A New Set of Orthonormal Modes for Linearized Meteorological Problems

ANTONIO NAVARRA
IMGA-CNR, Modena, Italy

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ABSTRACT

A new orthogonal decomposition based on the Schmidt decomposition approach has been applied to the barotropic equation linearized around the January 300-mb climatological flow. The Schmidt decomposition can be computed numerically performing a singular value decomposition of the numerical representation of the equation. The decomposition provides a set of positive real numbers whose minimum is linked to the singularity of the linear equation. A nonzero minimum singular value guarantees nonsingularity. Within the limits of the numerical precision and resolution used (R15 and R30) the nondivergent, global, barotropic equation linearized around the winter climatology is not singular, but it is very badly conditioned.

The Schmidt decomposition gives two sets of orthonormal basis functions, and a possible interpretation is offered by expressing the covariance matrix of forced responses in terms of Schmidt modes. An interpretation of the basis is obtained by showing that one set corresponds to the EOF of the responses forced by random sources and the second basis to the forcings that excite that particular EOF.

1. Introduction

The dynamical interpretation of teleconnections patterns is still controversial, but much effort has been devoted to trace their origin as a response of the atmosphere to some forcing—heating in the baroclinic case or derived vorticity sources in the barotropic case. Many authors (Hoskins and Karoly 1981; Webster 1972; Nigam et al. 1986; Navarra 1990; Navarra and Miyakoda 1988) have used linear models to obtain atmospheric responses to prescribed forcing, but the interpretation of the results of these models is difficult and their connection to the normal-mode analysis (free modes) is unclear. In this paper a new set of “normal modes,” that is, a new orthogonal decomposition, is used that allows one to establish a clear relation with forced responses to random forcing.

The models used in investigations of this kind were usually obtained by linearizing the physical equations, either in the barotropic version or in the baroclinic version, around some specified basic flow. Models of increased sophistication are obtained by allowing more and more structure in the base state. Early studies considered zonally averaged flows or even constant rotation, allowing separation of variables and an easy mathematical treatment of each longitudinal wave-number separately (Hoskins and Karoly 1981; Simmons 1982). Recently, wavy, longitudinally varying basic flows have become popular. Barotropic models linearized around wavy basic states have been consid-
The projection $x$ of the vector $y$ on the eigenvector system $V$ can be seen as the solution of the linear system
\[ Vx = y \]
where $V$ is the matrix of the eigenvectors. The projection does not exist if the system has no solution, that is, when some eigenvector depends linearly on some other; in this case the matrix cannot be diagonalized. Serious trouble also occurs in the case that $V$ is ill conditioned; namely, when the eigenvectors are independent in exact arithmetic but in finite numerical precision, they may be dependent. In order to avoid this difficulty, it would be very desirable to achieve a complete orthogonal decomposition of the linear equation so that the previous equation can be solved. The difficulty, of course, is not a trivial one and it lies in the very nature of the linearized barotropic/baroclinic equations.

In mathematical terms these kind of equations are classified as non-self-adjoint operators: that is, complex spectrum, oblique eigenvectors. Most of the nice properties eigenvalues/eigenvectors have are really limited to self-adjoint operators—operators that are identical to their adjoint and do not carry over necessarily to the more general case. It is possible, however, to find a complete orthonormal decomposition of a non-self-adjoint operator, based on a result by Schmidt (Gohberg and Krein 1969; Smithies 1970) that yields an orthonormal basis and a set of real numbers (the $s$ numbers) that play a role similar to the eigenvalues. The same decomposition yields two such bases, with slightly different properties, whereas in the EA two distinct decompositions must be performed, for the matrix and its adjoint.

The Schmidt decomposition (SD) can be obtained numerically through the singular value decomposition (SVD) of the numerical representation of the linear equation, as it will be shown in section 2. The SVD yields the series of $s$ numbers, also called singular values, and the two orthonormal bases, the $u$ basis and the $v$ basis (Golub and Van Loan 1989). One of the nicest properties of the SVD and the SVD is that they provide a clean way of analyzing singularity and rank deficiencies of linear equations. This and other main properties of the SVD are reviewed in the Appendix.

This paper presents the results of an analysis in terms of SVD on the barotropic nondivergent equation, linearized around the 300-mb January climatological flow. The results show that the barotropic inviscid equation is not singular at present levels of arithmetic precision and resolution (rhomboidal $< 30$), but that it is badly conditioned. Sensitivity experiments seem to indicate that this is a real property of the climatological flow. Attention, however, must be paid to the numerical detail, especially in the case of inviscid flow. Dissipation, especially Rayleigh dissipation, is proved to control conditioning very efficiently. A bad condition number means that any error in the definition of the matrix will result in an error $O(\text{cond})$ in the forced response.

The results discussed in section 2 can be seen as a generalization of results obtained by North (1984). In that paper North tried to find a relation between the EOF and the eigenmodes of a linear operator, but could only establish it in the case of normal operators. It will be shown in the following that the EOFs of responses forced by random forcings are connected with the Schmidt modes, and since the Schmidt modes reduce to the eigenmodes in the case of normal operators, the result of North (1984) is included as a special case.

In the following sections it will be shown how it is possible to attempt an interpretation of the $s$ numbers and of the $u$ and $v$ basis. Section 3 will present the application of the SD to the barotropic equation, extending it to different resolutions and examining the sensitivity to dissipation. Furthermore, it will be shown in section 4 how SD gives the opportunity of tracing a clear connection in the linear case between a dynamical property of the barotropic equations and the empirical orthogonal functions of stationary solutions.

2. The Schmidt decomposition for the barotropic vorticity equation

The model decomposition is based on the steady, nondivergent barotropic equation for the two-dimensional flow on a rotating sphere, namely,
\[ \nabla \cdot (v \zeta + f v) = F(\phi, \lambda) - K \nabla^2 \zeta - \epsilon \zeta \]  
\[ \nabla^2 \zeta = \zeta \]  
\[ u = -\frac{1}{a} \frac{\partial \zeta}{\partial \phi} \]  
\[ v = \frac{1}{a \cos \lambda} \frac{\partial \zeta}{\partial \lambda} \]
where $\zeta$ is the vertical component of the relative vorticity, $\nabla$ is the horizontal gradient operator in spherical coordinates, $F$ is an external prescribed forcing, $v = (u, v)$, is the horizontal velocity vector, $f$ is the Coriolis parameter defined as $2\Omega \sin \phi$, $\Omega$ is the earth angular velocity, and $\phi$ is the latitude.

Linearizing around a time-independent basic state, the following equation for the anomalies is obtained (basic-state variables are indicated by the suffix $b$):
\[ \nabla \cdot (v_{\zeta b} + v_{\delta \zeta} + f v) = F(\phi, \lambda) - K \nabla^2 \zeta - \epsilon \zeta \]
which is closed by the diagnostic relations between $v$ and $\zeta$. Dissipations $-K
abla^2 \zeta$ and $-\epsilon \zeta$ are routinely added. Physical justifications may be invoked to justify the dissipation terms, but in the following they will be regarded merely as empirical tuning parameters. The basic state is considered nondivergent so that $\nabla \cdot v_b = 0$, though many results have shown the importance of the divergent component of the wind for a realistic vorticity budget. It is perhaps worthwhile to understand
the properties of the Schmidt decomposition in a simple case before moving on to more detailed studies.

Equation (5) is solved in spherical coordinates for the whole globe, using a spectral technique (Bourke 1972). The variables are expressed in terms of spherical harmonics. Vorticity, for instance, can be written as

\[ \xi = \sum_{m=0}^{J} \sum_{n=|m|}^{J} [((\xi_{n}^{m})^{R}) \cos(m\lambda) + (\xi_{n}^{m})^{I} \sin(m\lambda)] P_{n}^{m}(\cos\phi). \]  

(6)

The numerical representation of equations (1)–(4) is a linear problem \( Ax = f \), where \( A \) is the spectral representation of the linear operator in (2) and \( x = [\xi_{0}^{0}, \xi_{0}^{0}, \cdots (\xi_{m}^{m})^{R}, \cdots (\xi_{m}^{m})^{I}] \) for \( m = 1, \cdots, 15 \) is the vector of the coefficient of the spherical harmonics. The matrix \( A \) can be computed one column at a time, with the technique used by Hoskins and Karoly (1981) and Navarra (1990), among others, starting from an existing spectral model.

The matrix \( A \) is the numerical representation of the variable coefficients introduced through the longitudinal and latitudinal variations of the basic state. It is dense and nonsymmetric. The basic state used in this paper (Crutcher and Meserve 1970) is a 300-mb January climatological flow and it is shown in Fig. 1 for R15 resolution. The characteristic signature of the jet streams is quite visible, and overall the main features of the climatological standing waves are present. Traditional EA consists of finding the eigenvalues and eigenvectors of \( A \) (Simmons et al. 1983; Branstator 1985b).

The spectrum of eigenvalues of \( A \) can be shown on a scatter diagram (top panel of Fig. 2) and the most unstable eigenvalue (MUE) stands out clearly on the right. Plotting the modulus of the eigenvalues (bottom panel of Fig. 2), the MUE (marked by the circle) has a rather nondescript position in the crowd of eigenvalues. The patterns of the first two most unstable modes are reproduced in Fig. 3 for R15 resolution. The growing rates are 10 and 15 days, respectively, and
\[ QQ^T = T = D + N \]  

where \( Q \) is an orthogonal matrix, \( D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \) is the diagonal eigenvalues matrix, and \( N \) is a strictly upper triangular matrix. When \( A \) is normal, \( N = 0 \) and so it can be diagonalized by an orthogonal transformation.

The departure of normality is defined as

\[ d_p = \sum_{i=1}^{n} \sigma_i^2 - \sum_{i=1}^{n} |\lambda_i|^2 \]

where \( \lambda \) are the eigenvalues and \( \sigma \) the \( s \) numbers. For self-adjoint and normal matrices the departure is strictly zero, since the \( s \) numbers and the eigenvalues coincide and in this case the eigenvectors form an orthonormal set. A highly non-self-adjoint matrix will have a large departure and eigenvectors that form arbitrary angles; a weakly non-self-adjoint matrix will have a small departure and quasi-orthogonal eigenvectors.

It is interesting to compute the departure from normality for the barotropic case to have an estimate of the degree of non-self-adjointness of the problem. Figure 4 shows a plot of the departure from normality for the barotropic equation. In order to have a calibration of the relative magnitude of the departure, the Rayleigh dissipation parameter is used parametrically to drive the barotropic case toward normality. The addition of dissipation has the effect of making the matrix \( A \) more diagonally dominant and hence makes it more normal. However, it is only for large, unrealistic dissipation that the departure from normality becomes low enough to allow orthogonal eigenvectors. Dissipation values routinely used in meteorological application modify, but do not eliminate, the non-self-adjointness. The important point to consider here is that for dissipation values that are in the range of 7–20 inverse days the equation is still non-self-adjoint, and therefore oblique eigenvectors are to be expected.

As is reported in the Appendix, the Schmidt decomposition can be achieved numerically by performing a singular value decomposition (SVD) on the matrix representation of the operator. The singular values are then an approximation to the \( s \) numbers and the left and right singular basis approximate the left and right Schmidt basis. In the following the numerical terminology (singular values and singular basis) will be used to stress their dependence on truncation.

The distribution of singular values for truncation \( R = 5 \) for the barotropic equation is shown in Fig. 5. The singular spectrum is similar in shape to the plot of the modulus of the eigenvalues, but the values are different, consistent with our estimation of departure from normality in Fig. 4. In particular, the smallest singular value is two orders of magnitude smaller than the eigenvalue of minimum modulus. Rayleigh dissipation does not affect the shape of the distribution overall. Nevertheless, at the tail of the distribution, the

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Fig. 2. Distribution of the eigenvalues for the January flow. Top panel: scatter diagram of the eigenvalues (units are per second). Unstable modes correspond to the positive real part on the right. Bottom panel: the distribution of the modulus of the eigenvalues. The circle marks the position of the most unstable mode.
smallest values are significantly affected. The tail of the distribution is important because of the role that it can play in the structure of forced solutions.

The SVD of the matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T$$

where $\mathbf{U}$, $\mathbf{V}$ are orthonormal matrices and $\Sigma$ is the diagonal matrix of the singular values. The column of the $\mathbf{U}$ and $\mathbf{V}$ matrices will be called in the following $u$ vectors and $v$ vectors.

A response to a prescribed forcing $f$ can be written as

$$x = \mathbf{V} \Sigma^{-1} \mathbf{U}^T f = \sum_{i=1}^{n} v_i \left( \frac{u_i f}{\sigma_i} \right)$$

where $\sigma_i$ are the singular values.

Fig. 3. Streamfunction of the most unstable eigenvectors for the barotropic case. Right panel: the first mode, left panel: the second mode. The contour interval is irrelevant.

Fig. 4. Departure from normality for inviscid (case 1) and various dissipation values. Dissipation is in day$^{-1}$; it varies from 10, 7, 1, 0.5, 0.1, 0.01, 0.001 for cases 2–8, respectively.

Fig. 5. Distribution of singular values at R15 truncation for several values of Rayleigh dissipation. Solid, dashed, dotted, and dash–dotted line correspond to the inviscid case, 20 day$^{-1}$, 10 day$^{-1}$, and 7 day$^{-1}$, respectively.
where the summation extends over the degrees of freedom \( n = 494 \) for R15 and the scalar product is indicated by \((\cdot,\cdot)\). Here \( \mathbf{U} \) is an orthonormal basis, and so we can see that the structure of the solution is going to be a linear combination of the \( \mathbf{v} \) vectors, each weighted by \( 1/\sigma_i \) times the projection of the forcing \( \mathbf{f} \) on the corresponding \( \mathbf{u} \) vector. The smallest singular value is then predominantly represented in the response, whereas the largest singular value has relatively less weight. It is important to examine the tail of distributions in Fig. 5 in detail. Figure 6 shows just the first 30 singular values. The effect of dissipation is clear. The smallest singular values are systematically modified, and, with rare exceptions, a larger dissipation corresponds to a larger singular value. The increase is not just an arithmetic shift because the singular values of the sum of a general matrix and a diagonal matrix, such as dissipation, are not the sum of the respective singular values.

The smallest singular value also gives important information regarding the nonsingularity of the operator. It is evident from Eq. (8) that if one of the singular values is zero, the forced problem has no solution.\(^1\) At the same time the matrix, and presumably the operator itself, is singular and noninvertible. Rigorous proofs of theorems show (Gohberg and Krein 1969) that it is the smallest singular value that gives the information on the nonsingularity of the matrix, not the minimum modulus eigenvalues. Obviously, it is impossible to obtain numerically a precise zero, so the question arises whether the values found are in fact zero, and therefore if the barotropic operator has to be considered numerically singular. If the magnitude of \( \sigma(\text{min}) \) is checked with the standard zero test involving machine precision, as for instance, the one given in Golub and Van Loan (1989), the barotropic inviscid equation is not singular for the machine precision used here (64 bits). A similar conclusion is reached by repeating the analysis at R30 truncation (the basic state was kept at R15). The overall R30 spectrum is otherwise similar to the R15 case and the tail behaves in a similar way under changes in dissipation (Fig. 7). This evidence is not conclusive, but it seems to point to the fact that the inviscid linear barotropic equation is not mathematically singular.

The condition number defined as \( \sigma(\text{max})/\sigma(\text{min}) \) is of \( 10^5 \), implying that a perturbation \( o(1) \) may produce an effect \( o(10^6) \). The sensitivity found by Anderson (1991) can therefore be explained by these results. Indeed, under these conditions it is even surprising that consistent results can be obtained at all. It is interesting to note that a very different conclusion would have been reached by computing with 32 bit precision. The barotropic equation is in fact numerically singular at this precision. Usage of single precision should be strongly discouraged for these kinds of computations.

3. S numbers and empirical orthogonal functions

If in the real atmosphere the teleconnections are caused by a response to forcing then the process must be repeated continuously and the same pattern must be excited for different forcings. Standard barotropic

\[1\] This is true only if we exclude the eventuality that \( \sigma_i = 0 \) and \((\mathbf{u}_i, \mathbf{f}) = 0\).
instability theory (Simmons et al. 1983) gives a pattern that resembles teleconnection patterns, in particular the PNA pattern, but the connection with a forced solution is not obvious. Branstator (1990) tried to use a large number of forced solutions to perform an EOF analysis to extract the most common response. In the following it will be shown how it is possible to connect the EOF of responses to random forcing and dynamical modes obtained directly from the barotropic equation with the Schmidt decomposition.

The stationary forced problem can be written as

\[ \mathbf{Ax} = \mathbf{f} \]

where \( \mathbf{f} \) is the forcing vector. Assume now that many \( \mathbf{f} \) are chosen such that a new matrix \( \mathbf{m} \times n \) is defined as

\[ \mathbf{F} = [f_1, \ldots, f_m] \]

and require them to be statistically independent (\( \mathbf{FF}^T = \mathbf{I} \)) where \( \mathbf{I} \) is the identity matrix. Then the \( m \) solutions, \( \mathbf{X} = [x_1, \ldots, x_m] \), can be written as

\[ \mathbf{X} = \mathbf{A}^{-1}\mathbf{F}. \]

In order to perform EOF analysis on the \( \mathbf{X} \), we have to form

\[ \mathbf{XX}^T = \mathbf{A}^{-1}\mathbf{FF}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^T\mathbf{A})^{-1}. \]

Substituting the Schmidt expansion (8), we get

\[ \mathbf{XX}^T = \mathbf{V}^2 \sigma^{-2} \mathbf{V}^T. \]

This equation is the key to the physical interpretation of the Schmidt expansion. In fact, many results are contained in (9). The first one is that the \( \mathbf{v} \) vectors are the EOFs for the barotropic linear responses forced by random sources; the second is that the percentage of variance explained by the \( j \)th \( \mathbf{v} \) vector is given by the singular values according to

\[ \frac{\sigma_j^2}{\sum \sigma_i^{-2}}. \]

The Schmidt decomposition through the \( \mathbf{v} \) vectors and singular values completely defines the covariance matrix. It is easy to show that covariances and one-point correlation maps can be derived from (9). Since the stationary solutions in Eq. (8) are weighted by the inverse of the singular values, it is not surprising that most of the variance involves the inverse of the singular values.

The requirement \( \mathbf{FF}^T = \mathbf{I} \) is probably not satisfied exactly in reality, because some degree of spatial correlation is present in the observed vorticity forcing. It is however, a useful idealization for the fundamental study of linear problems. For instance, it is satisfied by the large ensemble of pointwise, isolated forcings used by Branstator (1990) to identify the sensitivity of the baroclinic linear forced problem.

Combining Eqs. (9) and (8), we can also give a physical interpretation for the \( \mathbf{u} \) vectors: the \( \mathbf{u} \) vector \( i \) is the forcing distribution that excites the \( \mathbf{v} \) vector \( i \). In the following, the patterns resulting from a Schmidt analysis will be discussed at resolution R15 and R30. The first singular modes for the R15 barotropic equation are shown in Fig. 8. The inviscid response contains teleconnection-like patterns in both hemispheres, though a wave train is prominent over the Pacific and another one, more meridionally elongated, is visible over the Himalayan region. The inviscid response appears rather difficult to interpret physically, but it is, according to our preceding analysis, an acceptable Schmidt mode for the zero dissipation case, since it corresponds to a singular value that is nonzero within the numerical precision.

The introduction of dissipation produces more interesting modes. The response is concentrated in the Northern Hemisphere and in the Pacific region. In the 7 day \(^{-1} \) dissipation case a pattern resembling a phase of the unstable mode found by Simmons et al. (1983) (hereafter the SWB mode) is established over the Pacific with little response in the Southern Hemisphere. The forcing that preferentially excites these modes (Fig. 9) exhibits a large-scale structure with large variations from one case to the other. The pattern of Fig. 9 can be interpreted as a map of the efficiency with which the mode in Fig. 8 can be excited. Because of the large value of \( 1/\sigma_{\text{min}} \) the forcing need only have a small projection on the pattern in Fig. 9 to generate a response with a large component along the pattern in Fig. 8. For instance, in the case of 7-day dissipation, the response in Fig. 8 can be excited to some extent by isolated heating over the Indonesian region or, with reversed sign, over the Aleutians or even over Europe.

It is interesting to look at the higher Schmidt modes for the 7 day \(^{-1} \) case. The second mode (Fig. 10) is also a wavelike mode and the corresponding vorticity forcing again shows tropical centers along with the midlatitude pattern. The third mode (Fig. 11) is a well-developed teleconnection pattern. It seems to be made up of two distinct patterns, one extending from the Indonesian region and a weaker one from the central Atlantic. The two patterns interfere constructively over north Europe to reach a high amplitude locally. Because of the global nature of the mode that is resulting by these sensitivity analysis, the same pattern can be excited from several locations on the globe.

The first and second response mode are rather similar, describing a large-scale anomaly over the northern Pacific, slightly shifted in longitude. The vorticity sources are equally divided between the midlatitudes and the tropics. Strong centers exist over the Siberian coast and minor ones over North America and Europe. In the tropics, the strongest centers are situated in the Indonesian region and weaker centers are recognizable over Central America and the tropical Atlantic. It is important to note that the Schmidt modes are global modes and their structure is the result of the global
Fig. 8. Streamfunction for the first v vector of the Schmidt decomposition for several Rayleigh dissipations for the R15 case. Contour is arbitrary. Top left panel: 20 day$^{-1}$ dissipation, variance explained 97%; top right panel: 10 day$^{-1}$ dissipation, variance explained 99%; bottom left panel: inviscid case, variance explained 93%; bottom right panel: 7 day$^{-1}$, variance explained 43%.
Fig. 9. As in Fig. 8 but for the vorticity forcing, i.e., the \( u \) vectors.
forcing function. Though it is tempting to try to interpret physically some of the local features, it must be kept in mind that the response is always the result of the global pattern of vorticity sources.

The variance explained by these modes is very high in the low-dissipation cases. The response is dominated by one very small singular value that tends to preferentially excite the same pattern. The variance explained is very sensitive to the value of $\sigma_{\text{min}}$ and a significant amount of dissipation must be added before other modes can contribute significantly to the variance. In this case, a small group of singular values dominates the variance. The first singular mode for the 7 day$^{-1}$ case explains only 43% of the variance, the second 13%, and the third 4%. One way to interpret the variance explained is to consider it as the probability that a given pattern will emerge as a result of a randomly chosen forcing.

The Schmidt decomposition can be easily extended to higher resolution. Results from an R30 version of the barotropic model with the basic state kept fixed at R15 are shown in Figs. 12–15. The dimension of the matrix problem corresponding to this resolution is 1866. Once again a physical interpretation is possible only if enough dissipation is included. In these cases the patterns tend to resemble teleconnection patterns, whereas the first inviscid mode, explaining 73% of the variance, does not seem to have a particular resemblance to teleconnections (Fig. 12). Teleconnection modes begin to appear for 20 day$^{-1}$ dissipation (not shown), and they get more localized with increased dissipation. Figure 13 shows the first response mode and the corresponding vorticity forcing for the 7 day$^{-1}$ dissipation case. The variance explained varies greatly and in the high dissipation case the first mode explains only 15% of the variance. The vorticity forcing corre-
FIG. 11. As in Fig. 10 but for the third vector, variance explained 4%.

FIG. 12. Streamfunction for the first Schmidt $v$ vectors for R30 resolution. Inviscid case, variance explained 73%. The basic state has been left at R15. Contour is irrelevant.
The second mode (Fig. 14, top), for the 7 day$^{-1}$ dissipation, shows a marked alternating pattern, reminiscent of the PNA, with a strong maximum center of activity just west of the date line and a polar intensification. Weaker centers are located over the Atlantic and Europe. The third mode (Fig. 15, top) also has a wave train-like shape, but the maximum is reached east of the date line, over the Gulf of Alaska; and in the European sector a stronger structure appears.

The vorticity forcing corresponding to these modes (bottom of Figs. 14-15) is composed of tropical and midlatitude parts. In the tropics, the amplitude is almost entirely concentrated in the Indonesian region, with a zero-crossing line over Borneo. The wave trains dispersing from the different centers interfere to create the difference in the response patterns. As in other studies that have used nondivergent basic states, it must be noted that the absence of the divergent component of the basic state can overestimate the role of the Indonesian region as the preferred site of forcing of anomalies. Furthermore, the Indonesian heating will presumably project in reality on several forcing vectors, as in the case of the adjoint modes (Branstator 1985b).

Overall, the R30 experiments show consistent results with the R15 experiments, and dissipation is crucial, even at this resolution, to obtain interpretable solutions. The Schmidt modes are consistent with the lower resolution and a small number of modes can describe most of the variance.

4. Conclusions

This paper has demonstrated that the Schmidt decomposition, computed through the singular value decomposition of the numerical representation of a lin-
linearized equation, is a useful tool that provides new insight into linear meteorological problems. The Schmidt decomposition gives a rigorous estimation of the singularity of the linear equation, linking the smallest singular value to the distance from the set of singular equations. Results at resolution R15 and R30 have shown that the January climatological flow is not, strictly speaking, singular, if we assume machine precision for the elements of the matrix $\mathbf{A}$. The barotropic equation linearized around the January basic flow is, however, very ill-conditioned and highly nonnormal, suggesting that care must be exercised in the evaluation of functions and solutions of linear problems at low dissipation.

The Schmidt decomposition can also be used to give a powerful interpretation of repeated forced patterns by providing an easy link between the matrix $\mathbf{A}$ and the covariance matrix of solutions forced by random forcings. It has been shown that all the information on the systematically excited modes can be extracted from the algebraic structure of the linear barotropic equation, without the need to perform the solutions. Some caution must be exercised in the physical interpretation of the modes. Though some of the modes resemble teleconnection patterns, the orthogonality constraint of the Schmidt modes may force phase relations that are implied by their mathematical orthogonal structure but that are not necessarily present in reality.

The covariance matrix of the forced solutions can be rewritten in terms of the Schmidt modes and singular values without having to compute the actual solutions. The $v$ vector corresponds to the eigenvector of the covariance matrix (EOF) of the responses forced by random sources and the singular values correspond to the eigenvalues of the covariance matrix (variance explained). Easy-to-excite patterns and optimal forcings can be easily identified via a Schmidt decomposition.
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APPENDIX

The Singular Value Decomposition

The Schmidt decomposition is given by the following expansion of a linear operator \( \mathcal{A} \)

\[
\mathcal{A} = \sum_{i=1}^{n} \psi_i \sigma_i (\phi_i, \cdot )
\]

where \((\cdot, \cdot)\) is the scalar product in the space of the function \( \psi \) and \( \phi \). The rhs of this equation is an operator expression describing the effect of the Schmidt decomposition. It can be seen as an orthogonal projection (the scalar products with the \( \phi_i \)), then a stretching by the \( \sigma_i \), and a final linear combination of \( \psi_i \). The \( \sigma_i \) are called \( s \) numbers.

The Schmidt decomposition can be computed numerically from the numerical representation of the operator \( \mathcal{A} \) with a singular value decomposition (SVD). The SVD of a full-rank matrix \( \mathcal{A} \) is its decomposition in the product of three different matrices, that is,

\[
\mathcal{A} = U \Sigma V^T
\]

where \( U \) and \( V \) are real orthonormal matrices (i.e., \( U^T U = I \) and \( V^T V = I \)) of dimensions \( n \times n \) each, and \( \Sigma \) is a diagonal matrix, which is defined by \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n) \), and \( \sigma_1, \cdots, \sigma_n \) are real numbers organized to be in descending order and are called singular values. The members of the \( U \) and \( V \) basis are called \( u \) vectors and \( v \) vectors, respectively. If we put \( U = [u_1, u_2, \cdots, u_n] \) and \( V = [v_1, v_2, \cdots, v_n] \), the following relations hold:
\( \mathbf{A} \mathbf{v}_i = \sigma_i \mathbf{u}_i, \quad \mathbf{A}^T \mathbf{u}_i = \sigma_i \mathbf{v}_i, \quad i = 1, \ldots, n; \) \hspace{1cm} (1)

then we can relate the \( \mathbf{u} \) vectors “\( \mathbf{u}_i \)” to the \( \mathbf{v} \) vectors “\( \mathbf{v}_i \)” searching the corresponding \( i \)th singular values. Moreover, we can use them in the following representation of \( \mathbf{A} \):

\[
\mathbf{A} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T
\] \hspace{1cm} (2)

where \( r = n \). If \( r \) is not equal to \( n \)—namely, one or more of the \( \sigma_i \) is zero—then the matrix is singular and does not possess a complete set of eigenvectors. A detailed discussion of the mathematical properties of SVD can be found in Golub and Van Loan (1989).

REFERENCES


