly equal to that in the symmetric case.

Vertical profiles of the stable and the unstable steady solutions are shown in Fig. 12. For small values of $A_1$ the vertical profile is qualitatively similar to that in Fig. 9. However, above $A_1 = 0.08$, there exist two nodes with $U_j = 0$ ($j \neq 0$). As $A_1$ increases, the nodes descend downward. Note that this is not a classical shear instability of parallel flow with respect to any wave perturbation. It is, rather, apparently a stability problem with respect to a perturbation of the mean zonal flow; the gravity waves forced at (or below) the bottom boundary have a perturbed wave driving of the mean zonal flow through the amplification and the shielding terms. Therefore, it is impossible to deduce the stability from

FIG. 5. Bifurcation diagram for the symmetric forcing case; mean zonal flow at $z = 1$ for the parameter $A_1 = -A_2$ (wave momentum flux at the bottom). Stable steady solution is denoted by a solid circle; unstable one with complex conjugate eigenvalues of positive real part by X. Range of a periodic solution is shown by a vertical bar. Dotted line is a fitted curve of $U = \pm \sqrt{a A_1 - b}$ by using values at $A_1 = 0.09$ and 0.1.

FIG. 6. Time–height sections of four terms in the linearized system about the neutral eigenmode in Fig. 3a: (a) acceleration of mean zonal flow, (b) amplification term, (c) shielding term and (d) diffusion term. Contour interval is 0.01 if it is 0.1 in Fig. 3a. That for dotted lines in (a) is 0.002.
Figure 13 shows several examples of periodic solutions. The variation is not symmetric with respect to time (i.e., $U(t + T/2) \neq -U(t)$ except for a symmetric case (c)). Near the Hopf bifurcation point (a), there is no clear downward propagation of the phase of mean zonal flow oscillation, because the variation of the mean zonal flow is small compared with the time-average state (see also Fig. 14). Near the bottom boundary ($z \ll 0.5$) the mean zonal flow tends to have the same sign as the phase speed of the larger amplitude, while it has a longer period of the opposite sign above that level. The period of the oscillation becomes shorter as $A_1$ increases, except for (b).

Time-averages of the periodic solutions over one cycle are compared with the unstable steady solution for the same wave forcing in Fig. 14. Near the Hopf bifurcation point the time-average is close to the steady solution and the variation is small. As $A_1$ increases, the time-average deviates from the steady solution and the variation has a finite amplitude. The symmetric case (c) is an exception; the time-average corresponds to the steady solution $U = 0$ due to the symmetry of time-variation. For larger $A_1$ (d and e), the time-average is different from the steady solution; particularly in the upper layers ($z > 1.2$) they have an opposite sign.

Figure 15 shows some dynamical aspects for an example of nonsymmetric oscillation for $A_1 = 0.1$ and $A_2 = -0.15$. The fundamental characteristics are similar to those in the symmetric case (Fig. 7) except for the asymmetry of the time-variation. Namely, the acceleration is determined by a residual of three large terms: the two wave drivings of the mean zonal flow, and the vertical diffusion. There is a large cancelation between one of the wave driving terms and the vertical

![Figure 7](image)

**Fig. 7.** Time-height sections of each term in the nonlinear system for a periodic solution of $A_1 = 0.3$: (a) acceleration of mean zonal flow, (b) convergence of momentum flux due to a wave 1 with positive phase speed, (c) same as in (b) but for a wave 2 with negative phase speed, and (d) vertical diffusion term.

![Figure 8](image)

**Fig. 8.** Bifurcation diagram for a nonsymmetric forcing case; mean zonal flow at $z = 1$ for the parameter $A_1$. The other parameter of the wave momentum flux is fixed at $A_2 = -0.05$. All of the solutions are linearly stable.

the vertical profile of the mean zonal wind. Indeed the balance among the three terms on the rhs of Eq. (2.4) changes continuously around the critical point similar to the situation in Fig. 10.
Therefore, the equations for the mean zonal winds $U_1(t)$ and $U_2(t)$ are obtained directly from (2.7) and (2.8):

$$
\frac{dU_1}{dt} = \sum_{n=1}^{N} F_n(0) \frac{\alpha}{k_n(U_1 - c_n)^2} \times \exp \left[ -\frac{\alpha \Delta z}{2k_n} \left( \frac{1}{(U_1 - c_n)^2} + \frac{1}{c_n^2} \right) \right] + \frac{\Lambda}{(\Delta z)^2} (U_2 - 2U_1), \quad (5.1)
$$

$$
\frac{dU_2}{dt} = \sum_{n=1}^{N} F_n(0) \frac{\alpha}{k_n(U_2 - c_n)^2} \times \exp \left[ -\frac{\alpha \Delta z}{2k_n} \left( \frac{1}{(U_2 - c_n)^2} + \frac{2}{(U_1 - c_n)^2} + \frac{1}{c_n^2} \right) \right] + \frac{\Lambda}{(\Delta z)^2} (-U_2 + U_1). \quad (5.2)
$$

If $\bar{U}_j$ is a steady solution of (5.1) and (5.2), Eqs. (5.1) and (5.2) can be linearized around $\bar{U}_j$ for a small perturbation $U'_j$:

$$
\frac{dU'_1}{dt} = \sum_{n=1}^{N} F_n(0) \frac{\alpha}{k_n(\bar{U}_1 - c_n)^3} \times \exp \left[ -\frac{\alpha \Delta z}{2k_n} \left( \frac{1}{(\bar{U}_1 - c_n)^2} + \frac{1}{c_n^2} \right) \right] \times \left\{ -2U'_1 + \frac{\alpha \Delta z U'_1}{k_n(\bar{U}_1 - c_n)^3} \right\} + \frac{\Lambda}{(\Delta z)^2} (U_2 - 2U'_1), \quad (5.3)
$$

$$
\frac{dU'_2}{dt} = \sum_{n=1}^{N} F_n(0) \frac{\alpha}{k_n(\bar{U}_2 - c_n)^3} \times \exp \left[ -\frac{\alpha \Delta z}{2k_n} \left( \frac{1}{(\bar{U}_2 - c_n)^2} + \frac{2}{(U_1 - c_n)^2} + \frac{1}{c_n^2} \right) \right] \times \left\{ -\frac{2U'_2}{\bar{U}_2 - c_n} + \frac{\alpha \Delta z U'_2}{k_n(\bar{U}_2 - c_n)^3} + \frac{2\alpha \Delta z U'_2}{k_n(\bar{U}_1 - c_n)^3} \right\} + \frac{\Lambda}{(\Delta z)^2} (-U'_2 + U'_1). \quad (5.4)
$$

These correspond to Eq. (3.2) for the continuous model.

5. Two-level model

We develop a two-level model as an aid for understanding the results obtained in the previous section more easily. One of the advantages is that we can solve an eigenvalue problem analytically.

a. Model

A schematic diagram for the two-level model is shown in Fig. 16. It has the mean wind defined at four levels but two of these levels (top and bottom) are governed by the boundary conditions (2.9) and (2.10).

b. Linear analysis of the symmetric case

As in section 4a we consider a symmetric case for the two wave systems $F_1(0) = F_2(0) = A$. Equations (5.1) and (5.2) have a trivial steady solution $U_1 = U_2 = 0$. The linearized equations are given by
FIG. 10. Balance between three terms: (W1) convergence of momentum flux by wave 1 with positive phase speed, (W2) that by wave 2 with negative phase speed, and (D) vertical diffusion, in steady solutions. A symmetric case is (c).

\[
\frac{d}{dt} \left[ \begin{array}{c} U_1' \\ U_2' \end{array} \right] = \left[ \begin{array}{c} 4Ae^{-\Delta z} - 2A\Delta ze^{-\Delta z} - 2\lambda \\ -4A\Delta ze^{-2\Delta z} + \lambda \end{array} \right] \left[ \begin{array}{c} 2Ae^{-\Delta z} - 2 \end{array} \right] \left[ \begin{array}{c} U_1' \\ U_2' \end{array} \right],
\]

where \( \lambda = \Delta / (\Delta z)^2 \).

The coefficient matrix is similar to that in a multilevel model as shown in the Appendix. If we assume \((U'_1(t), U'_2(t)) = \text{Re}[(U^\uparrow, U^\downarrow)e^{\sigma t}] \), \( \sigma \) and \((U^\uparrow, U^\downarrow) \) are an eigenvalue and an eigenvector of the coefficient matrix of (5.5):

\[
\sigma = Ae^{-\Delta z}(1 + e^{-\Delta z})^2(2 - \Delta z) - \frac{3}{2} \lambda \pm \sqrt{(I)}.
\]

where

\[
(I) = A^2e^{-2\Delta z}(1 - e^{-\Delta z})^2(2 - \Delta z)^2 + Ae^{-\Delta z}
\times \{-2 + \Delta z + e^{-\Delta z}(2 - 5\Delta z)\} \lambda + \frac{5}{4} \lambda^2.
\]

Eigenvalues as a function of \( A \) are shown in Fig. 17 for particular values of \( \Delta z = 1.0 \) and \( \lambda = 0.03 \). These values were determined tentatively from Fig. 3a. The eigenvalues have the essential properties of the largest ones in the multilevel model (line 1 in Fig. 2). For \( A < 0.06 \) the eigenvalues are real (and negative) due to \( (I) > 0 \). While they are complex conjugate above that value. At \( A \approx 0.0894 \) real part of the eigenvalues changes the sign; namely a Hopf bifurcation takes place. The eigenvalues become real again for much larger \( A \) because \((I) \) is a quadratic function of \( A \), although this is out of range of the usefulness of the twollevel model.

The oscillatory eigenmode at the Hopf bifurcation point is shown in Fig. 18. The mean zonal flow in the lower layer \( U_1 \) has a larger amplitude than \( U_2 \) and has a phase lag of 103°. These are also essential properties of the first neutral eigenmode shown in Fig. 3a, if we assume that the two levels are near \( z = 0.6 \) and 1.6.

If we neglect the vertical diffusion (\( \lambda \rightarrow 0 \)) in Eq. (5.5), the eigenvalues become as follows:

\[
\sigma_1 = 2Ae^{-\Delta z}(2 - \Delta z)
\]

\[
\sigma_2 = 2Ae^{-2\Delta z}(2 - \Delta z)
\]

\[
\sigma_3 = 2Ae^{-\Delta z}(2 - \Delta z)
\]

\[
\sigma_4 = 2Ae^{-2\Delta z}(2 - \Delta z)
\]

\[
\sigma_5 = 2Ae^{-\Delta z}(2 - \Delta z)
\]

\[
\sigma_6 = 2Ae^{-2\Delta z}(2 - \Delta z)
\]
The steady solution is always unstable when \( \Delta z < 2 \). However, the perturbation grows exponentially; there is no periodic solution due to a Hopf bifurcation. This is the case of Fig. 3(a) in Plumb (1977). This is true even in a multilevel model. If \( \bf M_{\text{diffusion}} = 0 \) in the Appendix, the eigenvalues are obtained easily because the remainder is a triangular matrix:

\[
\sigma_j = 2A e^{-\Delta \tau} (2 - \Delta z), \quad j = 1, 2, \ldots, J - 1
\] (5.9)

It is clear all of \( \sigma_j \) are real and positive for any \( A \) if \( \Delta z < 2 \).

If we neglect the shielding effect, i.e., \( \Delta z \to 0 \) except for the exponential part, we obtain real eigenvalues for any \( A \):

\[
\sigma = 2A e^{-\Delta \tau} (1 + e^{-\Delta \tau}) - \frac{3}{2} \lambda
\]

\[\pm \left[ \left( 2A e^{-\Delta \tau} (1 - e^{-\Delta \tau}) - \frac{3}{2} \right) + \lambda^2 \right]^{1/2}. \] (5.10)

Again there is no Hopf bifurcation because the eigenvalue is real for any \( A \). In a multilevel model, we can also show that all of the eigenvalues are real when \( \bf M_{\text{shielding}} = 0 \) in (A1). Because the remainder of the coefficient matrix is real and symmetric (i.e., a real Hermitian matrix). Therefore, there is no possibility of Hopf bifurcation. From these results we can conclude that in addition to the amplification effect both the vertical diffusion and the shielding effect are necessary in order to get periodic solutions arising from a Hopf bifurcation.

Even for the simple model of Eq. (5.5) the analysis is not very simple to obtain an eigenvector because of the exponential factor. We make further simplifications by taking \( e^{-\Delta \tau} \to 1 \) but retaining the shielding term with a positive infinitesimal parameter \( s \). Eq. (5.5) then becomes as follows:

\[
\frac{d}{dt} \begin{bmatrix} U'_1 \\ U'_2 \end{bmatrix} = \begin{bmatrix} 4A - 2As - 2\lambda \\ -4As + \lambda - 4A - 2As - \lambda \end{bmatrix} \begin{bmatrix} U'_1 \\ U'_2 \end{bmatrix}. \] (5.11)

Eigenvalues and eigenvectors of the coefficient matrix are, respectively,

\[
\sigma = 2A (2 - s) - \frac{3}{2} \lambda \pm \left( \lambda \left( \frac{5}{4} - 2As \right) \right)^{1/2}, \] (5.12)

and

\[
(U^*, U^\#) = \left( 1, \frac{1}{2} \pm \left( \frac{5}{4} - \frac{4As}{\lambda} \right)^{1/2} \right). \] (5.13)
These still have the essential properties of the original two level model if the parameters are set appropriately. If $A > 5 \lambda / 16 s$, the eigenvalues are complex conjugate. And if

$$A = A_H = \frac{3 \lambda}{4(2 - s)},$$

(5.14)

they are pure imaginary, i.e., a Hopf bifurcation takes place. The period of the oscillation $T$ is given by

$$T = \frac{2 \pi}{\sigma_1} = 2 \pi \left[ \frac{\lambda}{2} \left( 4 \frac{A s}{\lambda} - \frac{5}{4} \right) \right]^{1/2}.$$  (5.15)

It decreases monotonically as $A$ increases (cf. Figs. 2b and 17b). The eigenvector indicates the precedence of the upper level with a phase between 0 and $\pi / 2$ depending on the ratio $A / \lambda$, because

$$U_2^* = \frac{1}{2} \pm i \left( \frac{4 A s}{\lambda} - \frac{5}{4} \right)^{1/2}. \quad (5.16)$$

It is clear from (5.13) that the eigenvector changes continuously around $A = A_H$. Namely there is no qualitative change of the eigenmode around the Hopf bifurcation point.

The two-level model has essential characteristics of the dynamics shown in Fig. 6. In the lower level $j = 1$, the amplification term in Eq. (5.3) is in phase with $U_1'$ while the shielding term is out of phase with it. Therefore, the vertical diffusion term is necessary to make an oscillation in the lower level. A phase difference of the oscillation between the two levels is indis-
pensable; if there is no phase difference between $U_1'$ and $U_2'$, the rhs of Eq. (5.3) is proportional to $U_1'$ and then $dU_1'/dt \propto U_1'$.

c. Periodic solutions in the nonlinear system

Periodic solutions are obtained by time-integrations of Eqs. (5.1) and (5.2). The bifurcation diagram is very similar to that of the multilevel model (Fig. 5). Some examples of the periodic solutions are shown in Fig. 19. Near the Hopf bifurcation point there exists a small amplitude periodic solution which is similar to the neutral eigenmode at the bifurcation point (Fig. 18). As the wave forcing $A$ increases, the oscillation has a large amplitude (b) and the higher harmonics of the oscillation have a finite amplitude (c and d). The oscillation in (d) is characterized by alternating appearances of two periods of gradual and rapid change. When the zonal flow at the lower level is reduced to a small value, the rapid-change period begins. It has the same sign as the zonal flow at the upper level. Rapid change at the upper level follows. Large amplitudes of the flow with the opposite sign are built up and the gradual-change period begins. All of the periodic solutions are symmetric with respect to time as in the multilevel model. Precedence in the upper layer is also clear.

6. Discussion

It is straightforward to apply the same analysis methodology to the original model by Holton and Lindzen (1972). In their model the wave drivings of the mean zonal flow are due to a Kelvin wave and a Rossby–gravity wave; $g_n(z, i)$ in (2.2) for these waves are given by

$$g_{\text{Kelvin}} = \frac{N\alpha(z)}{k_1(\bar{u} - c_1)^2},$$  \hspace{1cm} (6.1)

$$g_{\text{Rossby-gravity}} = \frac{N\alpha(z)}{k_2(\bar{u} - c_2)^2} \left\{ \frac{\beta}{k_2^2(\bar{u} - c_2)^2} - 1 \right\}$$  \hspace{1cm} (6.2)

where $N$ is the buoyancy frequency and $\beta$ is the equatorial beta parameter. Because the functional form is different between these waves, the system (2.1) is no longer symmetric. Moreover, different wavenumbers were chosen for the two waves (planetary zonal wave-number 1 for $k_1$ and 4 for $k_2$). Therefore, we conjecture that the mean zonal flow has a vertical profile similar to Fig. 9b, because the wave driving at the bottom is
over four times larger for the Kelvin wave (westerly acceleration) than for the Rossby-gravity wave (easterly acceleration) even though the functional form of the attenuation is different between these waves. Indeed, Plumb’s (1977) result of time integrations (Fig. 8(b) in his paper) shows a dominance of westerlies in the lower levels while easterlies dominate in the upper levels.

Preliminary results of time integrations of the original Holton and Lindzen model suggest that the steady solutions are unstable within a wide range of the wave-forcing amplitude for the same values of fixed parameters. A periodic solution is obtained for wave forcings as small as $\frac{k}{\lambda_0}$ of the original values they gave, although the period of the oscillation is over ten years. The periodic solutions do not have any symmetry with respect to time.

Appearance of the periodic solution is attributed to a Hopf bifurcation; a periodic solution branches off from a steady solution when the steady solution becomes unstable with respect to an oscillatory eigenmode. This instability is different from most of the well-known instabilities in dynamic meteorology. The perturbation has the same spatial symmetry as the basic state; in this problem both the perturbation and the basic state are the mean zonal flow. Contrarily, instability problems of a basic flow, such as baroclinic instability, assume a wave perturbation, for which the spatial symmetry is different from that of the basic mean zonal flow. Instability problems with the same symmetry are common to those of basic states in equilibrium with wave forcing, e.g., topographically-forced stationary Rossby waves in barotropic fluid (Charney and DeVore 1979) and those in the stratosphere (Yoden 1987b).

Another interesting point of this kind of instability is the character of the eigenvector around the critical value. Baroclinic instability in a conservative system is accompanied by a drastic change of the eigenvector at a critical value where the instability takes place. On the other hand, the eigenvector is continuous and there is no qualitative difference in the present instability of a forced-dissipative system.

![Diagram 16](image)

FIG. 16. A two-level model. Bottom ($U_0$) and top ($U_j$) levels are governed by the boundary conditions.
7. Conclusion

We investigated bifurcation properties of a simple equatorial QBO model, i.e., Plumb’s (1977) model for interactions between gravity waves and the mean zonal flow. If the wave forcing at the bottom is symmetric, namely, it is a standing wave, the system has a trivial steady solution of no mean flow. The steady solution becomes unstable with respect to an oscillatory eigenmode when the amplitude of the wave forcing exceeds a critical value. At the critical point a periodic solution bifurcates from the steady solution; a supercritical Hopf bifurcation takes place. The period solutions are symmetric with respect to time; opposite sign of a periodic solution is the same as the solution with a phase difference of \( \pi \). Higher harmonics of the oscillation become important due to the nonlinearity as the forcing is finitely increased from the bifurcation point.

If the wave forcing is not symmetric as in the case of a Kelvin wave and a Rossby–gravity wave, a steady solution with nonzero zonal flow is obtained for any amplitude of the forcing. If one component of the wave

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**Fig. 17.** As in Fig. 2 but for the two-level model.

**Fig. 19.** Periodic solutions in the symmetric case.

**Fig. 18.** Time variation of neutral eigenmode \( U_1(t) \) and \( U_2(t) \) at the Hopf bifurcation point \( A = 0.08942 \).
forcing is weak, the steady solution is stable regardless of the other component. It becomes unstable when both components of the wave forcing exceed their critical values. As in the symmetric case, periodic solutions appear due to a Hopf bifurcation. The periodic solutions are not symmetric with respect to time. For a finite amplitude oscillation, the time-average of the oscillation is different from the unstable steady solution with the same boundary values.

A two-level, prototype model was developed to solve an eigenvalue problem analytically and to get dynamical insights easily. The linearized system has similar characteristics to those for the first eigenmode in the multilevel model. Both the vertical diffusion of mean zonal flow and the shielding effect, which is an integrated effect of wave-momentum flux absorption from the bottom to a given level, are necessary for the Hopf bifurcation to take place and therefore for periodic solutions to exist. Precedence of the oscillation at the upper level is a fundamental character of the eigenvector.

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APPENDIX

As shown in section 3b, the coefficient matrix \( f' \) for small perturbation can be divided into three components:

\[
f' = M_{\text{amplification}} + M_{\text{shielding}} + M_{\text{diffusion}} \tag{A1}\]

If we consider the symmetric case of \( A_1 = -A_2 = A \), the three components are given by

\[
M_{\text{amplification}} = 4A\begin{bmatrix}
e^{-\Delta z} & 0 \\
e^{-2\Delta z} & 0 \\
\vdots & \vdots \\
e^{-(J-2)\Delta z} & 0 \\
0 & e^{-(J-1)\Delta z}
\end{bmatrix}
\tag{A2}
\]

\[
M_{\text{shielding}} = -2A\Delta z
\]

\[
\begin{bmatrix}
e^{-\Delta z} \\
2e^{-2\Delta z} \\
2e^{-3\Delta z} \\
\vdots \\
2e^{-(J-1)\Delta z}
\end{bmatrix}
\begin{bmatrix}
0 \\
e^{-2\Delta z} \\
e^{-3\Delta z} \\
\vdots \\
e^{-(J-1)\Delta z}
\end{bmatrix}
\tag{A3}
\]

\[
M_{\text{diffusion}} = \frac{\Lambda}{(\Delta z)^2}
\begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2 & 1
\end{bmatrix}
\tag{A4}
\]

REFERENCES


