The Raindrop Mean Free Path and Collision Rate Dependence on Rainrate for Three-Peak Equilibrium and Marshall–Palmer Distributions

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ABSTRACT

Using the scaling of the coalescence and breakup equation, it is shown that for equilibrium distributions, and in particular for three-peak equilibrium distributions (3PEDs), the mean free time ($\tau$) and mean free path ($\lambda$) between raindrop collisions are both inversely proportional to the rainrate and that the total number of raindrop collisions varies with the square of the rainrate. This quantifies previous results, which showed that equilibrium is more rapidly approached for more intense rainrates. For Marshall–Palmer (MP) distributions, more collisions occur for rainrates less than approximately 40 mm h$^{-1}$ than for 3PEDs and fewer for greater rainrates because the relative number of large drops increases with rainrate. For rainrates around 50 mm h$^{-1}$, there are barely any differences in the behavior of $\lambda$ and $\tau$ between MP distributions and 3PEDs.

1. Introduction

The results of laboratory raindrop collisions (McTaggart-Cowan and List 1975; Low and List 1982a, b), when applied in models of breakup and coalescence of raindrops, show that raindrop spectra approach equilibrium distributions (e.g., Srivastava 1971; Gillespie and List 1978; Valdez and Young 1985; List et al. 1987; Brown 1988; Feingold et al. 1988). The distribution, which encompasses all experimental collision data of 10 raindrop pairs, has three peaks in number density per logarithmic diameter and is called the three-peak equilibrium distribution (3PED).

The model used to study the time evolution of drop spectra (List and McFarquhar 1990a) is somewhat limited in that other size-dependent sources and sinks, such as accretion, size sorting, and evaporation, are not considered. However, List et al. (1987) found evaporation did not affect the equilibrium spectrum significantly, and List and McFarquhar (1990b) computed three-peak distributions for shafts having a nonsteady release of drops at the top. Steiner and Waldvogel (1987) and Zawadzki and de Agostinho Antonio (1988) measured peaks that may be related to the equilibrium peaks, but the observations may have some biases due to the instrumentation difficulties discussed by Sheppard (1990). Nevertheless, the 3PED’s properties merit further investigation.

List et al.’s (1970) calculations of the mean free path ($\lambda$) and the mean free time ($\tau$) and Rogers’ (1989) determination of the rate of collisions as a function of raindrop size, both for Marshall–Palmer, henceforth MP, distributions (Marshall and Palmer 1948), are expanded to 3PEDs and are compared to previous calculations. Simple relations exist between $\lambda$, $\tau$, the total collision rate, and the rainrate for 3PEDs because its shape is independent of rainrate (List 1988). The mean free time between breakup events and coalescences is also discussed to give a better indication of how the spectra change.

2. The mean free path and mean free time of raindrops

List et al. (1970) showed that if a drop of diameter $D_L$ with terminal velocity $V_L$ falls through a volume of still air occupied by drops of a smaller diameter $D_S$ with terminal velocity $V_S$, then the mean free path $\lambda(D_L, D_S)$, the average distance traveled by the large drop between collisions, is given by

$$\lambda(D_L, D_S) = \frac{4V_L}{N_S(V_L - V_S)(D_L + D_S)^2}$$

where $N_S$ is the number density of $D_S$ drops. Because the number of drops having a single size $D_S$ is zero (only the number of drops in a size interval is nonzero), a physically more useful result is the mean free path for collisions of drops of a certain size with drops in another size range. For example, $\lambda$ for drops of diameter $D_L$ colliding with smaller drops of diameter $D$, where $D_S \leq D \leq D_L$, is calculated by integrating the denominator of Eq. (1), yielding

$$\lambda(D_L, D; D_S \leq D \leq D_L) = \frac{4V_L}{\int_{D_S}^{D_L} N(D)\pi(V_L - V(D))(D_L + D)^2 dD}$$

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where $V(D)$ is the terminal velocity of raindrops of diameter $D$.

When an equilibrium distribution is present, List (1988) showed that the family of equilibrium raindrop number distributions, $N(D, R)$, can be defined by

$$N(D, R) = R \Psi_N(D),$$

(3)

where $R$ is the rainrate and $\Psi_N(D)$ the shape function for number concentration. Here $\Psi_N$ is uniquely determined for the three-peak equilibrium distribution that evolved from the coalescence and breakup of raindrops in the List and McFarquhar (1990a) model. Hence, the denominator of Eq. (2) has only a linear dependence on $R$ since all other factors in Eq. (2) depend only on the diameters of the colliding drops. Therefore, when an equilibrium distribution exists, $\lambda$ may be expressed by

$$\lambda(D_L, D; D_S \leq D \leq D_L) = \frac{\phi_N(D_L, D_S)}{R}$$

(4)

where $\phi_N$ is the factor dependent upon only those raindrops in the size interval $[D_L, D_S]$ that are geometrically swept out by the drop of size $D_L$. Since the mass density ($M$), number density ($N$), rainfall rate, and radar reflectivity ($Z$) are all mutually proportional for 3PEDs, $\lambda$ is also inversely proportional to $M$, $N$, and $Z$. The mean free time, $\tau$, that passes between collisions of drop $D_L$ and these drops is given by

$$\tau(D_L, D; D_S \leq D \leq D_L) = \frac{4}{\int_{D_S}^{D_L} N(D) \pi (V_L - V(D))(D_L + D)^2 dD}$$

$$= \frac{\phi_N(D_L, D_S)}{V_L R}.$$  

(5)

The inverse proportionality arises directly from the scaling of the equations governing the evolution of raindrop size distributions; Srivastava (1988) also showed that multiplying the initial distribution by a factor of $c$ reduced the time required to reach or to make a comparable approach to equilibrium by a factor of $c$.

Calculations suggest that the 3PED is the only equilibrium distribution since all simulations (e.g., Valdez and Young 1985; Feingold et al. 1988; List and McFarquhar 1990a) of warm rain using the Low and List (1982b) parameterization of laboratory raindrop collisions have approached similar equilibrium spectra. Minor differences in the 3PEDs of different authors exist because different numerical techniques were used to ensure mass conservation of the Low and List (1982b) parameterization. Even if some other equilibrium distribution was obtained using a different empirical parameterization of collisional breakup, the inverse proportionality discussed above would still be valid; it is not dependent on the form of the equilibrium distribution. However, the function $\phi_N$ would be different.

Figure 1 shows the mean free path and mean free time for collisions of $D_L$ drops [3.43 mm (1a) and 1.53 mm (1b)] with drops having diameters between $D_S$ (horizontal axis) and $D_L$. As $D_S$ decreases, $\lambda$ and $\tau$ decrease because the $D_L$ drop is colliding with a greater range of drop sizes, and hence the integrals in (2) and (5) include a larger concentration of drops. Cases are shown that correspond to both 3PEDs and nonequilibrium MP distributions with different rainrates. These distributions are shown in Figs. 2a and 2b, respectively. The MP distributions have nominal

![Graph](image-url)
rainrates of 5, 50, and 500 mm h\(^{-1}\), respectively [see Eq. (10.2) of Rogers and Yau 1989], but the actual rainrates obtained when the rainrate distributions are integrated are 5.91, 55.4, and 472 mm h\(^{-1}\), as indicated in the figure. For all calculations in this paper, the raindrops are classified into the same 40 categories that List and McFarquhar (1990a) used to describe the temporal evolution of drop spectra where there is an upper diameter limit of 5.4 mm; the rainrate corresponding to the integrations of the MP distributions using these size intervals are given by 5.85, 54.0, and 387 mm h\(^{-1}\). This indicates that very large drops (>5.4 mm) make large contributions to the rainrate only for very intense rain events because the slope of the exponential MP distribution becomes less negative as the rainrate increases. Although the largest drops experience the greatest number of collisions overall (Rogers 1989), \(\lambda\) is large for large \(D_S\) in Fig. 1 because large drops rarely collide with other large drops.

As a direct consequence of the scaling in Eqs. (4) and (5), the mean free path curves for 3PED cases have the same shape regardless of the rainrate (Fig. 1a and 1b). This shape, however, is different for each \(D_L\) because the geometric sweepout varies as a function of \(D_L\).

The number of large drops increases at a faster rate than the rainrate for MP distributions, so for \(D_L = 3.43\) mm, \(\lambda\) for large \(D_S\) decreases faster with rainrate for MP distributions compared to 3PEDs. For small \(D_S\), the opposite is true because its behavior is dominated by the population of smaller drops whose number increases at a slower rate than rainrate for MP distributions.

3. Total raindrop collision rate

The total rate of collisions between all drop sizes, \(C_T\), occurring in a volume of still air, which is independent of a large colliding drop diameter \(D_L\), is

\[
C_T = \int_0^\infty \int_0^{D_L} N(D_L) N(D_S) \frac{\pi}{4} \times (V_L - V_S)(D_L + D_S)^2 dD_S dD_L
\]

where the integration is performed over all possible pairs of drop collisions. An expression for \(C_T\) was evaluated by Rogers (1989) assuming an MP distribution. But, when an equilibrium distribution is present, the only dependence on rainrate comes from the presence of the number density, \(N\). Therefore, equation (6) may be rewritten:

\[
C_T = R^2 \int_0^\infty \int_0^{D_L} \Psi_N(D_L) \Psi_N(D_S) \frac{\pi}{4} \times (V_L - V_S)(D_L + D_S)^2 dD_S dD_L = R^2 \theta,
\]

where \(\theta\) is determined by considering the geometric sweepout of all possible raindrop pairs and weighted for occurrence. Only once does \(\theta\) need to be evaluated, indicating that the number of collisions varies with the square of the rainrate for any equilibrium distribution.

Figure 3 shows how the total collision rate varies with rainrate for both an MP distribution and a 3PED. Between 0.05 and 5.4 mm, there are 4106 drops m\(^{-3}\) in a 3PED with rainrate 50 mm h\(^{-1}\) and 4027 drops m\(^{-3}\) in an MP distribution with rainrate 55.4 mm h\(^{-1}\), suggesting why the values of \(C_T\) are not substantially different for these cases. For high rainrate MP distributions, there are more large drops and fewer small drops than for 3PEDs (Fig. 2), meaning fewer total drops and hence a lower interaction rate. For low rainrate MP distributions, the opposite is true. For MP distributions, the total number of collisions is approximately linearly dependent on rainrate. The ratio of the number of collisional breakup to the number of coalescences is a constant 1.11 for 3PEDs, but varies from 0.99 to 1.44 for MP distributions (1 to 387 mm h\(^{-1}\)), with higher values for higher rainrates. These ratios differ from Rogers' (1989) suggestion that only a small fraction of the total number of collisions causes breakup.

4. Contributions due to coalescence and different breakup types

Low and List (1982a,b) observed that colliding raindrops could either coalesce or breakup into filament, sheet, and disk configurations and parameterized the events in terms of \(D_L\) and \(D_S\). If \(R_f, R_s\), and \(R_d\) represent the ratio of the number of filament, sheet,
and disk breakups, respectively, to the total number of breakup collisions, their sum being unity, then the probability that any breakup event will result from a particular collision is obtained by multiplying the ratios by the breakup efficiency, $E_{bu} = 1 - E_{coal}$, where $E_{coal}$ is the coalescence efficiency,

$$E_{coal} = \begin{cases} a \left[ 1 + \frac{D_S}{D_L} \right]^2 \exp\left[ -\frac{baE_T^2}{S_C} \right], & E_T < 5.0 \, \mu J \\ 0, & E_T > 5.0 \, \mu J \end{cases}$$

(8)

where $a$ is the surface tension of water, $E_T$ is the total energy of coalescence, $S_C$ is the surface energy of the spherical equivalent of the united drop mass, and $a$ and $b$ are constants. Both $S_C$ and $E_T$ are functions of $D_L$ and $D_S$ only (see Low and List 1982a for details).

Figures 4a and 4b show, for a 3PED distribution ($R = 50\, \text{mm h}^{-1}$) and an MP distribution ($R = 55.4\, \text{mm h}^{-1}$), respectively, the mean free path $\lambda$ and mean free time $\tau$ between different breakups and coalescences for a large drop of diameter $D_L = 3.43\, \text{mm}$ colliding with smaller drops within a single bin, where the bins are defined as those used by List and McFarquhar (1990a). For this $D_L$, the mean collision times, labeled total, are close to the mean breakup times except for very small values of $D_S$. Only coalescence with $D_S$ drops smaller than 0.9 mm makes an important contribution to the mean free time; the coalescences that occur for $D_S$ larger than 2.3 mm are infrequent and do not occur for $D_S$ between 0.9 and 2.3 mm. Sheet breakup is the most frequent breakup type for collisions with drops between 0.8 and 2.0 mm. Filament breakup dominates collisions with smaller drops whereas disk breakup also plays a role for collisions with drops between 2.1 and 2.5 mm. Disk collisions, however, are the most scarce. These trends are the same whether an MP distribution or a 3PED is being studied. For different rainrates and an MP distribution, the same general conclusions are valid for the relative importance of the different breakup types, with the coalescence playing a marginally reduced role at higher rainrates because there are (fractionally) fewer small drops.

The mean free path for a raindrop of size $D$, colliding with all other drops may be calculated. The collision

FIG. 3. Total collision and breakup rates for raindrops of all sizes with each other for different rainrates, for MP distributions and 3PEDs.

FIG. 4. (a) Mean free path and mean free time between collisions, coalescences, breakups, and its breakup components; filament sheet and disk, for a large drop of diameter $D_L = 3.43\, \text{mm}$ colliding with smaller drops having diameters within a single bin size, for a 3PED with rainrate $50.0\, \text{mm h}^{-1}$. The same quantities for an MP distribution with rainrate $55.4\, \text{mm h}^{-1}$ are represented by (b).
rate, $C$, of raindrop $D_L$ with all other drops can also be obtained:

$$\begin{equation}
C = \frac{1}{\tau} \int_0^\infty \frac{1}{N(D) \pi (V(L) - V(D))(D_L + D)^2} dD
\end{equation}$$

(9)

The collision rate curves for 3PEDs do not substantially differ from those of MP distributions calculated by Rogers (1989) for any rainrate (figure not shown), suggesting $\lambda$ is not that sensitive to the change in the distribution. However, most collisions do not result in breakups as Rogers (1989) suggests (see also Fig. 3); coalescences play a major role, with their contributions increasing at lower rainrates in MP distributions because there are fractionally more small drops. Exact comparison with Rogers (1989) is impossible since he does not present his collision and breakup data separately. When only drops larger than 0.48 mm are included in the calculation (i.e., $D_L$ and $D_S > 0.48$ mm), only one quarter of the collisions result in coalescences instead of one half when all drop pairs are considered. This verifies that small drops make the most substantial contribution to the number of coalescences.

The total number of different breakups was also examined (figure not shown). More filament breakups occurred than any other breakup type, by almost a factor of ten, except for MP distributions with very high rainrates because, for those distributions, there are more large drops that experience the more energetic disk and sheet breakups more frequently. Details about which raindrop pairs experience which collisions are reviewed in List and McFarquhar (1990a) and Low and List (1982a,b).

5. Summary and Conclusions

Scaling arguments previously explored by Srivastava (1988) and List (1988) have been used to determine how the mean free path of raindrops, the mean time between raindrop collisions, and the total number of raindrop collisions vary with rainrate. When an equilibrium distribution is present, both $\lambda$ and $\tau$ are inversely proportional to the rainrate; the total number of interactions increases with the square of the rainrate.

The increase in the number of collisions with rainrate will cause a faster approach to equilibrium. This was quantitatively demonstrated by Srivastava (1988) using the same scaling arguments. The collision rate of raindrops calculated for 3PEDs did not substantially differ from those of Rogers (1989) for MP distributions. Thus, Rogers’ general conclusion that equilibrium is more apt to be approached in the heavier rainrate simulations is supported. In the field, Steiner and Waldvogel (1987) and Zawadzki and de Agostinho Antonio (1988) also found that the peaks were more apt to occur or become better defined for higher rainrates.

The relative contribution of coalescence and breakup to the mean free time between collisions and the total interaction rate was examined. Approximately half of the collisions resulted in breakups and the other half in coalescences.

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