Ageostrophic Effects on the Stratospheric Residual Circulation and Tracer Distributions

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ABSTRACT

We have examined an idealized, zonally averaged, nonlinear, ageostrophic circulation forced by differential heating and parameterized eddy mixing for a range of mixing values and boundary conditions. Using a simple f-plane channel model, we show that geostrophic and ageostrophic flows can have fundamentally different behaviors which may have important implications for the circulation and trace gas distributions in the stratosphere. Our main conclusions are: 1) As eddy forcing vanishes, an ageostrophic system is not constrained to approach radiative equilibrium (unlike a geostrophic system) and may tend to the limit of an inviscid diabatic circulation under a range of boundary conditions. 2) For stratospheric applications, we show that reduced eddy mixing in an ageostrophic model leads to a pronounced meridional contraction in the residual circulation (thereby limiting its poleward transport); as the eddy mixing vanishes, the circulation tends to an inviscid limit with a well-defined meridional width. This is characteristically different from the behavior of a geostrophic circulation, which vanishes in approximate proportions to the amount of eddy mixing. 3) Reduced eddy mixing in an ageostrophic model can ultimately lead to the steepening of the meridional slope of a long-lived tracer in the region between maximum rising and sinking motions. An implication of this study is that the comparatively weak wave driving in the Southern Hemisphere could produce a contracted residual circulation during the Southern winter, which may partially account for the asymmetry in the observed zonal-mean ozone column abundances between the Northern and Southern springs.

1. Introduction

It has often been assumed, based on linear, geostrophic dynamics, that in the absence of wave driving, the zonally averaged middle atmosphere would be in "radiative equilibrium" (the inviscid, 2-D solution with temperatures in radiative equilibrium, a zonal flow in thermal wind balance, and no meridional circulation). For geostrophic models, it has been shown that the departure from radiative equilibrium is proportional to the mixing of potential vorticity by planetary waves (Dickinson 1969) or frictional drag (Leovy 1964). In GCM studies, the net radiative heating has also been found to decrease with decreasing wave driving, which reinforces the notion that eddy mixing causes the middle atmosphere to deviate from radiative equilibrium (Mahlman et al. 1984; WMO/NASA 1985).

Holton (1986) recently analyzed the effects of wave driving on the stratospheric residual circulation and tracer distributions. For a zonally averaged, geostrophic flow forced by differential heating and parameterized eddy mixing in a β-plane channel, he obtained an analytic solution which reduces to the 2-D radiative equilibrium solution in the limit of vanishing eddy mixing. This result is, however, a direct consequence of the geostrophic assumption, which requires the meridional flow to vanish with eddy forcing.

In contrast to geostrophic models, there is no such constraint on a nonlinear, ageostrophic circulation. An example is the nonlinear Hadley circulations obtained by Schneider (1977) and Held and Hou (1980), who showed that a nonlinear, zonally symmetric system can support a nearly inviscid, thermally forced meridional circulation in the limit of weak diffusive mixing. These solutions were obtained for the troposphere, however, where surface friction places an important constraint on the surface zonal wind. Tung (1986) has raised the question whether an ageostrophic circulation would exhibit a similar behavior in the absence of any boundary drag.

The objective of this paper is twofold: First, we examine the behaviors of idealized geostrophic and ageostrophic circulations forced by differential heating and parameterized eddy mixing in the absence of boundary drag. Using a nonlinear version of Holton's channel model, we obtain analytic and numerical solutions to show that in the limit of vanishing eddy mixing, the ageostrophic solution can approach radiative equilibrium or a nonlinear thermal circulation, depending on model boundary conditions (section 3). Second, for conditions appropriate for the stratosphere we examine the effects of eddy mixing on the residual circulation and tracer distributions (section 4). We show that the effect of diminished eddy mixing is to contract the meridional extent of the residual circulation, thereby weakening its poleward transport and steepening the meridional slope of a long-lived tracer.
within the circulation. An implication of this study is that the comparatively weak wave driving in the Southern Hemisphere could produce a contracted residual circulation during the Southern winter, which may partially explain the observed subpolar column ozone maximum during the Southern spring and the asymmetry in column ozone between the Northern and Southern springs.

2. Linear and nonlinear models of zonally averaged circulations

a. Model equations

We consider a system of equations identical to that of Holton (1986), except for adding the nonlinear, ageostrophic terms (in brackets) in the zonal momentum equation; i.e.,

\[ \nabla \cdot (\nabla \vec{\Phi}) = -\sigma_0 \vec{K} \tilde{\alpha}, \]

the thermal wind relation,

\[ f \vec{u} = \frac{R}{H} \vec{T}_y, \]

the thermodynamic equation,

\[ \nabla N^2 = -\frac{\sigma_0 \alpha R}{H} (\vec{T} - \tilde{T}_E), \]

and the continuity equation,

\[ \vec{V}_y + \nabla z = 0, \]

where the notations are standard. The overbar denotes the zonal average; \( \vec{u} \) is zonal velocity, \( \vec{V} = (\vec{V}, \vec{W}) \) is the meridional mass flux vector in isentropic coordinates, where \( \vec{v} \) and \( \vec{w} \) are the horizontal and “vertical” velocities, respectively; \( \tilde{\alpha} = -\tilde{\alpha}_y - (f/N)^2 (\sigma_0 \tilde{u}_y)/(\sigma_0) \) is the quasi-geostrophic potential vorticity gradient of the mean flow on an \( f \)-plane (neglecting the variation of \( f \) with latitude, \( \beta \), does not alter the basic physics), and \( K \) is a constant eddy mixing coefficient; \( \sigma_0(z) = \sigma_x \exp(-z/H) \) is the mean density, \( \sigma_x \) is a reference density, \( z = H c_p R \ln \theta \) is the vertical coordinate, \( H = RT_\theta g \) is a mean scale height, \( R \) is the gas constant, \( T_\theta \) is a mean stratospheric temperature, \( g \) is the gravitational acceleration, \( c_p \) is the heat capacity of air at constant pressure, and \( \theta \) is the potential temperature. The quantity \( \tilde{T} \) is the deviation from a reference temperature \( T_0(z) \), and \( \tilde{T}_E \) is the perturbation radiative equilibrium temperature (i.e., \( \alpha \tilde{T}_E \) corresponds to the ozone heating rate with the global mean removed). The quantity \( M \) is the absolute angular momentum per unit mass divided by the earth’s radius (i.e., \( M = \vec{u} - f \tilde{y} \)), \( y \) is latitude, and \( f \) is the Coriolis parameter. The quantity \( N \) is a constant Brunt–Väisälä frequency given by

\[ \frac{R}{H} \left( \frac{dT_0}{dz} + \frac{\sigma T_0}{H} \right), \]

and \( \alpha \) is a constant radiative damping rate. The above system implicitly assumes the circulation does not alter the stratification significantly and the isentropic coordinate is well defined.

The transport equation for a trace constituent is

\[ \vec{V} \tilde{X}_y + \vec{V} \tilde{X}_z = \sigma_0 (K \tilde{X}_{yy} - \Gamma \tilde{X}), \]

where \( \tilde{X} \) is the trace gas mixing ratio, and \( \Gamma \) is the chemical loss frequency.

b. The limit of weak eddy mixing

The behaviors of geostrophic and ageostrophic models in the limit of weak eddy mixing can be qualitatively different. The geostrophic approximation assumes that the advection of relative angular momentum is small compared with the Coriolis torque (i.e., \( \tilde{\alpha} \approx f \)), so that (1) reduces to

\[ f \tilde{V} = \sigma_0 \tilde{K} \tilde{\alpha}, \]

the geostrophic velocity is thus directly proportional to \( \tilde{K} \) and tends to zero as \( \tilde{K} \to 0 \) (\( \tilde{\alpha} \) does not vanish since \( \tilde{u} \) is in thermal wind balance). The departure from radiative equilibrium being proportional to wave driving is, therefore, a necessary consequence of the geostrophic approximation.

In the case of an ageostrophic flow, as the eddy mixing vanishes, (1) reduces to the conservation of absolute angular momentum,

\[ \nabla \cdot (\nabla \tilde{M}) = 0, \]

which may be satisfied in one of several ways: One is that \( V \to 0 \) (i.e., the net diabatic heating vanishes and the system tends to radiative equilibrium); or \( \tilde{M} \) may tend to a constant in the presence of a nontrivial meridional circulation forced by the net diabatic heating; or, more generally, \( \tilde{M} \) can be a function of the meridional streamfunction (Tung 1986). The nonlinear ageostrophic system is therefore not constrained to approach radiative equilibrium as \( K \to 0 \).

3. Radiative equilibrium versus a nonlinear, inviscid thermal circulation

An example of how a nonlinear, ageostrophic system can approach an effectively inviscid limit distinctly different from radiative equilibrium is the nonlinear tropospheric Hadley circulation obtained by Held and Hou (1980) with surface drag. Here we obtain explicit examples to show that (i) the existence of such an inviscid thermal circulation does not necessarily require any surface drag, and (ii) whether an ageostrophic system approaches the 2-D radiative equilibrium solution as the eddy mixing vanishes depends on certain model constraints on the zonal wind.

For a channel with rigid lids; i.e., \( \vec{W} = 0 \) at \( z = 0 \) and \( D \) (the model height) and \( \vec{V} = 0 \) at \( y = 0 \) and \( L \) (the channel width), we can solve the linear and nonlinear versions of (1)–(4) to obtain the following closed-form solutions:
1) The 2-D radiative equilibrium solution. For \( K = 0 \) and a given \( \bar{T}_E(y, z) \), the nonlinear system (1)–(4) has an exact, inviscid radiative equilibrium solution with the temperature given by \( \bar{T}_E \),

\[
(\bar{V}_E, \bar{W}_E) = 0,
\]

\[
\bar{u}_E(y, z) = -\frac{R}{fH} \int_0^y \bar{T}_E dz^2 + G(y), \tag{6}
\]

where \( G(y) \) is an arbitrary function of latitude.

2) An exact ageostrophic solution: the well-mixed state. For any value of \( K \) and a given thermal forcing of the form \( \bar{T}_E(y, z) = \Delta_h F(z) Y(y) \), where \( F(0) = F(D) = 0 \), there is an exact, nonlinear solution to (1)–(4) for \( \bar{u}(L, z) = \bar{u}(0, z) + fL \). Specifically, for \( F(z) = \sin(\pi z/D) \) (crudely representing the stratospheric ozone heating) we obtain

\[
\bar{u}_m(y, z) = M_\ast + f\bar{y}, \quad \text{i.e.,} \quad \bar{M}_m(y, z) = M_\ast, \quad \text{and} \quad \bar{u}_m(y, z) = q_\ast \tag{7a}
\]

\[
\bar{T}_m(y, z) = \Delta_h \sin \left( \frac{\pi z}{D} \right) \left[ \frac{1}{L} \int_0^L Y(y) dy \right] \tag{7b}
\]

\[
\bar{W}_m(y, z) = \frac{\sigma_0 R \Delta_h}{N^2 H} \sin \left( \frac{\pi z}{D} \right) \left[ Y(y) - \frac{1}{L} \int_0^L Y(y) dy \right] \tag{7c}
\]

\[
\bar{V}_m(y, z) = -\frac{\sigma_0 R \Delta_h \alpha}{N^2 H} \left[ \frac{\pi}{D} \cos \left( \frac{\pi z}{D} \right) - \frac{1}{H} \sin \left( \frac{\pi z}{D} \right) \right]
\]

\[
\times \left[ \int_0^y \bar{y} Y(y) dy - \frac{1}{L} \int_0^L Y(y) dy \right], \tag{7d}
\]

where \( M_\ast \) and \( q_\ast \) are arbitrary constants. This solution corresponds to a thoroughly mixed state in absolute angular momentum per unit mass, \( \bar{M} \), and in potential vorticity, \( \bar{q} \). Note that (7) is independent of \( K \) since \( \partial \bar{u}_m/\partial y = 0 \).

3) The linear geostrophic solution. To obtain the counterpart of Holton’s (1986) geostrophic solution, we neglect the nonlinear terms in (1) and reduce (1)–(4) to two simultaneous equations:

\[
\bar{W}_z = \frac{K}{f} \left[ \sigma_0 \bar{u}_{yy} + \left( \frac{f}{N} \right)^2 (\sigma_0 \bar{u}_{yz}) \right], \tag{8a}
\]

\[
\bar{W}_y = \frac{\sigma_0 \alpha f}{N^2} (\bar{u}_z - \bar{u}_E). \tag{8b}
\]

Integrating (4) with respect to \( y \) using (9a) and substituting the resulting expression for \( \bar{V} \) into the linear version of (1) lead to

\[
\bar{u}(y, z) = U(z) \sin \left( \frac{\pi y}{L} \right) + C_y, \tag{9c}
\]

where

\[
U(z) = \frac{\lambda_0 \lambda_1}{\sinh m D} \left[ e^{-(D-z)/2H} \left( \frac{1}{2H} \sinh m z - m \cosh m z \right) - e^{z/2H} \left( m \cosh m (D - z) + \frac{1}{2H} \sinh m (D - z) \right) \right]
\]

\[
+ \left[ \frac{\lambda_0 (\lambda_1 - \lambda_2 \pi)}{H} + \frac{R \Delta_h}{N^2 H} \left( \frac{fL}{D} \right) \right] \cos \left( \frac{\pi z}{D} \right) + \left[ \lambda_0 (\lambda_2 \pi + \lambda_1 \pi) - \frac{R \Delta_h}{N^2 H} \left( \frac{fL}{\pi H} \right) \right] \sin \left( \frac{\pi z}{D} \right),
\]
and $C$ is arbitrary constant. The above solution reproduces Holton’s results; i.e., the residual circulation diminishes as $K \to 0$, and remains finite as $K \to \infty$. The approximate midchannel meridional tracer slope relative to the isentrope for a long-lived tracer is given by Holton (1986) as

$$\frac{d\tilde{z}}{dy_{\infty}^{col}} = -\frac{\mathcal{W}}{\sigma_0} \left[ K^2 + 1 \right]^{-1}, \quad (10)$$

where $l$ is the meridional wavenumber. From (9a) and (10), we see that the tracer slope attains a maximum at some intermediate value of $K$ and vanishes as $K \to 0$ and $K \to \infty$. Notice that except for $\tilde{u}$, the geostrophic solutions for $\tilde{v}$, $\tilde{W}$, $\tilde{V}$, $\tilde{T}$, and the tracer slope are independent of the parameter $C$, which controls the meridional structure of the zonal wind and the total zonal mass flux in the system.

a. Comparison of geostrophic and ageostrophic solutions

To illustrate the contrasting properties of geostrophic and ageostrophic solutions, we compare (6), (7) and (9) for identical forcing functions and boundary conditions. Specifically, we set $Y(y) = \cos(\pi y/L)$ and $F(z) = \sin(\pi z/D)$ in (7), $G(y) = -R \Delta_k D / (fHL) \sin(\pi y/L) + C_y$ in (6), together with boundary conditions $\tilde{W}(y, 0) = \tilde{W}(y, D) = 0$, $\tilde{V}(0, z) = \tilde{V}(L, z) = 0$, $\tilde{M}_k = 0$, and $\tilde{u}(L, z) = \tilde{C}_L$. For direct comparisons of these solutions, the simplest example is that of $C = f$ and $\sigma_0 = \sigma = \text{constant}$, which reduces (6), (7), and (9) to the more tractable solutions given below:

1) The radiative equilibrium solution (for $K = 0$):

$$\tilde{T}_E(y, z) = \Delta_k \sin \left( \frac{\pi z}{D} \right) \cos \left( \frac{\pi y}{L} \right), \quad (11a)$$

$$\tilde{u}_E(y, z) = -\frac{R \Delta_k}{fH} \left( \frac{D}{L} \right) \cos \left( \frac{\pi z}{D} \right) \sin \left( \frac{\pi y}{L} \right) + f \bar{y}, \quad (11b)$$

$$\tilde{w}_E(y, z) = 0; \quad (11c)$$

2) The well-mixed nonlinear solution (for any $K$):

$$\tilde{T}_m(y, z) = 0, \quad (12a)$$

$$\tilde{u}_m(y, z) = f \bar{y}, \quad \text{i.e.,} \quad \tilde{M}_m(y, z) = 0, \quad (12b)$$

$$\tilde{w}_m(y, z) = \frac{\alpha R \Delta_k}{N^2 H} \sin \left( \frac{\pi z}{D} \right) \cos \left( \frac{\pi y}{L} \right); \quad (12c)$$

3) The linear solution (for any $K$):

$$\tilde{T}_l(y, z) = \frac{\alpha \Delta_k}{\alpha + \beta \gamma} \sin \left( \frac{\pi z}{D} \right) \cos \left( \frac{\pi y}{L} \right), \quad (13a)$$

$$\tilde{u}_l(y, z) = -\frac{R \Delta_k}{fH} \left( \frac{D}{L} \right) \frac{\alpha}{\alpha + \beta \gamma} \times \cos \left( \frac{\pi z}{D} \right) \sin \left( \frac{\pi y}{L} \right) + f \bar{y}, \quad (13b)$$

$$\tilde{w}_l(y, z) = \frac{\alpha \Delta_k}{N^2 H} \frac{\alpha \delta \gamma}{\alpha + \delta \gamma} \sin \left( \frac{\pi z}{D} \right) \cos \left( \frac{\pi y}{L} \right). \quad (13c)$$

The above solutions satisfy the same boundary conditions and may be directly compared to illustrate the following points:

(i) For $K = 0$, the ageostrophic model has two distinct solutions, (11) and (12), which are qualitatively different. The well-mixed solution (12) provides an explicit example of the inviscid, nonlinear thermal circulation anticipated in section 2b; it satisfies the same energy and angular momentum constraints as the nonlinear Hadley solution of Held and Hou (1980), but requires no boundary friction, showing that the latter is not requisite for such a nonlinear inviscid circulation to exist.

(ii) The well-mixed solution is valid for all values of $K$. Even though it is special in that the strength of the circulation is invariant with $K$, it remains a valid example showing that as $K \to 0$, the ageostrophic system tends to an inviscid limit distinct from radiative equilibrium.

(iii) In this particular example (i.e., $C = f$), the geostrophic solution is not an acceptable approximation of the ageostrophic solution: Substituting (13) into the geostrophic zonal momentum equation (1) shows that the nonlinear terms are not small compared with the Coriolis torque, so that the geostrophic approximation is not self-consistent.

(iv) As $K$ and $\delta \to 0$, the geostrophic solution reduces to the radiative equilibrium solution, but the ageostrophic solution does not. For small values of $K$ (note that $\delta$ is proportional to $K$), the geostrophic velocity decreases linearly with $K$, while the ageostrophic velocity remains constant.

Figure 1 shows the maximum vertical velocity and the midchannel tracer slope as a function of $K$ given by the analytic solutions for the numerical values listed in Table 1. The plots show that in this example the geostrophic and ageostrophic solutions are characteristically different, except for large values of $K$. At large values of $K$, the agreement does not mean that the geostrophic approximation becomes valid, as discussed in iii. In the limit of $K \to \infty$, the effect of eddy mixing is to reduce the second derivatives of the zonal wind. For the boundary conditions $\tilde{u}_e = 0$ at $z = 0$ and $D$, $\tilde{u}(0, z) = 0$ and $\tilde{u}(L, z) = fL$, this leads to $u_e \to 0$ (i.e., $\tilde{T}_r \to 0$) and $\tilde{u}_r \to f$, which happens to agree with the well-mixed solution.

b. A specific example of radiative equilibrium as an isolated inaccessible state

We have in (12) an analytic example of an ageostrophic solution which does not approach radiative equilibrium as $K \to 0$; but this does not rule out the possibility that there may be another solution to (1)-(4) which does tend to (11). To determine whether there is an ageostrophic solution close to (11) for arbitrarily small values of $K$, we constructed a finite-difference model based on a time-dependent, constant
A self-consistent requirement for the geostrophic approximation is that the nonlinear advection of zonal momentum be small compared with the Coriolis torque; i.e., \((\bar{u})_y/f \ll 1\) (see Held and Hou 1980). Using the analytic geostrophic solution for an arbitrary \(C\), we obtain the following order-of-magnitude estimate
\[
\frac{(\bar{u})_y}{f} = \frac{C}{f} + \frac{R \Delta_k \pi D}{f H L} \frac{\alpha}{\alpha + \delta \gamma},
\]
which shows that the validity of the geostrophic approximation depends on eddy mixing, \(\delta(K)\), and the zonal wind parameter, \(C\). Evaluating this ratio using Table 1 gives \((\bar{u})_y/f \approx C/f + 0.55 \alpha / (\alpha + \delta \gamma)\); thus the geostrophic solution is valid for \(C/f \ll 1\) and \(\delta \gg \alpha / \gamma\) (large eddy mixing).

The geostrophic solution (12) is valid only for the boundary condition \(\bar{u}(L, z) = f L\). The corresponding geostrophic solution for \(\bar{u}(L, z) = C L\) must be obtained numerically. To compare the geostrophic and geostrophic solutions for an arbitrary \(C\), we constructed finite-difference models based on the linear and nonlinear versions of (1)–(4) for a constant \(\alpha_0\). Except for adopting the parametric boundary condition \(\bar{u}(L, z) = C L\), the model parameters are as given in section 3a.

We obtained steady state solutions for \(C/f = 1, 0.5\), and 0 and \(K\) ranging from \(10^8\) to \(10^3\) m^2 s^{-1}. For all values of \(K\), numerical solutions to the geostrophic and geostrophic models converge, respectively, to their analytic solutions (where available, i.e., for any \(C\) for the geostrophic solution and \(C/f = 1\) for the geostrophic solution). Numerical calculations show that as the value of \(K\) decreases, the geostrophic solution for \(C/f = 0.5\) also approaches a nonlinear, inviscid limit distinct from radiative equilibrium (as in the case of \(C/f = 1\)), while the solution for \(C = 0\) virtually coincides with the geostrophic solution, showing that it is possible for the geostrophic system to approach radiative equilibrium under favorable conditions.

### Table 1. Model constants.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L)</td>
<td>6000 km</td>
</tr>
<tr>
<td>(D)</td>
<td>50 km</td>
</tr>
<tr>
<td>(N^2)</td>
<td>(4 \times 10^{-8}) s^{-2}</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(10^{-6}) s^{-1}</td>
</tr>
<tr>
<td>(f)</td>
<td>(10^{-4}) s^{-1}</td>
</tr>
<tr>
<td>(\Delta_k)</td>
<td>32 K</td>
</tr>
<tr>
<td>(T_0)</td>
<td>250 K</td>
</tr>
<tr>
<td>(\Gamma)</td>
<td>(5 \times 10^{-3}) s^{-1}</td>
</tr>
<tr>
<td>(c_p)</td>
<td>1000 m^2 s^{-1} K^{-1}</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>(2/7)</td>
</tr>
<tr>
<td>(R)</td>
<td>(c_p \Gamma = 285.7) m^2 s^{-2} K^{-1}</td>
</tr>
<tr>
<td>(H)</td>
<td>(R T_0/\Gamma = 7.289) km</td>
</tr>
</tbody>
</table>

The values of \(f, N^2, \alpha, \Gamma\) are taken from Holton (1986) and those for \(L, R, \Delta_k,\) and \(T_0\) are chosen to correspond to Holton's \(L = R \Delta_k \pi/(f H L) = 6.57 \times 10^{-2}\) s^{-1}.  

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**c. The dependence on boundary conditions**

By replacing the term \(f y\) by \(C y\) in (11) and (13), we obtain the radiative equilibrium and geostrophic model solutions for any value of \(C\), the parameter which controls the meridional structure of \(\bar{u}\) and the total mass flux.
As \( K \to 0 \), the ageostrophic model can respond by allowing \( V \) or \( \nabla m \to 0 \), as discussed in section 2b. In the case of \( C = 0 \), the zonal wind must satisfy the boundary condition \( \bar{u}(0, z) = \bar{u}(L, z) = 0 \), which prevents \( \bar{u} \) in the ageostrophic model from deviating appreciably from a sine-of-latitude profile (which happens to be the geostrophic solution) and from approaching the angular momentum-conserving \( \bar{u}_m = fy \) (which corresponds to \( C/f = 1 \)) throughout the model domain as \( K \) vanishes. From the work of Held and Hou (1980), we see that one way an ageostrophic system could develop a nontrivial, inviscid circulation is for the circulation to contract as \( K \to 0 \). This would require that at some latitude the vertically averaged temperature must vary more rapidly than the radiative equilibrium temperature; i.e., a strong vertical zonal wind shear.

The present model is prevented from developing strong vertical shears by the condition \( \tilde{w} = \bar{u}_z = 0 \) at the upper and lower boundaries. In this case, surface drag is essential for obtaining the low-latitude, nonlinear Hadley solution of Held and Hou.

In summary, the results of this section show that (i) an ageostrophic model may or may not approach radiative equilibrium as the eddy mixing decreases, depending on the constraint on the zonal wind; (ii) the existence of an inviscid, nonlinear circulation does not require any boundary drag under certain conditions; and (iii) in the absence of surface drag, an ageostrophic model can approach radiative equilibrium if its zonal wind is prevented from approaching the angular-momentum conserving profile as \( K \to 0 \), in which case the only recourse is for the meridional circulation to vanish in order to satisfy the conservation of absolute angular momentum.

4. Results for the stratosphere

Due to the artificial zonal wind boundary conditions, none of the examples in the previous section is directly applicable to the stratosphere. For stratospheric results we performed finite-difference model calculations using the following boundary conditions: The zonal wind in the lower stratosphere is assumed to be a continuation of the tropospheric wind, which is governed by tropospheric processes not included in our model; an appropriate boundary condition for the stratospheric circulation is to set \( \bar{u} \) at \( z = 0 \) to the zonal wind at the tropopause. Specifically, we take \( \bar{u}(y, 0) = cfy \), where \( c = 0.1 \), corresponding to a tropopause jet of 60 m s\(^{-1} \) over 6000 km. [The solution is insensitive to \( \bar{u}(y, 0) \), so long as \( \bar{u}(y, 0) < \bar{u}_m \), see section 4b.] With \( \bar{u}(y, 0) \) specified, the model must be allowed to determine \( \bar{W}(y, 0) \), \( \bar{u}_z \), and \( \bar{u}_m \) from the thermodynamic equation, or the system would be overdetermined. Additionally, zonal winds at the lateral boundaries are allowed to vary, instead of given by \( C \), as in the previous section. The model conditions are otherwise as described in section 3c.

a. Geostrophic and ageostrophic solutions for the stratosphere

Figure 2 shows the calculated streamfunction from geostrophic and ageostrophic models for \( K = 10^3 \), \( 10^4 \), and \( 10^3 \) m\(^2\) s\(^{-1} \) and Fig. 3 shows the corresponding vertical velocities. A major difference between the two solutions is that the eddy mixing decreases, the geostrophic circulation decreases roughly linearly with \( K \) without appreciable structural changes, while the ageostrophic circulation undergoes a pronounced meridional contraction and becomes localized to the maximum heating region. For \( K = 10^3 \) m\(^2\) s\(^{-1} \), the geostrophic and ageostrophic results are similar and the circulation extends over the width of the channel. As \( K \) decreases to \( 10^4 \) m\(^2\) s\(^{-1} \), the ageostrophic circulation contracts (thereby reducing its poleward transport) and the maximum sinking motion migrates equatorward. As \( K \) is reduced to \( 10^3 \) m\(^2\) s\(^{-1} \), the circulation further contracts and intensifies locally in the maximum heating region (near the maximum \( T_0 \) level), leaving a weak circulation elsewhere in the domain. In contrast to this, as \( K \) decreases, the geostrophic solution approaches radiative equilibrium without significant structural changes. The lack of structural changes in the geostrophic solution is a consequence of the \( f \)-plane approximation. The spherical model results of Held and Hou (1980) and Garcia (1987) show that in the limit of weak dissipation, the geostrophic model response becomes confined to the equatorial region, reflecting the vanishing of the Coriolis parameter and the breakdown of geostrophy at the equator. Outside the tropics, the structural change in the geostrophic solution is comparatively modest and qualitatively different from the well-defined contraction of an ageostrophic flow, a point brought out by the channel model results. The channel model may therefore be regarded as being representative of the extratropics.

b. The nearly inviscid limit

In the limit of \( K \to 0 \), the ageostrophic circulation does not vanish, but approaches an effectively inviscid limit, with its meridional width given by the idealized model of Held and Hou (1980). Held and Hou showed that the key to this inviscid circulation is the conservation of absolute angular momentum; i.e., as \( K \to 0 \), \( \bar{u}(y, D) \to \bar{u}_m = fy \) along the poleward branch of the circulation. Figure 4 shows that this is indeed the case for the ageostrophic solution. As \( \bar{u} \) tends to \( \bar{u}_m \) and the circulation contracts, a zonal jet develops at the poleward extent of the circulation, beyond which the solution tends to radiative equilibrium as \( K \to 0 \). Note that the solution may approach radiative equilibrium locally (at high latitudes) but not globally.

Based on the conservation of absolute angular momentum, the thermal wind balance, and global energy requirements, the width of the inviscid circulation may
Fig. 2. Meridional streamfunctions of geostrophic and ageostrophic models with realistic stratospheric boundary conditions for different values of $K$. The units are m$^3$ s$^{-1}$. The circulation of the geostrophic model decreases roughly linearly with $K$, without significant structural changes, while that of the ageostrophic model undergoes a pronounced meridional contraction and becomes confined to the fluid interior.
Fig. 3. As in Fig. 2 except for vertical velocities.
Fig. 4. Zonal winds at the top of the ageostrophic model at a function of latitude. The eddy mixing coefficient, $K$, is given in units of $m^2 s^{-1}$. For comparison, the zonal wind due to an air parcel moving poleward while conserving its angular momentum is shown as $\bar{u}_m$. The meridional width of the inviscid circulation as determined by (16) is indicated by $y_\ast$.

be determined as follows (see Held and Hou 1980): Integrating the thermal wind relation (2) vertically gives

$$\bar{T}_y = \frac{1}{D} \int_0^D \bar{T}_y dz = -\frac{H}{RD} [\bar{u}(y, D) - \bar{u}(y, 0)].$$

Substituting $\bar{u}(y, D) = \bar{u}_m = fy$ and $\bar{u}(y, 0) = \epsilon fy$ into the above expression and integrating with respect to $y$ give

$$\bar{T} = T_c - \frac{(1 - \epsilon)f^2 H}{2RD} y^2,$$  \hspace{0.5cm} (14)

where $T_c$ is a constant of integration, which may be evaluated by assuming that $\bar{T}(y_\ast) = \bar{T}_E(y_\ast)$ (where $y_\ast$ defines the meridional width of the circulation):

$$T_c = \frac{2\Delta_h}{\pi} \cos \left( \frac{\pi y_\ast}{L} \right) + \frac{(1 - \epsilon)f^2 H}{2RD} y_\ast^2.$$  \hspace{0.5cm} (15)

Substituting (14) and (15) into the vertically averaged thermodynamic equation and integrating over the extent of the circulation from $y = 0$ to $y_\ast$ lead to a single equation for $y_\ast$:

$$\frac{2\Delta_h}{\pi} \left[ -\frac{L}{\pi} \sin \left( \frac{\pi y_\ast}{L} \right) - y_\ast \cos \left( \frac{\pi y_\ast}{L} \right) \right] - \frac{(1 - \epsilon)f^2 H}{3RD} y_\ast^3 = \frac{1}{D} \int_0^D \int_0^D \bar{W}N^2 dzdy = 0.$$  \hspace{0.5cm} (16)

For the values in Table 1, (16) can be solved to give $y_\ast/L \approx 0.45$. The averaged vertical velocity may then be diagnosed as

$$\bar{W} = \frac{\alpha \Delta_h R}{N^2 H} (\bar{T}_E - \bar{T})$$

$$= \frac{2\alpha \Delta_h R}{\pi N^2 H} \left[ \cos \left( \frac{\pi y}{L} \right) - \cos \left( \frac{\pi y_\ast}{L} \right) \right]$$

$$- \frac{(1 - \epsilon)\alpha f^2}{2N^2 D} (y_\ast^2 - y^2),$$

for $0 \leq y \leq y_\ast$.  \hspace{0.5cm} (17)

Since $\epsilon \ll 1$, the above results are insensitive to $\bar{u}(y, 0)$. From the derivation of (14) we see that this is generally true as long as $\bar{u}(y, 0) \ll \bar{u}_m$.

Equation (17) shows that the strength of the circulation is a function of the imposed differential heating, $\Delta_h$, but not of $K$. As the eddy driving decreases, thermal forcing takes over, rendering the ageostrophic circulation "diabatically controlled" (as opposed to "eddy controlled"). As shown earlier, this is possible only for an ageostrophic flow, which can satisfy the requirement $\nabla \cdot (\bar{V} \bar{M}) \rightarrow 0$ by allowing $\nabla \bar{M} \rightarrow 0$ rather than $\nabla \bar{V} \rightarrow 0$. The inclusion of the ageostrophic terms ensures that the angular momentum conservation is not violated. Numerical results show that locally these ageostrophic terms need not be large compared with the Coriolis torque to alter the structure or the strength of the circulation. Figure 5 shows that the maximum vertical velocity of an ageostrophic circulation, as it becomes localized, can actually increase as $K$ decreases, while that of the geostrophic flow decreases linearly with $K$. It should be noted that a small amount of dissipation is required in order for a steady state circulation to close its streamlines (Hou 1984) and to prevent symmetric instability from developing in a finite-difference model (Held and Hou 1980); hence the term "nearly

Fig. 5. Maximum vertical velocities for different values of $K$ taken from Fig. 3. The geostrophic model results are indicated by circles. The ageostrophic model results lie in the shaded region, with maximum rising velocities marked by crosses and maximum sinking velocities by triangles.
inviscid”, even though the idealized model solution is independent of eddy mixing.

c. Distribution of long-lived tracers

From (10) we can obtain the effective meridional slope of a long-lived tracer such as N₂O between the maximum rising and sinking motions: we can estimate the vertical velocity from Fig. 5 as the average strength of the rising and sinking motions, the meridional wave number based on the distance between the maximum rising and sinking motions, and the chemical damping rate $\Gamma$ from Holton (1986) as $5 \times 10^{-8} \text{s}^{-1}$ (for N₂O at 35 km). The resulting meridional tracer slopes are shown in Fig. 6 as a function of $K$. Unlike the geostrophic model solution, the tracer slope in an ageostrophic model does not vanish as $K$ decreases, but ultimately steepens as a result of both the meridional contraction and the local intensification of the contracted circulation. Furthermore, as the circulation contracts, the sinking branch of the circulation moves equatorward (Fig. 3), which may also have important implications for the latitudinal structures of long-lived trace gases in the stratosphere.

5. Conclusions

We have examined an idealized, zonally averaged, ageostrophic circulation forced by differential heating and parameterized eddy mixing for a range of mixing values and boundary conditions. Specific examples were obtained to show that geostrophic and ageostrophic flows can have qualitatively different behaviors which may have important implications for the stratospheric residual circulation and trace gas distributions. The main results of this study are:

1) As eddy forcing vanishes, an ageostrophic system is not constrained to approach radiative equilibrium (unlike a geostrophic system) and may instead tend to an inviscid limit with a nontrivial diabatic circulation, depending on the boundary conditions. This is possible because in the limit of vanishing eddy mixing, an ageostrophic system can satisfy the conservation of absolute angular momentum by requiring either the residual circulation or the gradient of absolute angular momentum to vanish, as illustrated in section 3.

2) For conditions appropriate for the stratosphere, we show that as the eddy mixing decreases, the ageostrophic circulation undergoes a pronounced meridional contraction and may intensify locally, while in contrast to this, the geostrophic circulation decreases proportionally to the eddy mixing coefficient. In the limit of small eddy mixing, the meridional width of the resulting thermal circulation may be determined from the idealized Hadley circulation model of Held and Hou (1980). These results suggest that as the wave forcing decreases, the zonally averaged stratosphere may tend to radiative equilibrium locally (at high latitudes) but not globally. Since the analysis of section 4b can be carried over to a globe, these results are expected to be qualitatively valid for a sphere. These issues are being investigated using a 2-D primitive-equations model of the stratosphere.

3) The above findings have important implications for distributions of long-lived trace gases: (i) The fact that the stratospheric residual circulation does not vanish in the absence of eddy forcing means that as the eddy mixing decreases, the mean meridional slope of a long-lived tracer can eventually steepen (relative to the isentropes) in the region between maximum rising and sinking motions. (ii) The meridional contraction of the residual circulation resulting from weak wave forcing can lead to reduced transport of trace gases into the high latitudes, particular for long-lived tracers. In view of the generally weaker wave driving in the Southern Hemisphere (Newman et al. 1988), we speculate that this could result in a contracted stratospheric transport circulation during the Southern winter, which may provide at least a partial explanation for the observed subpolar column ozone maximum during the Southern spring and the asymmetry in the column ozone between the Northern and Southern springs. We are currently using a time-dependent 2-D coupled radiative-chemical-dynamical model to examine the meridional contraction resulting from an asymmetric wave driving on a sphere and the extent to which the result may be mitigated by the presence of the annual heating cycle and mesospheric drag.
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