Weather Regimes, Low-Frequency Oscillations, and Principal Patterns of Variability: A Perspective of Extratropical Low-Frequency Variability

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ABSTRACT

The dynamical basis of extratropical low-frequency variability (LFV) is investigated using a quasigeostrophic model on a sphere with realistic Northern Hemisphere topography. The model is driven by Newtonian relaxation to an axisymmetric radiative equilibrium temperature. Two versions of the model are used: one with two vertical levels and horizontal T15 resolution and the other with five levels and T21 resolution. In previous investigations, by the authors, the former model has been found to possess multiple attractors in a stable range of the model parameters and to wander irregularly among attractor ruins for unstable parameter sets. A similar behavior is found for the higher-resolution model as well.

Three aspects of LFV are considered in this paper. The first two are intermittent appearances of quasi-stationary weather regimes and low-frequency oscillations. The third is the dominance of a few principal patterns of variability that show red noise-like temporal behavior. In the real atmosphere, the first two can be found by careful examinations in the general predominance of the last aspect. The model is able to simulate these aspects with a certain level of realism.

It is found that the first two are associated with the existence of multiple attractors and oscillations intrinsic to them. As for the two-level model, attractors that used to be confined to small regions in phase space correspond to quasi-stationary weather regimes and those located in regions where the phase space structure is flat support the oscillations with sizable amplitudes at a more turbulent stage. The five-level model shows a more complicated behavior, but the relevance of multiple attractors and associated oscillations has been confirmed as well.

It is found in the realistic range of parameters that singular modes of linearized equations with time-mean basic states form the basis of principal spatial patterns for which low-frequency temporal variations dominate. The singular vector with the smallest singular value roughly coincides with the first mode of empirical orthogonal function (EOF) for the lower-resolution model. The smallness of the singular value guarantees the small rate of time change so that the system spends a long time along the linear axis in phase space. The singular vector is found to be relatively insensitive to the changes in the basic state for the linearization. The relevance of the singular vector is more controversial in the higher-resolution model. The similarity with the leading EOFs is lost considerably. However, it is found that leading singular vectors still play a role in determining a low-dimensional linear subspace with which most of the low-frequency variance is associated. The interaction between singular modes and forcing due to transients is suggested to be responsible for the deviation of the principal patterns from singular modes.

1. Introduction

By now, it is widely recognized that the dynamics internal to the extratropical atmosphere, in addition to external forcings, is an important source of low-frequency variability (LFV) beyond the timescale of baroclinic waves. This paper attempts to interpret internally induced LFV of a numerical model from a dynamical systems viewpoint, that is, in terms of behavioral changes of the system with respect to changes in model parameters.

Observational studies show that the extratropical LFV is characterized by intermittent appearances of weather regimes and low-frequency oscillations in a chaotic background. The former is characterized by the persistence of quasi-stationary and geographically fixed anomaly patterns (e.g., Mo and Ghil 1988; Molteni et al. 1990; Kimoto and Ghil 1993a,b), and the latter by weak periodicity on the intraseasonal timescales (Branson 1987; Kushnir 1987; Ghil and Mo 1991; Plaut and Vautard 1994). The understanding of these coherent variabilities is a quite challenging task.

In addition to these intermittent features, it is known
that relatively few spatial patterns dominate the extratropical LFV. These patterns can be effectively extracted by empirical orthogonal function (EOF) analysis (e.g., Molteni et al. 1988), and the linear combinations of leading EOF patterns can represent well-known large-scale teleconnection patterns (Wallace and Gutzler 1981; Barnston and Livezey 1987). Temporal coefficients of EOFs, or principal components (PCs), usually show symmetrical distributions with respect to the climatological mean with no preferred amplitude or periodicity to the first approximation (e.g., Molteni et al. 1988). In fact, the PCs are sometimes modeled by the first-order Markov process, or red noise. This predominance of a few principal patterns is the third aspect of LFV we would like to address in this study. The first two intermittent features, that is, weather regimes and low-frequency oscillations, are detected in general by careful examinations of data that are predominated by the last aspect.

We have recently investigated the first two aspects of LFV produced by internal dynamics of a quasigeostrophic model with realistic topography (Itoh and Kimoto 1996, 1997; hereafter referred to as IK1 and IK2, respectively). Changes in the model’s behavior with respect to incremental changes in external parameters have been examined carefully for a large set of long-term integrations.

It has been found that the model possesses multiple attractors for parameter sets not very far from the one that gives a realistically turbulent state; that is, the system settles in one of the attractors that exist for fixed parameter values, depending on the initial condition. Some of the attractors consist of a large-scale quasi-stationary pattern, high-frequency baroclinic waves, and very weak or no low-frequency oscillations. These occupy small regions in phase space. Some other attractors accompany stronger low-frequency oscillations, occupying larger phase spatial extents.

Among the model parameters, the larger north–south gradient of radiative equilibrium temperature values and the smaller vertical stability/horizontal diffusion values give the more turbulent states. As one of these parameters is changed toward the turbulent side, the stability of attractors is lost one by one; that is, the system can no longer stay within them indefinitely and ends up being attracted by another one of the stable attractors. In the final, realistic stage, the stability of all the attractors is lost, and an attractor that occupies a large domain in phase space results. Beyond this bifurcation, the system wanders irregularly among the ruins of the attractors that exist at nearby parameter settings. The traces of the original attractors can be identified even at this turbulent stage since the system tends to reside there for long periods of time and transitions are swift. Therefore, the simulated LFV is characterized by intermittent appearances of quasi-stationary weather regimes and low-frequency oscillations as in the real atmosphere. The former occurs when the system visits a small attractor ruin that has consisted of a quasi-stationary large-scale pattern and high-frequency baroclinic waves, while the latter appears when the system sojourns in an attractor ruin that has had a strong low-frequency oscillation. A similar phenomenon to this has recently been found in several physical systems and is called “chaotic itinerancy” (e.g., Ikeda et al. 1989; Kaneko 1990). Weak order has been found in the chaotic itinerancy of our model; that is, there are frequent and infrequent transitions just as in the statistics of weather regime transitions found in the real atmosphere (Mo and Ghil 1988; Molteni et al. 1990; Kimoto and Ghil 1993b). A preferred transition corresponds to the macroscopic order in phase space trajectories that manifests when an attractor loses stability to get attracted to another (IK2).

The model in IK1 and IK2 employed two vertical levels and horizontal triangular 15 (T15) spectral resolution (T21 has also been used in some experiments). The resolution is higher than some earlier mechanistic studies of extratropical LFV (e.g., Charney and DeVore 1979; Legras and Ghil 1985; Mukougawa 1988) but may not be sufficient in view of realism. We shall examine whether the relevance of multiple attractors and chaotic itinerancy can be carried over to a higher-resolution version of the model (a five-level, T21 denoted as L5T21).

The dominance of a few large-scale anomaly patterns in LFV can be interpreted as the attractor of the extratropical atmosphere being confined in a relatively low-dimensional subspace, represented by the leading EOFs. North (1984) discussed the relation between EOFs and dynamics in linear dynamical systems. When the linear operator is self-adjoint, the eigenanalysis (EA) of the operator can be used to understand the relation. However, when it is not self-adjoint, as in the case of ordinary meteorological models, the orthogonality among eigenmodes is broken and the interpretation using them is not straightforward. Needless to say, EOFs of a nonlinear system are harder to be understood by EA. Branstator (1990) pointed out the difficulty of EA and tried to understand EOFs of anomalies in terms of modes excited by a random forcing applied to the climatological mean state.

Dymnikov (1988) and Navarra (1993) proposed to use singular value decomposition (SVD) as an alternative to EA. The latter showed that it is useful to understand the steady-state response of a linear system to random forcing. Associated with the leading singular modes are very small singular values that guarantee slow time evolutions. Appendix A of this paper summarizes the relation between EA and SVD, and shows advantages of SVD over EA for the present problem. Metz (1994) reported a good correspondence between leading modes of EOF of a general circulation model (GCM) and leading modes of SVD (hereafter referred to as singular modes) based on a nondivergent barotropic model linearized about the 500-hPa or equivalent barotropic basic state of the GCM. On the other hand,
an attempt by Marshall and Molteni (1993) to relate leading EOFs with singular modes in a mechanistic model did not show straightforward results. Bladé (1996) concluded negatively on the correspondence between EOFs of a GCM and singular modes of a linearized barotropic model. Therefore, the utility of SVD for nonlinear cases is controversial. This paper attempts to clarify the utility and limit of SVD in a nonlinear system.

The quasigeostrophic model of IK1 and IK2 is used in this paper to investigate the above-mentioned three aspects of LFV. The radiative equilibrium temperature profile to force the model is set to mimic the wintertime conditions as in the previous studies. A higher-resolution version is also adopted. The organization of the present paper is as follows. In the next section, the model used in this study is briefly described. Sections 3 and 4 give results for the two-level version (L2T15) of the model that are relatively straightforward. Features of chaotic itinerancy are presented in section 3, and the dynamical basis of weather regimes and low-frequency oscillations in a “realistic” parameter range is understood from differences in structures of attractors in phase space when parameters produce stable behavior. The relationship between principal patterns of LFV and singular modes in the two-level model is described in section 4. The leading mode of EOF obtained by the covariance matrix of time integration data coincides very well with the first SVD mode of a linearization matrix of the governing equations with respect to the “climatological” mean basic state. Slight departure of the EOF from the singular mode is understood as the modification due to the presence of a quasi-stationary weather regime. In section 5, the results of the five-level model are presented. It turns out that the baroclinic wave activity is much higher and more realistic in the LST21 model than in the L2T15, and that the extension of the lower-resolution results is not straightforward. However, we shall manage to show, by introducing a subsidiary parameter that controls the baroclinic waves’ activity, that realistically subtle signatures of weather regimes and low-frequency oscillations can be traced back to the parameter range that enjoys multiple attractors. The relationship between principal patterns and singular modes also becomes much more complicated and realistic in the five-level model. The one-to-one correspondence between the leading EOFs and singular modes is no longer possible. However, the role of a set of leading singular modes will be clarified in defining the low-dimensional subspace of EOFs, in which the system prefers to reside. The first few EOFs should be determined by the interactions between the leading singular modes and feedback from the high-frequency disturbances, the details of which are left for a future study. Section 6 gives conclusions and remarks.

2. Description of the model

The model used in this study is identical to that in IK1 and IK2, and is called the linear balance model (Lorenz 1960). The equation system is composed of the vorticity, thermodynamic, and thermal wind equations. They are expressed as follows, when longitude (λ), latitude (φ), and pressure (p) are used as the space coordinates:

\[ \frac{\partial}{\partial t} \nabla^2 \psi = -J(\psi, \nabla^2 \psi) - J(\psi, f) + \nabla \cdot \left( f \frac{\partial X}{\partial p} \right) \]

\[ - \nu_f \left( \nabla^2 - \frac{2}{a^2} \right) \nabla^2 \psi + F_x, \]  \hspace{1cm} (1)

\[ \frac{\partial T}{\partial t} = -J(\psi, T) + \frac{p}{R} \nabla^2 X - \nu_h \nabla^4 T - \mu (T - T_p), \]  \hspace{1cm} (2)

\[ \nabla^2 T = -\frac{p}{R} \nabla \cdot \left( f \frac{\partial \psi}{\partial p} \right), \]  \hspace{1cm} (3)

where \( \psi \) represents the streamfunction, \( T \) the temperature, \( p \) the pressure, \( S \) the static stability, \( \omega = \nabla^2 X \), \( \omega \) the vertical \( p \) velocity (=d\( p \)/d\( t \)), \( f \) the Coriolis parameter, \( C_t \) the specific heat at constant pressure, \( R \) the gas constant for air, \( \nu_h \) the horizontal diffusion coefficient, \( F_x \) the vertical diffusion term for the streamfunction, \( T_p \) the radiative equilibrium temperature, \( \mu^{-1} \) the time constant of the Newtonian heating or cooling, and \( a \) the radius of the earth. Here \( J \) and \( \nabla \) are the Jacobian and nabla operators, respectively, defined on the sphere. The Newtonian heating and cooling scheme is substituted for the calculation of the radiation, as is shown in Eq. (2). Thus, the motion is driven by the difference between the model temperature and the radiative equilibrium temperature; the latter is assumed to be longitudinally uniform and to represent the pole-to-equator temperature gradient. See Itoh (1993) for the explicit expression of the vertical diffusion term \( F_x \). Since baroclinic waves play an important role in the midlatitude dynamics, they should be realistically simulated. In order that baroclinic waves work effectively even at relatively low resolutions, a scale-selective, \( \nabla^4 \) form is used for the horizontal diffusion.

The two-level model with triangular truncation at wavenumber 15 is the same as in IK1 and IK2, and is referred to as the L2T15 model. The five-level model introduced in this study is described as follows. The upper and lower boundaries are placed at the 0- and 1000-hPa levels, respectively. Equation (1) is applied at the 100-, 300-, 500-, 700-, and 900-hPa levels, while Eq. (2) is used at the 200-, 400-, 600-, and 800-hPa levels. The upper and lower boundary conditions are

\[ X = 0 \quad \text{at } p = 0 \text{ hPa}, \]  \hspace{1cm} (4)

\[ \nabla \cdot (f \nabla X) = \rho J(\Phi_s, \psi_s) \quad \text{at } p = 1000 \text{ hPa}, \]  \hspace{1cm} (5)

respectively, where \( \Phi_s \) stands for surface heights multiplied by \( g \), the gravitational acceleration, \( \rho \) the air density at the earth's surface, and \( \psi_s \) the streamfunction at the 800-hPa level, which is the same as in L2T15 model. The streamfunction at the 800-hPa level is cal-

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culated as averages of the 700- and 900-hPa level streamfunction. Equation (5) expresses an approximation that the vertical motion is forced by the horizontal wind blowing over uneven surfaces. In the horizontal, triangular truncation at wavenumber 21 is employed. This model will be called the LST21 model.

The LST21 model has two options, in addition to the standard model. In the first option, stochastic perturbations are added to the model, as in the L2T15 model in IK1. The formulation is the same as that in IK1; namely, thermal perturbations are generated in the Tropics randomly, but on the average once in 20 days, lasting for 5 days. The vertical profile is $1/\sqrt{2}$, 1, $1/\sqrt{2}$, and 0 for the 800-, 600-, 400-, and 200-hPa levels, respectively. This model will be called the stochastically perturbed model and will be used to force the system to visit attractor ruins when there exists only one attractor in a limited region in phase space.

The second is the option by which baroclinic waves are artificially damped. In the LST21 model, baroclinic waves are much more active than those in the L2T15 model; therefore, the interpretation of results from the LST21 model is not straightforward. Using this option, similar results to those of the L2T15 model can be obtained, and the interpretation becomes much easier. In this option, Rayleigh damping with a timescale of 16 days (or 32 days in some experiments) is applied to zonal wavenumbers 5–8. Amplitudes of baroclinic transients are the largest in this wavenumber range. By suppressing baroclinic developments in this way, the forcing due to them is weakened. On the other hand, baroclinic waves with smaller wavenumbers may be amplified slightly and the forcing is expected to become less localized. In fact, a comparison of the variance due to transients of less than the 10-day timescale with the standard model shows smaller geographical peaks and smoother distributions (figure not shown). The model with this option will be called the model with “damped baroclinic waves” or “damped” model for short.

As in IK1 and IK2, only the Northern Hemisphere with realistic topography is treated assuming asymmetry of the streamfunction with respect to the equator. Thus, the numbers of the (real) dependent variables are 240 for the L2T15 model and 1155 for the LST21 model.

Computations are carried out in the nondimensional form of the above equations. Nondimensional units for length and time are $a$ and $\Omega^{-1}$, respectively, where $\Omega$ is the angular velocity of the earth’s rotation. All results in the following will be shown in nondimensional forms, except for time. Results are based on sufficiently long time integrations, which vary from 1100 to 10 000 days, depending on what we want to see. As for the initial condition, we use the resting atmosphere or final values in experiments for nearby parameter values.

The control parameter for tracing chaotic itinerancy is the static stability $S$ in this study. The standard value for the two-level model is $1.2 \times 10^{-6}$ m$^2$ s$^{-2}$ kg$^{-1}$. In the following, the unit, $\times 10^{-6}$ m$^2$ s$^{-2}$ kg$^{-1}$, will be omitted for simplicity. On the other hand, the standard values of 2, 2, 3, and 20 from the bottom level are used for the five-level model. The standard value for the two-level model is slightly smaller than the observed. This value is chosen to strengthen the baroclinic waves. Because the resolution of T15 is coarse, baroclinic waves are not strong enough with the observed value of the static stability (IK2). The values for the LST21 model are comparable to the observed.

Other standard values of parameters are the same as in IK1 and IK2. For these standard values and for the profile of realistic topography, the reader is referred to IK1. While the profile of the radiative equilibrium temperature $T_r$ in the two-level model is displayed in IK1, that in the five-level model is shown in Fig. 1 here. These profiles produce a realistic wintertime circulation of the Northern Hemisphere.

In all the presentations in the following, high-frequency components of the PCs are filtered out to focus on lower frequencies. In order to do so, the 10-day low-pass filter used by Blackmon (1976) is adopted in this study.

3. Weather regimes and low-frequency oscillations

Results of the L2T15 model are presented in this and the next sections. The following analysis methods will be used to elucidate characteristics of simulated LFV for given parameter sets: EOF analysis, probability density plot, and spectral analysis. The EOF analysis not only picks up principal patterns of variability, but also provides the spatial basis functions to define a two-
dimensional phase subspace. The probability density function on this subspace is computed by using daily values of the leading two PCs. Local maxima in density are thought to reflect those regions in phase space where trajectories make frequent and persistent visits (cf. Kimoto and Ghil 1993a). When domains surrounding local maxima are small, they may be called quasi-stationary weather regimes. The spectral analysis is used to detect low-frequency oscillations.

Three attractors have been found in IK1 with the static stability parameter $S = 2.00$ at which a systematic search for time-dependent solutions in phase space was performed. We follow the previous study to call these three attractors, and attractor ruins that originate from them, X, M, and P. Among them, X accompanies a low-frequency oscillation, while M and P have only high-frequency, synoptic-scale oscillations. When the parameter $S$ is decreased from 2.00, chaotic itinerancy is initiated (IK2); that is, the system starts wandering around all the solution branches spontaneously. Even at this stage, three regimes corresponding to the original attractors are unambiguously identified in IK2 by several methods including probability density plots in phase space. At least in this model the attractor ruins are the dynamical basis of weather regimes.

When the parameter $S$ is changed to further unstable values, one of the three regimes, X, enlarges its domain, and P and M become smaller and difficult to recognize. Results of a 10 000-day integration (after a 1000-day spinup) at the standard value of this study, $S = 1.20$, are shown in the following.

First, an EOF analysis is applied to the time series. Although almost the same results can be obtained by using the streamfunction data of the upper-level only, both of the two levels are simultaneously used here in the EOF analysis for the convenience of later discussions. The upper-level streamfunction patterns of the first (EOF1) and the second (EOF2) modes are shown in Fig. 2. Neither of these patterns corresponds to the counterparts of the observed wintertime atmosphere shown, for example, in Molteni et al. (1988) or in Kimoto and Ghil (1993a). In particular, the model’s EOF1 has large amplitude spread over the hemisphere and may not appear very realistic. At present, however, we emphasize the conceptual understanding of LFV and no further discussion on the pattern dissimilarity is attempted.

The probability density distribution is calculated as the two-dimensional histogram with a bin width of 0.25. The result on the plane spanned by EOFs 1 and 2 is shown in Fig. 3. The density maxima corresponding to regimes M and P are visible as well as regime X, which is located near the origin. However, the maximum value of M is considerably lower than that of X, and regime P is almost buried in X. Therefore, when using the raw data without applying the low-pass filter, the density peak corresponding to P becomes invisible, since it is masked by the noise due to the synoptic transients (not shown).

In this study, we would like to rely more on the consistency and continuity among the different parameter
values rather than on statistical tests in judging dynamical significance of weak signals buried in chaos. However, an exercise of a Monte Carlo test as described by Kimoto and Ghil (1993a) shows that these three peaks are more than 95% significant (figure not shown), although the most significant region associated with X is located near \((-0.4, 1.0)\), which is slightly off the center of the density maximum.

Anomaly patterns within regime X have considerable variability since this regime extends over a large domain near the origin. Therefore, it may not be appropriately called a quasi-stationary weather regime. The broadening of X’s domain as \(S\) decreases results from the fact that it originally is located in a region of phase space where the structure is relatively “flat.” This can be seen in Fig. 4, which shows time series of the \(\psi_{1,1}\) mode (the streamfunction with zonal wavenumber 0 and total wavenumber 1) at \(S = 2.00\) at which the multiple stable attractors coexist. Three curves correspond to integrations with three different initial conditions. It is seen that convergence of the trajectory to attractor X is much slower than that from an attractor ruin to P. The slow convergence to X is seen for many other initial conditions (figures not shown). In contrast to X, domains of regimes P and M as seen in Fig. 3 are small and the large-scale anomaly patterns are quasi-stationary. Furthermore, the peaks in density are neither so high nor distinct. Such phase space structure of regimes P and M corresponds well to the observations (e.g., Kimoto and Ghil 1993a).

It has been noted that there exist preferred routes in transitions among observed weather regimes (e.g., Mo and Ghil 1988; Molteni et al. 1990; Kimoto and Ghil 1993b). The present model also shows this property and IK2 has shown that this can be understood by global bifurcations associated with chaotic itinerancy.

Figure 5 shows the sum of power spectra over all the
spherical harmonic coefficients in the time integration at the standard values, $S = 1.20$, and in attractor $X$ at $S = 1.86$. The value $S = 1.86$ is just before $X$ becomes unstable (IK2). Both spectra have peaks in low frequencies, at periods of about 25 days for $S = 1.86$ and of about 20 days for $S = 1.20$. By changing $S$ bit by bit, it has been proved that the former changes continuously to the latter (with little change in spatial patterns; not shown). Therefore, the low-frequency oscillation intrinsic to $X$ is seen to survive even at the chaotically itinerant stage.

To conclude, weather regimes and a low-frequency oscillation seen in a chaotic background in this model are dynamically based on attractors occupying small regions in phase space and an attractor with a low-frequency oscillation over flat structure, respectively, when the system is stable.

### 4. Principal patterns of low-frequency variability

Power spectra of the time coefficients of the leading EOFs resemble those of red noise dominated by low frequencies (figure not shown) except for the peak at 20 days. Furthermore, as can be inferred from Fig. 3, the amplitude histograms are close to a normal distribution and can be regarded as symmetric between positive and negative values to a first approximation. In this section, we consider the dominance of principal patterns of variability and their temporal behavior.

Considering the above apparently linear behavior of the temporal variability, we take equations linearized about the time-averaged state, written symbolically as

$$\frac{\partial \Psi}{\partial t} + A\Psi = f,$$

where the right-hand side represents the forcing including the nonlinear terms. Here $A$ is the matrix of linearization with respect to the zonally varying basic state. Because of the slowness in LFV, the time tendency will be neglected in the following and we consider a steady response problem. Given the forcing the response is obtained by solving the simultaneous equations. In order to investigate the solution, SVD turns out to be effective. Details are given in appendix A. In summary, when the forcing is white in space (precisely speaking, when the inner product between $\mu$ vector, i.e., left singular vector, of $A$ and $f$ is white), $v$, (ith $\nu$ vector or right singular vector) with a small singular value $\sigma$, dominates the response. Even when the time tendency is retained, it is negligible for patterns with very small singular values because $A\nu = \sigma\mu$. Therefore, as long as the forcing changes slowly with respect to time and remains spatially white (in the above-mentioned sense), the trajectory may stay close to the subspace spanned by the leading singular modes along which the trajectory speed remains slow.

Singular value decomposition has been applied to the matrix $A$ linearized about the time-mean state of the integration at $S = 1.20$. The product of decomposition, namely, the $\nu$ and $\mu$ vectors, has patterns at both upper and lower model levels. The use of the two levels in the EOF analysis in the previous section has been to enable direct comparison with SVD. The distribution of singular values is shown in Fig. 6. It is seen that the first mode has a very small singular value of $5.87 \times 10^{-4}$, the inverse of which amounts to 1704 days of damping time. The spatial pattern of this mode is expected to dominate the response. This is a resonant excitation of a stationary wave. The upper-level streamfunction pattern of the $\nu$ vector of the first singular mode (hereafter abbreviated as SVD V1) is shown in Fig. 7a and is almost identical to EOF1 shown previously in Fig. 2a. The spatial correlation between the two patterns is as high as 0.846, where the correlation is calculated after the hemispheric means are subtracted.

On the other hand, the spatial pattern in EA of the matrix $A$ with the smallest eigenvalue is shown in Fig. 7c. The overall feature is not dissimilar to EOF1, but there are discrepancies especially in comparison with SVD V1. The correlation coefficient with EOF1 is 0.540. Furthermore, the eigenvalue is $3.90 \times 10^{-3}$, seven times larger than the first singular value. In appendix A, it is proved that none of the eigenvalues is smaller than the smallest singular value of the same matrix. Thus, the advantage of SVD over EA in understanding the principal patterns of variability is demonstrated.

The fact that the time change of the EOF1 pattern is slow is confirmed by examining the correlation between the distance in phase space from the EOF1 axis and the trajectory speed. Figure 8a shows the scatter diagram between the two. The trajectory speed has been computed by the low-pass filtered data in order to remove

![Image](http://example.com/image.png)

**FIG. 6.** Distribution of singular values. The basic state of linearization is the time-mean field of the experiment with $S = 1.20 \times 10^{-4}$ m$^2$ s$^{-2}$ kg$^{-2}$. 
the effect of synoptic-scale transients. It is seen in the figure that the closer to EOF1, the slower the trajectory is. The correlation coefficient between the distance from EOF1 and trajectory speed is 0.465.

The distance is computed from SVD V1 instead of EOF1 in Fig. 8b. Because of the resemblance in the spatial pattern the overall distribution appears similar. However, the major difference from Fig. 8a is that more points scatter in large distance but with small speed. Most of these points are found within the attractor ruin M. When the points within M are removed, the correlation increases from 0.303 to 0.383. Therefore, EOF1 in this case can be interpreted as SVD V1 modified by the existence of an attractor ruin that is localized in phase space.

The fact that EOF1 obtained from the results of time integration of the nonlinear system is close to SVD V1 of the linear analysis should be complemented by the flatness of the phase space structure in the vicinity of the basic state used for the linearization. Therefore, it
FIG. 8. Scatter diagrams represented on the plane of the speed of low-pass filtered trajectories and the distances from (a) EOF1 and (b) SVD V1 in phase space in the experiment for $S = 1.20 \times 10^{-2} \text{m}^2 \text{s}^{-2} \text{kg}^{-1}$. The marker ○ indicates points within attractor ruin M, which is defined as time periods when the trajectory is in the area bounded by the two straight lines in Fig. 3.

should be confirmed that the analysis in the linear subspace at the basic state, which is nothing but a point in phase space, is valid over a broader region. This is to be examined in several ways. First of all, as seen in Fig. 8 there is no point that is very close to SVD V1. Even the closest points are a finite distance from the singular vector. The fact that the distance–speed relation holds for such distant points implies that the direction specified by the linear analysis about a basic state may have more universal validity than it would appear. Second, that the $\nu$ vector is not very sensitive to the basic state can be seen by direct computations for different basic states. Figure 9 shows correlation coefficients between the SVD V1 pattern shown in Fig. 7a and $\nu$ vectors with basic states constructed by the time mean plus a linear combination of EOFs 1 and 2. It seen that, within a region about one standard deviation of EOF1 time coefficients from the time-mean state, the original pattern is preserved and remains to be the first mode. When the second and higher modes are allowed for the comparison (shaded regions in Fig. 9 are where maximum correlation is found between the original SVD V1 and the higher modes for the constructed basic states), the domain of similarity is extended considerably.

The similarity between EOF1 and SVD V1 in this experiment is not by chance. We have calculated EOF1s and SVD V1s for many different parameter sets for the L2T15 model, finding striking resemblances between them (not shown).

The singular vector appears to be dynamically significant for the L2T15 model. However, it should be remarked that this conclusion changes when the linear analysis is made based on equations different from the original. For example, even when EOF1 has equivalent barotropic structure, SVD V1 of the linearized baro-
tropic equations may not resemble it. To illustrate this, Fig. 10 shows SVD V1 of the barotropic equation with the upper-level time-mean field of $S = 1.20$ as the basic state. Clearly it differs from Fig. 7a. The results are similar when the average of the two levels is used as the basic state or when the Rayleigh damping is introduced (figures not shown). The analysis here supports Blade’s (1996) conclusion that barotropic singular modes do not explain EOFs of a GCM and is consistent with Branstator’s (1990) finding that EOFs and modes obtained by random excitation of linear models correspond only when baroclinic equations are used for the linearization. Unlike SVD, EOF is not sensitive to the use of single or multiple levels. This may be because the covariance matrix constructed from the time integration data contains the barotropic components generated by the nonlinear interactions.

In contrast to the first mode, patterns of EOF2 and SVD V2 differ considerably (Figs. 2b, 7b). Similarity among higher modes is not found, either. Orthogonality constraints on the higher modes may have an effect, but other factors such as the modification of forcing by baroclinic transients may also play a role. Since the discrepancy between EOFs and singular patterns is found even for the first mode in the higher-resolution, L5T21 model, this issue will be discussed further in the next section.

5. Results of the five-level T21 model

Results of the L5T21 model are described in this section. With the large number of degrees of freedom in this model, it is very difficult to perform a systematic search for attractors with continuation algorithms as has been done in IK1. However, it is possible to find multiple attractors and to follow a route to chaotic itinerancy by carefully changing the static stability, the control parameter of this study, in time integrations. Methods for obtaining multiple attractors in this model are described in appendix B.

The organization of this section is similar to that of the previous two sections. However, as a result of active baroclinic waves, results of the L5T21 standard model appear very different from those of the L2T15 model, which makes the interpretation difficult. Therefore, first in this section, we present results of the L5T21 model with damped baroclinic waves. The time constant of the Rayleigh damping is taken as 16 days (hereafter referred to as $R = 16$ days). The 32-day time constant gives similar results. It is shown that results of time integration at parameter values that give turbulent states can be understood as chaotic itinerancy based on the existence of multiple attractors in less turbulent ranges of the parameter. Low-frequency oscillations of the original attractors will be shown to survive in turbulent states. Then, principal patterns of variability and their relation to singular modes of the time-mean state are discussed. All these results are similar to those for the L2T15 model. Next, an experiment is performed, using the L5T21 standard model. The interpretation is made successfully, using the above results as a guide, even though baroclinic waves are active.

In this section, $\gamma$ is used as a control parameter, which represents the ratio of the static stability used in each experiment to the standard value in the lower three levels. The stability in the stratosphere (the uppermost level) is large and is held fixed in all the experiments. The system is less turbulent for greater $\gamma$'s because the greater stability suppresses the developments of baroclinic waves.

a. Results of the five-level T21 model with damped baroclinic waves

1) WEATHER REGIMES AND LOW-FREQUENCY OSCILLATIONS

A time integration of 10 000-day length is performed, taking $R = 16$ days and the static stability at the standard value (see section 2). Days between 1000 and 10 000 are analyzed. The time-mean field is not shown but is similar to that of the L5T21 standard experiment [Fig. 16 in section 5b(1)].

To detect weather regimes and low-frequency oscillations, the same calculations as in section 3, the EOF analysis, probability density plot, and spectral analysis are made also here.

An EOF analysis is carried out for the streamfunctions of all the model levels together. Similar patterns are obtained when single 500-hPa level data are used. The
Fig. 11. Pattern of the 500-hPa streamfunction of EOF1 in the experiment of the standard static stability value $\gamma = 1.0$ and $R = 16$ days for the L5T21 model. Others are the same as in Fig. 2.

reason for using all the levels is the same as before: to make the comparison with SVD straightforward. The pattern of EOF1 is illustrated in Fig. 11. Although the patterns of EOFs 2–4 are not presented here, they are similar to EOFs 2–4 of the standard experiment (shown in Fig. 17). Unlike EOF patterns of the L2T15 model, rather realistic EOF patterns are obtained. That is, EOF2 is similar to the Pacific–North American (PNA) pattern, while EOF4 resembles the eastern Atlantic (EA) pattern (Wallace and Gutzler 1981). In EOF1, a pattern with almost the same sign over the hemisphere is dominant with a wave train embedded in it. This corresponds to no counterpart in the real atmosphere but is frequently seen in perpetual simulations (e.g., Branstator 1990).

Figure 12 shows the distribution of the probability density function on the plane of EOFs 1 and 2 for this experiment. At least two maxima designated as A and B can be identified. Although the latter is not well separated from the former for this parameter, it will be shown that they are continued from distinct two maxima at less turbulent stages (see Fig. 14).

The sum of power spectra over all the spherical harmonic modes is shown along $R = 16$ days in Fig. 13, which is prepared to examine the continuity of the spectral peaks for changing parameters. The power spectra multiplied by frequency similar to Fig. 5 are plotted for experiments with different values of parameters, which are denoted in the abscissa; $R$ changes to the left of the vertical solid line with $\gamma$ fixed at unity and $\gamma$ changes to the right of the line with $R$ fixed at 16 days. The system gets more turbulent as the parameter moves to the left of the figure as can be seen by the increase in the power level. In this section, the part to the right of the vertical line is discussed. The left part will be discussed in the next section.

At $R = 16$ days and $\gamma = 1.0$, that is, along the vertical line of Fig. 13, two peaks of 50–60 days and about 20 days are discernible in the low-frequency band, while peaks exist at about 4 and 4.5 days in the synoptic timescale. Note that the periodicity of 20 days is comparable to the Branstator–Kushnir oscillation (Branstator 1987; Kushnir 1987).

Next, the origin of the probability density maxima and the low-frequency oscillations observed in this model is clarified. Special attention is paid to the relation between the two low-frequency oscillations and the two density maxima.

At least two attractors have been found at $\gamma = 1.70$ (not shown). One is a doubly periodic solution, that is, a torus with periods of 30 and 5 days, while the other is a solution with a single period of about 4 days. These will be called attractors Y and Z, respectively.

The solution branches become unstable with the decrease of $\gamma$. The torus (attractor Y) loses its stability at $\gamma = 1.65$. Therefore, it cannot be followed by ordinary time integrations. But, by using the stochastically perturbed model, one can force the system to visit a previously collapsed attractor and can confirm that the low-frequency periodicity of the attractor ruin survives, albeit with slight changes to shorter periods.

The periodic solution with the period of 4 days (attractor Z) bifurcates at $\gamma = 1.30$, resulting in a torus that has another period of about 80 days. The result of the spectral analysis from this value to $\gamma = 1.00$ is shown in Fig. 13. After several bifurcations, this attractor becomes chaos at $\gamma = 1.25$, which has dominant periodicities at 80, 10, 6.5, and 4 days, as can be seen
in Fig. 13. In the following, the periods of 10 and 6.5 days will not be referred to for simplicity.

When $\gamma$ becomes 1.24, the attractor $Z$ suddenly enlarges its size by an explosive bifurcation (Thompson and Stewart 1986), to include periods of 20 and 4.5 days. The peaks of 20 and 4.5 days are clearly seen in addition to the peaks of 80 and 4 days in Fig. 13. It is natural to conjecture that these oscillations result from the shortening of the periods in attractor $Y$ (30 and 5 days). In other words, a new attractor includes both of old attractor $Z$ and attractor ruin $Y$. This is shown by a detailed analysis in appendix C. These four main periods (80, 20, 4.5, and 4 days) continue to the experiment at $\gamma = 1.00$, as can be clearly seen in Fig. 13, although these periods except 20 days become a bit shorter with decreasing $\gamma$.

Figure 14 shows the distribution of the probability density function on the plane spanned by EOFs 1 and 2 at $\gamma = 1.20$. There are two maxima, A and B. The significance test as in section 3 shows that these two maxima are slightly less than 90% significant (not shown). They are not statistically significant in the ordinary sense. However, this comes from the two-dimensional probability density plot. In fact, the three-dimensional plot using EOF 3 shows that the two maxima are well separated at more than 95% significance (not shown). It is clear from Fig. C2 that maximum A
corresponds to attractor ruin \( Z \), while maximum \( B \) comes from attractor ruin \( Y \).

It is seen that the density maxima \( A \) and \( B \) in Fig. 14 continue to those at \( \gamma = 1.00 \) (Fig. 12), by following the change of the probability density function with decreasing \( \gamma \) (figures not shown). Thus, it has been verified that the maxima \( A \) and \( B \) in Fig. 12 originate from the two attractors when the system is stable, that is, \( \gamma \) is large, although they do not correspond to attractors themselves but to parts of the long-period oscillations (cf. Fig. 14 of IK1).

Thus, the dynamical basis of weather regimes and low-frequency oscillations is multiple attractors and oscillations intrinsic to them when the parameter takes stable values, as in the L2T15 model.

2) Principal patterns of low-frequency variability

Principal patterns of variability in the L5T21 model with damped baroclinic waves are considered. SVD of the linearized matrix with respect to the time-mean basic state is performed as for the two-level model. Again the minimum singular value has a very small value, \( 3.72 \times 10^{-4} \), corresponding to a damping time of 2688 days. Singular values are smaller than \( 1 \times 10^{-3} \) up to the fourth mode.

The first mode of SVD is shown in Fig. 15. Compared with the EOF1 pattern in Fig. 11, the similarity between the two patterns can be recognized, except for some features over the Eurasian continent. The pattern correlation between the two is 0.726. The direct correspondence between the first modes of EOF and SVD \( \nu \) vector carries over to this experiment.

b. Results of the five-level T21 standard model

1) Weather regimes and low-frequency oscillations

A time integration with 10 000-day length at the standard parameter value of \( S \) (see section 2) using the L5T21 standard model is performed. Days between 1000 and 10 000 are analyzed. The time-mean geopotential height field at the 500-hPa level is shown in Fig. 16. The standard deviation of the transients is also presented in that figure. Despite slight northward shifts of the jet stream, its maxima are realistically located off the east coasts of Asia and North America. The standard deviation also shows realistic maximum values close to 140 m.

An EOF analysis is carried out for the streamfunctions of all the model levels simultaneously. The patterns of EOFs 1–4 at the 500-hPa level are shown in Fig. 17. The fractional variances associated with the two leading modes, 20.7% and 10.6%, are smaller than those of the L2T15 model and are closer to, though still slightly larger than, those of the observed. The pattern of EOF1 has almost the same sign over the hemisphere, even more so than EOF1 for the \( R = 16 \) day experiment. The patterns of EOFs 2 and 4 are similar to the PNA and EA patterns, respectively (Wallace and Gutzler 1981).

Next, using time coefficients of EOFs 1 and 2, the probability density function is depicted in Fig. 18. The two peaks near the origin can be traced from the finite values of \( R \) but are no longer found separately in this experiment. Apart from the origin, however, we find an isolated peak near amplitude 4 of EOF1, although the value of the probability density is very small. The dynamical basis for this regime can be clarified in a manner similar to the previous section. The detail is shown in appendix D.

The result of a spectral analysis is shown along \( R = \infty \) in Fig. 13. There are two peaks around 50–70 days and at about 20 days in the low-frequency band. Furthermore, a broad peak can be seen in the synoptic timescale. These peaks continue from \( R = 16 \) days, although spectra become larger with increasing \( R \), as expected. Hence, it can be concluded that the origin of the low-frequency oscillations in the standard experiment is the same as that at \( R = 16 \) days.

It is found that the spatial pattern of the 50–70-day oscillation is represented well by EOFs 1 and 2 (Figs. 17a,b) in such a way that EOF2 precedes EOF1 by a quarter-period. Figure 19 shows the phase-quadrature patterns of the 20-day oscillation; here they are represented by the two leading EOFs of a bandpass-filtered dataset retaining 10–40-day periods. The time evolution is as follows: Fig. 19a \( \rightarrow \) Fig. 19b \( \rightarrow \) \(-1 \times \) Fig. 19a \( \rightarrow -1 \times \) Fig. 19b. The spatial pattern and the time...
2) Principal patterns of low-frequency variability

Principal patterns of variability in the L5T21 standard model are considered. Again as in the L5T21 model with damped baroclinic waves, the minimum singular value has a very small value, 3.51 × 10^{-4} (corresponding damping time 2848 days), and singular values are smaller than 1 × 10^{-3} up to the fourth mode.

However, the direct correspondence between EOF and SVD patterns is not so excellent as in the cases examined so far. Comparing the first modes shown in Figs. 17a and 20a, both patterns have almost the same sign over the hemisphere, but one-to-one correspondence of anomaly centers is not possible. The patterns of EOF2 and SVD V2 (Figs. 17b, 20b) resemble each other in the sector from the mid-Pacific to the western Atlantic, but dissimilarity is also conspicuous in other parts.

Since the similarity between the EOF1 and SVD V1 patterns has been recognized for the model with damped baroclinic waves, the cause of the dissimilarity here is considered to be the forcing by strengthened synoptic-scale transients. Furthermore, baroclinic waves with smaller wavenumbers, say, 3–4, may be weakened slightly, therefore the forcing is expected to become more localized. When the forcing is localized and the localized components acquire large amplitudes, the forced component may predominate the response despite the smallness of the leading singular values. In other words, the forcing by the nonlinear interaction terms works to deviate EOFs from the leading $\nu$ vectors. Thus, it would be difficult to find direct correspondences between the individual modes of leading EOFs and $\nu$ vectors in general cases.

Nevertheless, it is possible to demonstrate that the $\nu$ vectors with small singular values do play some roles in forming the principal patterns of low-frequency variability in the standard model. First of all, the EOF1 pattern can be represented to a large extent by a linear combination of SVD V1 and V2. Representing EOF1 as a sum of $\nu$ vectors $\mathbf{v}$, with coefficients $\alpha_1$, $\alpha_2$, and $\alpha_3$ divided by the sum of $\alpha_i$ are 0.685 and 0.163, respectively, and all the others are smaller than 0.05. This implies that EOF1 is roughly parallel to the linear subspace spanned by the two leading singular modes.

The cumulative fractions of the total variance represented by the $\nu$ vectors up to the 20th mode are shown in the central column of Fig. 21 for the standard experiment. The first nine $\nu$ vectors have equivalent barotropic structure and the higher modes do not. This is
Fig. 17. Patterns of the 500-hPa streamfunction of EOFs (a) 1, (b) 2, (c) 3, and (d) 4 in the standard experiment of the L5T21 model. Fractional variances associated with EOFs 1, 2, 3, and 4 are 20.7%, 10.6%, 6.7%, and 5.6%, respectively. Other conventions are the same as in Fig. 2.

found by defining equivalent barotropy to exist when the pattern correlation between 500 hPa and each of the other four levels is at least 0.10. For comparison, cumulative fractional variances associated with the EOF modes are also shown in the left column of Fig. 21.

While the first nine modes of EOFs represent 60% or more of the total variance, the same number of singular vectors represents a smaller but significant amount of 40%.

To check the significance of the fractional variance
Fig. 18. Distribution of the probability density as in Fig. 3 but on the plane of EOFs 1 and 2 in the standard experiment of the L5T21 model. The arrows indicate the direction of trajectories near the weather regime existing around EOF1 = 4.

associated with the first nine singular modes, it is compared with randomly chosen orthonormal bases. An example is shown in the rightmost column of Fig. 21. In this case the random basis is defined by exchanging the real and imaginary parts of the spherical harmonic expansions of the \( \psi \) vectors. Therefore, the equivalent barotropy is maintained. Since the first singular mode is dominated by the zonally averaged component, the exchange of the wave part affects it little. Except for this mode, however, variances associated with the random modes 1–9 are smaller than those with the singular modes. This result holds for other random bases constructed in different ways (e.g., by changing the sign of imaginary parts, etc.). Thus, the \( \psi \) vectors with small singular value play some roles in forming the principal patterns of LFV.

6. Conclusions and remarks

By using the linear balance model, a variant of the quasigeostrophic system, dynamics of low-frequency variability (LFV) in the extratropical atmosphere has been investigated. An attempt is made to interpret simulated LFV mainly from a viewpoint of changes in macroscopic behavior of the system with respect to parameter changes. The model adopts realistic topography and is driven by the Newtonian relaxation to an equator-to-pole radiative equilibrium temperature profile. Two versions of the model have been used: a two-level model with triangular 15 horizontal truncation (L2T15) and a five-level model with T21 resolution (L5T21). The L5T21 model has the option that Rayleigh damping is added to modes with zonal wavenumbers 4–8, which is called the model with damped baroclinic waves. The L2T15 model and the L5T21 model with damped baroclinic waves give relatively straightforward results. Guided by them, the complicated results of the L5T21 standard model have been successfully interpreted.

Extratropical LFV has at least three aspects, namely, quasi-stationary weather regimes, low-frequency oscillations, and principal spatial patterns of variability for which there is no preferred periodicity and the low frequency predominates as in red noise. The dynamical basis of these three aspects has been clarified as follows.
The dynamical basis of weather regimes and low-frequency oscillations, as has been given basically in Itoh and Kimoto (1996), is the existence of multiple attractors. They can coexist for fixed sets of model parameters at which the system’s behavior is not very turbulent. Most of them become unstable at more realistically turbulent settings of the parameters, but their traces are still recognizable. At this stage, the system trajectories not only visit frequently regions in phase space where the attractors existed (attractor ruins), but also wander among them spontaneously (chaotic itinerancy). Quasi-stationary weather regimes are formed by relatively small attractor ruins or by stagnation regions of larger attractor ruins. Attractors that accompany low-frequency oscillations of sizable amplitudes necessarily have finite extent in phase space.

The two-level model results show that an attractor ruin expands its domain after chaotic itinerancy sets in with destabilizing changes in parameters. This can happen when the original attractor is located in a region of phase space with a flat or smooth structure. As a result, the spatial patterns undergo considerable changes within the attractor ruin so that it may not appropriately be identified as a quasi-stationary weather regime. On the other hand, other attractor ruins remain confined within relatively small regions in phase space and form quasi-stationary and persistent regimes. The low-frequency oscillation associated with the enlarged attractor shows a realistically large amplitude. In this way, the weather regimes and low-frequency oscillations of the L2T15 model are associated with attractors with relatively fine and flat phase space structures, respectively. The L5T21

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model shows more complicated behavior, but the relevance of multiple attractors and associated low-frequency oscillation has been verified.

Low-frequency variations associated with principal spatial patterns without predominant periodicity are interpreted in a relatively straightforward manner for models with insufficient baroclinic wave development; namely, singular modes with the smallest singular values of equations linearized with respect to the time-mean state appear as predominant patterns. Thus, the predominance is ascribed to linear resonance. Small rates of time change of the patterns are guaranteed by the smallness of singular values. It is also found that the leading singular mode is not very sensitive to the basic state of linearization. However, when the baroclinic waves become more active and get more localized, the principal patterns of LFV deviate from the singular modes. Namely, the effects of baroclinic waves in modifying the forcing of the linearized equations are no longer negligible, and the predominant patterns are determined by the interactions between singular modes and the organized effects of baroclinic eddies. In addition, the existence of weather regimes is shown to affect the statistically determined principal patterns of variability.

Table 1 summarizes the three aspects of the low-frequency variability and their dynamical basis. Figure 22 shows schematic scatterplots for explaining the dynamical basis of the low-frequency variability.

Among the three aspects of LFV, principal patterns of variability dominate the other two, intermittent features. We have given a liberal account on the role of singular modes in interpreting the principal patterns of LFV. Time means have been adopted to give the basic state for the linearized analysis. However, the time-mean state is the outcome of the temporal variation of the system, and we have been using the result of variation to interpret variation itself. Therefore, our approach at present lacks the logical consistency. Explaining the time mean itself and clarifying the interaction between singular modes and the forcing are left for future studies.
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APPENDIX A

The Relation between SVD and Conventional Eigenanalysis

We consider the following equation:

$$Ax = f.$$  \hfill (A1)

In this model, $A$ is a matrix obtained by linearizing the original system of equations with respect to the time-mean basic state, $x$ is the streamfunction, and $f$ is the forcing that includes nonlinear interaction terms. We assume in the following that eigenvectors are normalized.

The SVD of $A$ is defined as follows:

$$A = USV^T,$$  \hfill (A2)

where $U$ and $V$ are orthonormal matrices, $V^T$ is the transpose of $V$, $\Sigma$ is a diagonal matrix, and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$. Here $\sigma_i$'s are real numbers that are numbered with an increasing order, $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots$, and are called singular values. Columns of the matrices $U$ and $V$ are called $u$ vector (or left singular vector) and $v$ vector (or right singular vector), respectively. Singular values correspond to positive roots of the eigenvalues of matrix $A^TA$ (the same as those of $AA^T$). The $i$th column of $V$ and $U$ corresponds to the $i$th eigenvectors of $A^TA$ and $AA^T$, respectively, with the common eigenvalue $\sigma_i$. Using these, one obtains the following expression:

$$x = \sum_{i=1}^{n} \frac{v_i(u, f)}{\sigma_i},$$  \hfill (A3)

where $(y, z)$ represents the inner product between $y$ and $z$. It follows that $v_i$ is the most predominant pattern when the forcing is white. Furthermore, when $\sigma_i$ is much smaller than others, $v_i$ predominates irrespective of details of the forcing.

Next, we consider conventional EA. Eigenvalues of $A$, which will be denoted by $\lambda_i$, are, in general, complex. Then, eigenvalues of $A^T$ are $\lambda_i^{\ast}$ (complex conjugate). Associated eigenvectors $d_i$ and $e_i$ satisfy biorthogonal relation. Therefore, it follows that

$$x = \sum_{i=1}^{n} d_i(e, f) \frac{1}{\lambda_i},$$  \hfill (A4)

where $(e, d_i) = 1$ is assumed.

These relations are described in detail by Branstator (1985), Dymnikov (1988), Navarra (1993), and Metz (1994).

It is problem dependent whether one should use SVD or EA. Although EA is relevant for some problems, for example, in which there is growth, SVD has some advantages thanks to the facts that singular values are real numbers and that $u$ and $v$ vectors satisfy orthogonality. Furthermore, $\sigma_1 \leq |\lambda|$ can be proven as follows where $|\lambda|$ is the smallest absolute value among $\lambda_i$'s. First, it is apparent that

$$\sigma_i^2 \leq \sigma_1^2 y_i^2 + \sigma_2^2 y_i^2 + \cdots + \sigma_n^2 y_i^2$$  \hfill (A5)

holds, where $\Sigma y_i^2 = 1$. The right-hand side can be rewritten as

$$\text{rhs of (A5)} = y^T A^T A A T y = (A T y, A T y),$$  \hfill (A6)

with appropriate choice of an orthogonal matrix $T$, where $y^T = (y_1, y_2, \ldots, y_n)$. On the other hand, since $A d_i = \lambda d_i$ by choosing $y = T^{-1} d_i$, the right-hand side of (A6) becomes

$$(A d_i, A d_i) = (\lambda d_i, \lambda d_i) = |\lambda|^2.$$  \hfill (A7)

Hence, $\sigma_1 \leq |\lambda|$ ($\leq |\lambda|$). The equality holds for a special case where $y_i = 1$, $y_2 = \cdots = y_n = 0$; therefore, in general, $\sigma_1 < |\lambda|$. Thus, it is easier to interpret principal patterns by using SVD. This is another advantage of SVD.

APPENDIX B

Simple Methods for Obtaining Multiple Attractors

The simplest algorithm for obtaining multiple attractors is to perform the time integration from many initial values. However, when systems have large degrees of freedom, much computational time is needed, and even so, there are many cases in which all initial values converge into only one attractor. Hence, this algorithm is not practical for systems with large degrees of freedom. Here, two simple and practical algorithms suitable for such systems are proposed. Using these algorithms, multiple attractors have been obtained for the L5T21 model.

In the first algorithm, many time integrations are performed from the resting atmosphere for various values of a control parameter. Time series, EOFs, power spectra, and so on are checked for each experiment. As a result, discontinuity in these indicators may be seen between two parameter values. It is assumed that discontinuity occurs between $\gamma_1$ and $\gamma_2$ ($\gamma_1 < \gamma_2$). Then, a time integration is carried out at $\gamma_1$, setting the final value of the experiment at $\gamma_2$ to an initial value. We may obtain a different attractor from the original one in the experiment at $\gamma_1$ starting from the atmosphere at rest. Then, experiments are performed at smaller pa-
rameter values in succession. Similarly, an experiment is carried out at $\gamma_2$, setting the final value of the experiment at $\gamma_2$ to an initial value. Also in this case, a different attractor may emerge from the attractor in the experiment started from the rest. Then, experiments are performed at larger parameter values one after another. Thus, multiple attractors can be obtained at some range of a control parameter.

In the second algorithm, two time integrations are performed, using the stochastically perturbed model and the standard model, at a certain parameter value from the resting atmosphere. From these results, two probability density functions are constructed. Since trajectories in the stochastically perturbed model move over larger areas in phase space than those in the standard model, other maxima in the probability density function may be formed, in addition to those seen for the non-perturbed case. These maxima possibly correspond to other attractors or attractor ruins. Therefore, the time integration of the standard model starting from an initial value within these maxima might give other attractors.

APPENDIX C

Details of the Explosive Bifurcation in the L5T21 Damped Model

A detailed analysis near the explosive bifurcation ($\gamma = 1.24$) gives a great insight into the result at $\gamma = 1.00$. We take $\gamma = 1.20$ for this purpose. At larger values, trajectories do not frequently visit the place of the 20-day oscillation, and leading EOF patterns are considerably different from those at $\gamma = 1.00$, which makes the continuation to $\gamma = 1.00$ difficult.

Figure C1 shows “local” power spectra for the $\gamma = 1.20$ experiment; the power spectra (multiplied by frequency) summed over all the spectral coefficients are computed for consecutive 128-day intervals in order to see changes of the spectral shape with respect to time. Only the first half (1000–5000 days) of the time integration is shown for clarity. The latter half (5000–10 000 days) also exhibits a similar feature. It can be seen that there are two types of periods that have different spectral characteristics; one is composed of the terms of 2050±2150, 2550±2650, and 3450±3650 days (hereafter term B), which have spectral peaks at 20 and 4.5 days and no peak at 4 days. The other is almost the rest. Here, for simplicity, all the rest is treated as term A, although several small parts may belong to term B, for example, around 1550 days. Since the latter has a strong peak at a period of 4 days, it may be interpreted as the term when trajectories visit attractor ruin Z. (Long periods cannot be resolved in this analysis.) The former appears to reflect attractor ruin Y, because the two spec-
central peaks are considered to be the result that the original peaks in attractor Y shorten.

In order to confirm this, scatterplots are depicted on a plane spanned by principal EOF modes, and correspondence of scattered points is examined. The EOF analysis at $\gamma = 1.20$ is made between days 1000 and 10 000. The low-pass filter is applied before the analysis in order to suppress dominance of small-scale, baroclinic waves in the leading modes. Figure C2a shows a scatterplot for term B by color and term A by black on the plane spanned by EOFs 1 and 2, while that for attractor Y at $\gamma = 1.67$ and attractor Z at $\gamma = 1.26$ on the same plane is illustrated in Fig. C2b. Plotted areas of term B and attractor Y roughly coincide, and those of term A and attractor Z also show good correspondence. The coincidence of spectral peaks and scattered points leads us the conclusion that behaviors in terms B and A reflect attractor ruins Y and Z, respectively.

**APPENDIX D**

**Further Evidence of Multiple Attractors and Chaotic Itinerancy Appearing in the L5T21 Standard Model**

One local maximum of the probability density in Fig. 18 is identified along the abscissa near amplitude 4 of EOF1. An examination of trajectories near this peak shows movements as indicated in the figure by arrows. It is also found that the trajectories slow down in the vicinity of this peak to verify persistence (figure not shown). Therefore, there is evidence for the existence of another quasi-stationary weather regime for the L5T21 model. Although analyses of transitions are not carried out in this paper, this type of regime is likely to show preferred routes of onsets and exits (cf. Kimoto and Ghil 1993b), thus providing a mechanism for inhomogeneous transition statistics other than the one related to global bifurcations as described in IK2 and mentioned in section 3 of this paper.

Two attractors, called X and W, are found at $\gamma = 3.1$, a fairly large value. Trajectories of X and W are plotted in Fig. D1 on a plane spanned by EOFs 1 and 2 of the standard experiment ($\gamma = 1.0$). While attractor W is confined in a small region near the origin, X extends along the positive direction of EOF1. The extent of scatter and the direction of rotation of attractor X agree well with the regime identified in Fig. 18 for $\gamma = 1.0$.

At $\gamma = 3.0$ attractor X loses its stability. Trajectories starting from the vicinity of X converge to W after a while (figure not shown). At a smaller value of $\gamma = 1.05$, W becomes unstable and the trajectory gets expanded to visit X intermittently (figure not shown) as in the chaotic itinerancy seen for the L2T15 model in IK1 and IK2 and for the L5T21 model with damped baroclinic waves in section 5a. Again a global, explosive bifurcation appears responsible for the enlargement of the attractor. The dynamical basis of the weather regime is the chaotic itinerancy among the attractor ruins also in this model, although this regime originates not from a quasi-stationary attractor but from a stagnation region of a very slow oscillation with a period of order 100 days.

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