Parameter Sweep Experiments on Spontaneous Gravity Wave Radiation from Unsteady Rotational Flow in an $f$-Plane Shallow Water System

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ABSTRACT

Inertial gravity wave radiation from an unsteady rotational flow (spontaneous radiation) is investigated numerically in an $f$-plane shallow water system for a wide range of Rossby numbers, $1 \leq Ro \leq 1000$, and Froude numbers, $0.1 \leq Fr \leq 0.8$. A barotropically unstable jet flow is initially balanced and maintained by forcing so that spontaneous gravity wave radiation is generated continuously. The amount of gravity wave flux is proportional to $Fr$ for large $Ro(\geq 30)$, which is consistent with the power law of the aeroacoustic sound wave radiation theory (the Lighthill theory). In contrast, for small $Ro(\leq 10)$ this power law does not hold because of the vortex stabilization due to the small deformation radius. In the case of fixed $Fr$, gravity wave flux is almost constant for larger $Ro(> 30)$ and decreases rapidly for smaller $Ro(< 5)$. There is a local maximum value between these $Ro(\sim 10)$. Spectral frequency analysis of the gravity wave source shows that for $Ro = 10$, while the source term related to the earth’s rotation is larger than that related to unsteady rotational flow, the inertial cutoff frequency is still lower than the peak frequency of the dominant source. The results suggest that the effect of the earth’s rotation may intensify spontaneous gravity wave radiation for $Ro \sim 10$.

1. Introduction

Gravity wave radiation from unsteady rotational flows is one of the most fascinating topics in atmospheric science, from the theoretical (Zeitlin et al. 2003; Vanneste and Yavneh 2004), observational, experimental (Williams et al. 2005), as well as numerical (Schecter and Montgomery 2004; Dritschel and Vanneste 2006) viewpoints. It is well known that gravity waves play an important role in the middle atmosphere by driving general circulation (Holton et al. 1995; Fritts and Alexander 2003). There are many studies on gravity waves, and several sources for these waves (topography, convection, jets, fronts, cyclones, and so on) have been identified (Fritts and Nastrom 1992; Sato 2000). From several observational studies it has been suggested that inertial gravity waves are radiated from strong rotational flows, such as polar night jets (Yoshiki and Sato 2000), subtropical jets (Uccelini and Koch 1987; Kitamura and Hirota 1989; Sato 1994; Plougonven et al. 2003), and cyclones (Pfister et al. 1993; May 1996). However, these observational studies of limited cases have reported mainly the characteristics of radiated inertial gravity waves and background flow fields. The radiation processes of gravity waves (hereafter we refer to inertial gravity waves as gravity waves) from unsteady rotational flows are not fully understood.

One possible radiation mechanism is that gravity waves are spontaneously radiated from initially balanced rotational flows. Since gravity waves are radiated from the unsteady motion of an initially balanced state, Ford et al. (2000) called this radiation process “spontaneous emission” or “Lighthill radiation.” The process is different from “geostrophic adjustment” for the classi-
cal initial value problems (Rossby 1938; Gill 1977). In the process of geostrophic adjustment, initial unbalanced states are supposed. Then, gravity waves are radiated in the time evolution from initial unbalanced states toward final balanced states. As a result, final balanced states are free from gravity wave radiation. In contrast, in the process of spontaneous emission, gravity waves are observed at any stage. Using general circulation models or mesoscale models, several numerical investigations of spontaneous gravity wave radiation have been made for the case of idealized baroclinic waves (O’Sullivan and Dunkerton 1995; Zhang 2004; Plougonven and Snyder 2007) and polar night jets (Sato et al. 1999). The purpose of these studies is to investigate particular geophysical phenomena, such as cyclogenesis, and only a few case studies in various geophysical situations are adopted, because of limited computational resources. A simpler configuration can be investigated by considering shallow water rather than a continuously stratified fluid.

To investigate spontaneous gravity wave radiation in detail, we use an $f$-plane shallow water system. This is the simplest system in which both rotational flows and inertial gravity waves can exist. The equations of this system are equivalent to those of a two-dimensional adiabatic gas system with a specific heat ratio of $\gamma = 2$, if the effect of the earth’s rotation is negligible. Thus, gravity waves in shallow water systems are analogous to sound waves in compressible gas systems. In the field of aeroacoustic sound wave radiation, there is a well-known sixth power law for the Mach number $M (M = U/C_a; U$ is the velocity and $C_a$ is the phase speed of sound wave) related to the amplitude of sound wave radiation in small $M$ (Lighthill 1952). As for $f$-planes, Ford (1994) investigated spontaneous gravity wave radiation from an unsteady rotational flow as a first step. Using the acoustic analogy of Lighthill, he introduced a source of gravity waves and showed the power law of the Froude number Fr for gravity wave amplitude in case of small Fr. His study treated only restricted Rossby number (Ro) parameter values, 0.1 to 20, in eight cases. Since he used a nonforcing system with an initial unstable vorticity strip, gravity waves were radiated in an early instability stage only, and further radiation ceased as the flow became stable. In the real atmosphere, however, observational studies suggest that gravity waves are continuously radiated from unsteady rotational flows.

In the present study, we investigate continuous gravity wave radiation, which is spontaneously radiated from an unsteady rotational flow, for a wide range of parameter values of Ro as well as Fr. We use a barotropic unstable jet flow as a basic state and add forcing to maintain this flow. By choosing a forcing parameter, we maintain an unsteady rotational flow that changes periodically with time. Spontaneous gravity wave radiation from an unsteady rotational flow is then generated continuously. We have investigated continuous gravity wave radiation from an unsteady rotational flow using a similar system with a fixed parameter value (Sugimoto et al. 2007b). In the previous study, we introduced a source approximation for spontaneous gravity wave radiation in both near and far fields, using the analogy of the aeroacoustic sound wave radiation theory. Recently, one of the authors developed a new shallow water numerical code (Ishioka 2007b). In this model, we use a projection method with very dense grid points set in the jet region but also set coarse grid points far from jet region, where gravity waves propagate. This model allows us to save much computational time for numerical calculations and makes it possible to estimate gravity wave amplitude quantitatively in a wide parameter range. Using this model, we have investigated the Fr dependence of the gravity wave amplitude for fixed Ro values (Sugimoto et al. 2005) and the validity of the source approximation in a wide parameter range (Sugimoto et al. 2007a). In the present study, we discuss the Ro and Fr parameter dependence of the conditions of spontaneous gravity wave radiation.

This paper is organized as follows. Section 2 describes basic equations and the setup of the nonlinear numerical experiment. Section 3 introduces the Ford–Lighthill equation for the forced-dissipative $f$-plane shallow water system, in which the source of gravity waves is defined. We also define gravity wave flux and introduce the power law of Fr for gravity wave flux using a scaling analysis. The results of numerical experiments for a wide parameter range of Ro and Fr are shown in section 4. Here, we also analyze the spectral frequency of the source term in order to explain the results. Section 5 gives a summary and discussion of our findings.

### 2. Experimental setup

We consider gravity wave radiation from an unsteady zonal jet flow in a rectangle domain, using the following $f$-plane shallow water system:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fu &= -g \frac{\partial h}{\partial x}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu &= -g \frac{\partial h}{\partial y}, \\
\frac{\partial h}{\partial t} + u \frac{\partial (h - h_b)}{\partial x} + v \frac{\partial (h - h_b)}{\partial y} + (h - h_b) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0.
\end{aligned}
\]
Here $t$ is time, and $f$ and $g$ express the Coriolis parameter and the gravitational acceleration, respectively. Cartesian coordinates $(x, y)$ are chosen so that $x, y$ are the longitudinal and latitudinal direction, respectively. Also, $u, v$ are the horizontal velocity components, and $h$ is the surface level. We also introduce a bottom topography $h_b$. The velocity $(u_0, v_0)$ of the steady state and the surface level $h_0(y)$, which is in geostrophic balance with $u_0(y)$ ($f u_0 = -g dh_0/dy$), are unstable zonal jet flows (Hartmann 1983):

$$[u_0(y), v_0] = [U_0 \text{sech}(y - y_0/B), 0] \quad \text{and} \quad h_0(y) = -(fB U_0/g) \arctan[\exp(2y/B)] + H_0,$$

(4)

where $U_0$, $y_0$, and $B$ are parameters to determine the intensity, position, and width of the jet, respectively. Since we want to investigate spontaneous gravity wave radiation in a simplified experimental setting, we set a bottom topography $h_b$ so that the flow field is symmetric with respect to the center of the jet $[h_b = -(fB U_0/g) \arctan \exp(2y/B)]$. The total depth of the fluid $H_0(=h_0 - h_b)$ is then constant. Using this topography, we can investigate the case of shallow depth without difficulty. In the present study, the nondimensional parameters $Ro$ (the ratio of advective to Coriolis term) and $Fr$ (the ratio of flow velocity to phase speed of gravity wave) are determined by the basic state as

$$Ro = \frac{U_0}{fB} \quad \text{and} \quad Fr = \frac{U_0}{\sqrt{gH_0}}.$$

We fix $U_0$ and $B$ for the numerical experiments; $Ro$ and $Fr$ are then nondimensional reciprocals of the frequency $f$ and the phase speed of gravity waves $\sqrt{g H_0}$, respectively. Gravity wave radiation from the time evolution of the fluid initiated from this basic state is investigated for a wide range of $Ro$ and $Fr$. Figure 1 shows an example of the latitudinal profiles of the zonally symmetric basic state of the jet flow $u_0(y)$ and the latitudinal gradient of the potential vorticity $dq_0(dy)/dy$.

![Figure 1](https://example.com/figure1.png)

*Fig. 1. Latitudinal profiles of the zonally symmetric basic state (Ro = 100 and Fr = 0.3). (left) The jet flow $u(y)$ and (right) the latitudinal gradient of the potential vorticity $dq_0(dy)/dy.$

(9)

Using the basic state of (4)–(5), the disturbance equations of (1)–(3) are rewritten as follows:

$$\frac{\partial u'}{\partial t} = -u' \frac{\partial u'}{\partial x} - v' \frac{\partial u'}{\partial y} + fu' - g \frac{\partial \eta'}{\partial y} - \alpha u', \quad \text{(7)}$$

$$\frac{\partial v'}{\partial t} = -u' \frac{\partial v'}{\partial x} - v' \frac{\partial v'}{\partial y} + fu' - g \frac{\partial \eta'}{\partial y} - \alpha v', \quad \text{and}$$

$$\frac{\partial \eta'}{\partial t} = -u' \frac{\partial \eta'}{\partial x} - v' \frac{\partial \eta'}{\partial y} - (H_0 + \eta') \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right), \quad \text{(8)}$$

where $u'(=u - u_0)$ and $v'$ are disturbance from the basic state, and $\eta'$ is a free-surface displacement from the depth of the fluid $H_0$. The forcing terms with relaxation rate $\alpha$ are added in (7)–(9) and act to maintain the unstable zonal jet. Since we want to estimate gravity waves far from the jet region, the forcing rate $\alpha$ is set as follows only near the jet region ($y = 0$):

$$\alpha(y) = A_f e^{-B_f y^2}. \quad \text{(10)}$$

Here, $A_f$ and $B_f$ are parameters to give the amplitude and width of the forcing, respectively. Note that $h_0$ of the basic state is excluded in (7)–(9) explicitly by intro-
ducing the bottom topography. For numerical experiments, we fix $U_0 = 10\pi$ and $B = \pi/10$. We also choose $A_f = 12$ and $B_f = 1/10$ so that the jet flow maintains an unsteady motion. Setting these values of the forcing term, we have no separation of the vortex in the nonlinear phase of the barotropic instability, and there is not much damping for propagating gravity waves. Note that we have used several different parameter values to check the robustness of the results.

The domain for numerical experiments is set to be periodic in the longitudinal ($x$) direction and approximately infinite in the latitudinal ($y$) direction. Calculation for the $y$ direction is performed by use of the mapping

$$y = 2r \tan(\theta/2) \quad \text{and} \quad \frac{d}{dy} = \frac{\cos^2(\theta/2)}{r} \frac{d}{d\theta}$$

so that $\theta \in (-\pi, \pi)$ is projected to $y \in (-\infty, \infty)$ (Ishioka 2007b). Here, $r$ is the scaling parameter to determine the ratio of grid intervals for the $x$ and $y$ direction. This projection method allows us to save the number of grid points far from the jet region, where gravity waves propagate, while taking small grid intervals in the jet region.

For the numerical calculation the mapped shallow water equations are computed using a pseudospectral transform method (Ishioka 2007a). Namely, the dependent variables are expanded as

$$W(x, \theta, t) = \sum_{k=-K}^{K} \sum_{l=-L}^{L} s_{kl}(t)e^{ikx}e^{il\theta},$$

where $W(x, \theta, t)$ represents $u'$, $v'$, and $\eta'$, and $K$ and $L$ are the truncation wavenumbers for the $x$ and $\theta$ direction, respectively. The time evolution of $u'$, $v'$, and $\eta'$ in physical space is written as

$$\frac{\partial W}{\partial t} = Z(x, \theta, t),$$

where $Z$ represents the right-hand side of (7)–(9). The time evolution of the coefficient $s_{kl}$ is determined by the forward transform as

$$\frac{ds_{kl}}{dt} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{0}^{2\pi} Z(x, \theta, t)e^{-ikx}e^{-il\theta} dx d\theta.$$  

In the present study, we set $r = 4$. The domain size is set to be $K = 21$, $L = 336$ (64 $\times$ 1024 grids) in order to prevent aliasing error from the calculation in nonlinear terms. By setting these resolutions, grid intervals for the $y$ direction ($\Delta y$) in the jet region are $\Delta y \sim 0.024$, and the farthest grid point is positioned at $y \sim 5125$. The fourth-order Runge–Kutta method is used for the time integrations with an increment of 0.0002. We also introduce a pseudo-hyperviscosity term,

$$\nu \left( \frac{\partial^2}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^3 W(x, \theta, t),$$

(15) to smooth numerical behavior. Since

$$\frac{\partial^2}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial x^2} + \left[ 1 + \left( \frac{y}{2r} \right)^2 \right] \frac{\partial^2}{\partial y^2}$$

$$+ \frac{y}{2r^2} \left[ 1 + \left( \frac{y}{2r} \right)^2 \right] \frac{\partial}{\partial y},$$

(16) (15) corresponds to the normal hyperviscosity near the jet region ($y \sim 0$) and plays a role in a wave absorbing layer far from the jet region (for large $y$), where gravity waves propagate. The parameter for this pseudo-hyperviscosity $\nu$ is fixed to $\nu = 5 \times 10^{-7}$. This value is chosen to be as small as possible so as not to collapse the numerical calculation. We have checked for insensitivity of the results to the pseudo-hyperviscosity by changing both the parameter value for pseudo-hyperviscosity and grid numbers.

3. The Ford–Lighthill equation and the parameter dependence of gravity wave flux

To investigate spontaneous gravity wave radiation, we introduce the source of gravity waves by analogy with the theory of aeroacoustic sound wave radiation (the Lighthill theory). Rewriting the shallow water (1)–(3) in the flux form, we obtain the Ford–Lighthill equation (Ford 1994) in the forced-dissipative system with topography as follows:

$$\left( \frac{\partial^2}{\partial t^2} + f^2 \frac{\partial^2}{\partial y^2} \right) \frac{\partial h}{\partial t} = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} - F_1 - F_2,$$

(17) where $x_1 = x$, $x_2 = y$, and $c_0 = \sqrt{gH_0}$ is the phase velocity of the fastest gravity wave. The left-hand side is the wave operator of a linear inertial gravity wave, and the right-hand side can be regarded as the source of inertial gravity waves. Here, $T_{ij}$ is written as

$$T_{ij} = \frac{\partial}{\partial t} (uh_x \mu_k) + \frac{f}{2} (\epsilon_{ijk} h u_x \mu_k + \epsilon_{ijk} h u \mu_k)$$

$$+ \frac{g}{2} \frac{\partial}{\partial t} (h - H_0)^2 \delta_{ij},$$

(18) and

$$F_1 = \alpha \frac{\partial}{\partial x} \left[ \frac{\partial (h - u_x)}{\partial x} + \frac{\partial (hu)}{\partial y} \right]$$

$$+ \alpha f \left[ \frac{\partial (hu)}{\partial x} - \frac{\partial (h - u_x)}{\partial y} \right],$$

(19) and

$$F_2 = \frac{\partial}{\partial y} \left( g \frac{\partial h}{\partial x} \frac{\partial h}{\partial t} + f g \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right),$$

(20)
Here, \( \epsilon_{12} = \epsilon_{21} = -1, \epsilon_{11} = \epsilon_{22} = 0, \) and \( u_1 = u, u_2 = v \). \( F_1 \) and \( F_2 \) are the source terms related to the zonal forcing and the bottom topography, respectively.

Averaging (17) in the \( x \) direction, we obtain the equation

\[
\frac{\partial \overline{\tilde{h}(y,t)}}{\partial t} = \frac{\partial^2}{\partial y^2} \overline{T_{22}(y,t)} - \overline{F_1(y,t)} - \overline{F_2(y,t)},
\]

(21)

where \( \overline{\cdot} \) represents \( x \)-averaged values. Solving (21), we calculate the zonally (\( x \)) averaged temporal variation of \( h(\partial \overline{\tilde{h}}/\partial t) \) far from the source region (Sugimoto et al. 2007b):

\[
\frac{\partial \overline{\tilde{h}(y,t)}}{\partial t} = \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y^-}^{y^+} dy' J_0 \times \left[ f \sqrt{(t-t')^2 - \left( \frac{y-y'}{c_0} \right)^2} \bigg] \overline{N}(y', t'),
\]

(22)

where \( y \pm = y \pm c_0(t - t') \). The equation \( \overline{N}(y', t') = \frac{\partial^2 \overline{T_{22}(y', t')}}{\partial y^2} - \overline{F_1(y', t')} - \overline{F_2(y', t')} \) is the source term averaged in the \( x \) direction. Integrating the right-hand side of (22) yields the equation

\[
\frac{\partial \overline{\tilde{h}(y,t)}}{\partial t} = \frac{1}{2c_0} \int_{t_0}^{t} dt' [\overline{J}(y_+, t') - \overline{J}(y_-, t')] + D,
\]

(23)

where \( \overline{J} \) is a function that satisfies \( \overline{\tilde{N}}(y, t) = \overline{\tilde{h}(y, t)}/\partial y \) and \( D \) represents the effect of dispersion. If we know the source term \( \overline{\tilde{N}} \) near the jet, we estimate \( \overline{\tilde{h}}(\partial \overline{\tilde{h}}/\partial t) \) far from the jet region from (23).

In addition, from (21), we estimate the scaling of gravity wave amplitude \( \overline{\tilde{h}} \) far from the jet region (Ford 1994). Since gravity waves are radiated mainly from an unsteady rotational flow (Sugimoto et al. 2007a,b), source terms related to the zonal forcing and the bottom topography can be negligible (\( \overline{F_1}, \overline{F_2} \approx 0 \)). If the first term in (18) is dominant and depends on \( t \) sinusoidally, then \( \overline{T_{22}} \) is scaled as

\[
\overline{T_{22}(y)} \sim \omega \overline{\tilde{u}^2(y)} H_0,
\]

(24)

where \( \omega \) is the frequency of the source and \( \overline{\tilde{u}^2} \) is the Fourier transform of \( u^2 \) with respect to \( t \). Here \( \omega \) is determined by the dominant frequency of an unsteady rotational flow. If the source is localized near the jet region, then (21) is scaled as

\[
io \overline{\tilde{h}(y)} \sim \frac{(\omega^2 - f^2)^{1/2}}{2c_0} \left( e^{i(\omega^2 - f^2)/2y/c_0} \right) \int_{-B}^{B} \overline{T_{22}(y)} dy,
\]

(25)

where \( B(\sim B) \) is the width of the jet region. Using the scale for \( \overline{T_{22}} \) from (24), we obtain

\[
\overline{h}(y, t) \sim H_0 \left( \frac{U_0}{c_0} \right)^2 \left[ \left( \frac{\omega^2 - f^2}{c_0} \right) - B \right].
\]

(26)

To measure gravity wave amplitude, we introduce pseudoenergy and its flux in the shallow water equations. Far from the source region, where gravity waves propagate and the value of the potential vorticity is zero, pseudoenergy \( A_e \) and its flux \( F_e \) are defined as (Ford 1994)

\[
A_e = \frac{1}{2} h(u'^2 + v'^2) + u_i u' h + \frac{1}{2} gh'^2 \quad \text{and} \quad (27)
\]

\[
F_e = u A_e + H_0 u_0 \cdot u' u' - \frac{1}{2} H_0 u'^2 u_0 + \frac{1}{2} gh'^2 u
+ g H_0 h' u'.
\]

(28)

Here, \( u = u_0 + u' \), \( u_0 = (u_0, 0) \), \( u' = (u', v') \), and \( h = H_0 + h' \). Therefore, using the scaling for \( h' \) from (26), letting \( h' \approx \overline{\tilde{h}} \), and assuming that the second term in (27) and the first term in (28) are dominant, the pseudoenergy flux far from the jet region is scaled as

\[
\|F_e\| \sim \frac{\omega B^2 U_0^3 \text{Fr}(\omega^2 - f^2)^{1/2}}{g}.
\]

(29)

We fix \( B \), \( U_0 \) in the numerical experiments. If \( \omega \) is constant and the effect of the earth’s rotation is negligible (\( f \approx 0 \)), (29) shows that gravity wave flux \( F_e \) is proportional to \( \text{Fr} \). As in the case of the Lighthill theory, where \( \omega B \approx U_0 \) and \( \text{Fr} \approx U_0 \), \( F_e \) is proportional to the sixth power of \( U_0 \).

### 4. Results

**a. Typical gravity wave radiation (\( \text{Ro} = 100, \text{Fr} = 0.3 \))**

First, the results for the case of \( \text{Ro} = 100 \) and \( \text{Fr} = 0.3 \) are shown. In a previous study we adopted these parameter values using the numerical model in a doubly periodic domain (Sugimoto et al. 2007b). In the present study, since the numerical model has no gravity wave reflection at the boundary, it is easy to see the characteristics of spontaneous gravity wave radiation. Figure 2 shows the time evolution of the total enstrophy for each zonal wavenumber (\( k = 1 \rightarrow 4 \)) in the nonlinear phase (8.0 \( \leq t \leq 12.0 \)) of barotropic instability.
Initially each wavenumber grows linearly owing to barotropic instability (not shown), then saturates to finite amplitude. In the nonlinear phase, an interaction with each wave motion and the zonal forcing causes an unsteady rotational flow, and each wavenumber changes periodically with time (Sugimoto et al. 2007b). The dominant wavenumber is \( k = 2 \), and time evolutions of all wavenumbers have two types of oscillation. One is a low-frequency mode, which has relatively large amplitude and intermittency (around \( t = 11.0 \)). The other is a high-frequency mode, which has relatively small amplitude and changes quasi-periodically. The frequency of this mode, \( \nu_j \), is about 5–6.

Figure 3 shows the \( t-y \) cross section of \( \partial H / \partial t \) for \( 8.0 \leq t \leq 12.0 \) and \( -50 \leq y \leq 50 \). The characteristics of \( \partial H / \partial t \) in (17) indicate linear gravity wave propagation if there is no source term. As the jet fluctuates quasi-periodically in time, gravity waves are radiated continuously from the jet region (around \( y = 0 \)) along with the phase speed of the fastest gravity wave, \( \sqrt{gH_0} = 100\pi/3 \). Since there is no reflection of gravity waves at the boundary, it is clearly observed that spontaneous gravity wave radiation is generated continuously. The dominant frequency of gravity waves, \( \nu_g (\sim 5–6) \), is in good agreement with that of an unstable rotational flow (\( \nu_g \sim \nu_j \)) as required by the Lighthill mechanism. This indicates that the dominant frequency of an unsteady rotational flow determines the frequency of gravity waves. Since we introduce the bottom topography so that the total depth of the fluid is constant, gravity wave radiation is almost symmetric with respect to the center of the jet. Hereafter we focus mainly on the upper-half region (\( y \geq 0 \)).

As shown in our previous study (Sugimoto et al. 2007b), spontaneous gravity wave radiation from nearly balanced rotational flow is generated continuously. Sugimoto et al. (2007c) studied an unstable jet in an \( f \)-plane shallow water system with the use of linear stability analysis and nonlinear numerical simulation and showed that quasigeostrophic approximation not only gave a good estimation of maximum growth rate for large \( \text{Ro} \), but also was valid even in the nonlinear phase of instability for large \( \text{Ro} \) as long as \( \text{Fr} \) was small. Thus, the balanced rotational flows were thought to be dominant for small \( \text{Fr} \) of the present case. Note that this radiation process of gravity waves does not correspond to the classical Rossby adjustment problem.

Figure 4 is an example of the time evolution of vor-
Fig. 4. The time evolution of (top) vorticity $\zeta$, (middle) $\partial \Phi / \partial t$ (contour) and the source calculated from (17) (color), and (bottom) geopotential height $\Phi$. Time is (a) $t = 6.0$, (b) $t = 6.1$, (c) $t = 6.2$, and (d) $t = 6.3$. Vertical and horizontal axes are $x$ and $y$, respectively. In the middle figures, negative and positive regions of the source are painted in blue and red, respectively. Contours are 0 and $\pm 10^i$, ($i = -3, -2, \ldots, 8$), and positive values are denoted by solid lines, while negative values are denoted by broken lines.
ticity $\zeta$, $\partial \Phi/\partial t$ and source term calculated from (17), and geopotential height $\Phi(=gh)$. The source region is localized in a region of strong vorticity. The time evolution of the gravity wave source is associated with the development of an unstable jet flow and causes spontaneous gravity wave radiation continuously. While a wavenumber of 2 for the $x$ direction of gravity waves is found in the jet region, which corresponds to the wavenumber of the source, only zonal (wavenumber 0) gravity waves are observed far from the source region. This is because gravity waves having wavenumber 2 are evanescent in the dispersion relation of gravity waves (Sugimoto et al. 2007b).

Figure 5 shows the time evolution of $\partial \Phi/\partial t$ for each latitude (5–40) and the theoretical estimation of (23) without dispersion. It is clearly observed that gravity waves are radiated continuously from the jet region along with the phase speed of the fastest gravity wave. The theoretical estimations are in good agreement with the results of numerical simulation, so that the forcing and pseudo-hyperviscosity hardly affect gravity wave radiation and propagation in these regions. Since the correspondence of each line is quite good, we have to show each line individually (cannot use a superposed one). In our previous study, $\partial \Phi/\partial t$ of numerical simulation coincides with the theoretical estimation in a limited region of a short period, because of the reflection of gravity waves at the boundary (Sugimoto et al. 2007b). Since we use a special technique for the numerical simulation, there is high accuracy in the estimation of gravity wave amplitudes within the region $5 \leq y \leq 40$. Far from the jet ($y \geq 40$), pseudo-hyperviscosity damps the outward propagating waves, accordingly not abruptly (not shown).

We also check the time evolution of gravity wave flux averaged in the $x$ direction for each latitude. The result shows that gravity wave flux is well conserved far from the jet region $20 \leq y \leq 40$. In section 4b, we investigate the dependence of nondimensional parameters, using gravity wave flux at $y = 40$. Note that we have checked the dependence of the resolution, doubled the grid numbers for both directions (half grid intervals), and obtained the same results.

b. Parameter dependence

In this section, we show the results of the parameter sweep experiments. We performed a systematic series of parameter sweep experiments with $Ro = 1–1000$ (1, 3, 5, 7, 10, 15, 20, 30, 50, 70, 100, 500, 1000), 13 values, and $Fr = 0.1–0.8$ (0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.6, 0.7, 0.8), 12 values. Since our main goal is to investigate the parameter dependence of spontaneous gravity radiation for a wide range of parameter space thoroughly, we use large Ro and Fr cases, although these values are not realistic for phenomena in the real atmosphere.

Figure 6 shows the $Ro$–$Fr$ parameter dependence ($Ro = 1, 5, 10, 100; Fr = 0.1, 0.3, 0.5, 0.7$) of the time evolution of the enstrophy for each wavenumber ($k = 1–4$) in the $x$ direction. There are two regimes in the parameter space. The first regime is everything except for the cases of $Ro = 1$ and large Fr ($0.5, 0.7$). In this regime, there are two dominant modes in an unsteady rotational flow. One has intermittency and large fluctuation, and the other has high frequency and small fluctuation. These results are qualitatively the same as the previous result, as shown in Fig. 2. Thus, the unsteady motion of the flow field is not affected by either changes in $f$ (caused by Ro) or $H_o$ (caused by Fr) in all parameter ranges. On the other hand, in the second regime for the cases of $Ro = 1$ and large Fr ($0.5, 0.7$), there is no unsteady rotational flow. Since decrease of the deformation radius ($\sqrt{gH_o/f} = RoL/Fr$) inhibits
the interactions of rotational flow, the flow becomes stable.

Figure 7 shows the Ro–Fr parameter dependence (Ro = 1, 5, 10, 100; Fr = 0.1, 0.3, 0.5, 0.7) of the t–y cross section of $\partial h / \partial t$. Time and y are $8.0 \leq t \leq 12.0$, $-10 \leq y \leq 50$, respectively. Although the time evolution of an unsteady rotational flow does not change in the first regime in Fig. 6, the gravity wave radiation caused by an unsteady jet flow differs. There is a great difference between Ro = 1 and Ro = 5 in the first regime. For Ro = 1, gravity waves are no longer observed, not only for the cases of large Fr (0.5, 0.7) (second regime), where there is no unsteady rotational flow, but also for the cases of small Fr (0.1, 0.3) (first regime). Fluctuations in Fr = 0.1 are not gravity waves, but balanced modes associated with an unsteady jet flow, since the phase speed of these fluctuations differs from that of the fastest gravity wave. Since $\omega^2 - f^2$ is negative in (29) for small Ro (high $f$), gravity waves do not propagate (the frequency of a gravity wave can never be negative). Note that results for Ro = 3 are almost the same as those for Ro = 5 (not shown).

For Ro = 5, gravity waves are radiated continuously from the jet region ($y = 0$). The phase speed of the fastest gravity wave, $\sqrt{\rho H_0} = u_0 / Fr$, is in good agreement with the results of the numerical simulation, since the frequency of an unsteady jet flow is almost constant for the change of Fr, as shown in Fig. 6. Gravity waves propagate far from the source region in the same time interval in the case of small Fr (large $H_0$), so that the wavelength is proportional to Fr$^{-1}$. It is also observed that $\partial H / \partial t$ has large amplitude for larger Fr.

Next, we discuss the parameter dependence of gravity wave amplitude. The Ro dependence of gravity wave flux for the cases of Fr = 0.1 and Fr = 0.7 are shown in Fig. 8. Here, we show the time averaging ($5.0 \leq t \leq 15.0$) for the zonal component of gravity wave flux at $y = 40$. For all Fr, while gravity wave flux is almost constant for Ro = 30, gravity wave flux decreases suddenly for smaller Ro = 10. In addition, gravity wave flux has local maximum for the case of Fr = 0.1 (Fr = 0.5) around Ro = 10. For large Ro, in which $f$ is high, the effect of the earth’s rotation is negligible, so that gravity wave flux will be constant. For small Ro, in which $f$ is low, a sudden decrease of gravity wave flux may be found because of the negative...
The result of a local maximum for \( \omega^2 - f^2 \) in (29). The result of a local maximum for \( \text{Ro} = 10 \) is interesting, since in (29) gravity wave flux will decrease for smaller \( \text{Ro} \) (higher \( f \)). This result implies that the earth’s rotation may intensify spontaneous gravity wave radiation for \( \text{Ro} \approx 10 \).

For the case \( \text{Ro} \geq 5 \), spontaneous gravity wave radiation is observed. Figure 9 shows the \( \text{Fr} \) dependence of gravity wave flux for different \( \text{Ro}(=5, 10, 50, 100) \). For large \( \text{Ro}(=50, 100) \), where \( f \) is so small that the effect of the earth’s rotation is negligible, gravity wave flux is proportional to \( \text{Fr} \), which is consistent with the theoretical estimation of (29) based on the Lighthill theory. On the other hand, gravity wave flux is not proportional to \( \text{Fr} \) for relatively small \( \text{Ro}(=5, 10) \), so that the effect of the earth’s rotation is important. Note that for \( \text{Ro} = 10, 50 \) curves are not very smooth. This may be caused by insufficient time averaging.

As we have seen, though the time evolution of the flow field is qualitatively similar for different parameter values, gravity wave radiation for each experiment has

![Fig. 7](image1)

**Fig. 7.** The \( \text{Ro} - \text{Fr} \) parameter dependence of the \( t-y \) diagram of \( \frac{\partial \Phi}{\partial t} \) time and \( y \) are \( 8.0 \leq t \leq 12.0, -10 \leq y \leq 50 \), respectively. Vertical axis is \( \text{Fr}(=0.1, 0.3, 0.5, 0.7) \), and horizontal axis is \( \text{Ro}(=1, 5, 10, 100) \). Color and contour intervals are the same as in Fig. 3.

![Fig. 8](image2)

**Fig. 8.** The \( \text{Ro} \) dependence of the zonally averaged gravity wave flux at \( y = 40 \) for (left) \( \text{Fr} = 0.1 \) and (right) \( \text{Fr} = 0.7 \). The value of the flux is time averaged for \( 5.0 \leq t \leq 15.0 \).
several different features. In section 4c, we analyze the source of gravity waves in detail and show why these results are obtained.

c. Analysis of the gravity wave source

Several of our results for the parameter dependence of gravity wave flux are not anticipated by the classical Lighthill theory. Gravity wave flux

\[ S = \int_{-B_j}^{B_j} T_{22} \, dy \]

where \( B_j(=8) \) is the width of the jet region. We also divide \( T_{22} \) into the following three terms to investigate which terms are dominant:

\[
S = \int_{-B_j}^{B_j} T_{22} \, dy \\
= \int_{S_1}^{S_2} \frac{\partial(hu)}{\partial t} \, dy - \int_{S_1}^{S_2} \frac{\partial hu}{\partial t} \, dy + \int_{S_1}^{S_2} \frac{g}{2} \frac{\partial h^2}{\partial t} \, dy.
\]

(30)

Figure 10 shows the Ro dependence of the gravity wave source for the cases of \( \text{Fr} = 0.1 \) and \( \text{Fr} = 0.7 \). Since \( S_3 \) is small for all cases, we neglect this term. While \( S_2 \) (solid line) is proportional to \( \text{Ro}^{-1} \), since \( f \sim \text{Ro}^{-1} \), \( S_1 \) (broken line) is almost constant for each Ro. This is because \( U_0 \) and \( H_0 \) are constant and an unsteady jet flow is qualitatively the same for all cases in the first regime, as shown in Fig. 6. The shift of the dominant source term, which agrees with a previous laboratory study (Williams et al. 2005), may cause a local maximum of gravity wave flux for \( \text{Ro} = 10 \). However, it is supposed that gravity wave flux will be larger for smaller Ro, since \( S_2 \) is proportional to \( \text{Ro}^{-1} \). Nevertheless, gravity wave flux decreases suddenly for \( \text{Ro} \leq 10 \).

Figure 11 shows the dependence of \( \frac{\text{Ro}}{H_0} (5, 10, 50, 100) \) for the spectral frequency of the gravity wave source (30) at \( \text{Fr} = 0.1 \). In a spectral analysis, we sample 50,000 points in the time series of \( t = 5.0 \) to \( t = 15.0 \). We apply a cosine window to both the initial and final 1/10 length of full data points to obtain a smooth periodical data series. In addition, we use three-point averaging to smooth the curves. The frequency \( f/2\pi \) denoted by the "cutoff" in Fig. 11 is the critical frequency in (29), above which gravity waves can propagate. While the peak amplitude of \( S_1 \) (broken line) in (30) is almost

Fig. 9. The Fr dependence of the zonally averaged gravity wave flux at \( y = 40 \) for \( \text{Ro} = 5, 10, 50, 100 \). Gravity wave flux is estimated by the same method as in Fig. 8. The thin line indicates the power law of Fr.

Fig. 10. The Ro dependence of the gravity wave source for (left) Fr = 0.1 and (right) Fr = 0.7. Maximum values of the zonally averaged source integrated in the jet region (\( S = \int_{-B_j}^{B_j} T_{22} \, dy \)) in the time interval \( 5.0 \leq t \leq 15.0 \) are shown. The broken line and solid line correspond to \( S_1, S_2 \) in (30), respectively.
constant at the same frequency $[\nu_f^{(1)} \sim 5]$ for different Ro, that of $S_2$ (solid line) in (30) is larger for smaller Ro. The dominant frequency of $S_2$ is also almost the same $[\nu_f^{(2)} \sim 2.5]$ for different Ro and smaller than that of $S_1 [\nu_f^{(2)} < \nu_f^{(1)}]$. Since the inertial cutoff frequency $f/2\pi$ increases at smaller Ro, gravity waves do not propagate at $2.5 \ll f/2\pi$, where $Ro \approx 6.36$, while the amplitude of $S_2$ in (30) increases for smaller Ro. Furthermore, at $Ro \approx 3.18$, the peak frequency of $S_1$ in (30) is lower than the inertial cutoff frequency, so that gravity wave radiation from an unsteady rotational flow is also suppressed. Thus, we conclude that the sudden decrease of gravity wave flux (Fig. 8, left) is caused by the inertial cutoff of gravity wave propagation. The local maximum of gravity wave flux around $Ro = 10$ is caused by two effects: the dominance of the source term related to the earth’s rotation and the low inertial cutoff frequency, which is sufficiently smaller than the peak frequency of the dominant source. Note that there is no local maximum for the case of $Fr = 0.7$ (Fig. 8, right). This may be related to the breakdown of the power law of Fr for $Ro = 10$. We investigate this next.

The breakdown of the power law of Fr for $Ro \leq 10$ is not explained by the parameter dependence of the gravity wave source. Figure 12 shows the $Fr$ (0.1 and 0.7) dependence of the spectral frequency of the gravity wave source for different Ro($=5, 10, 50, 100$). Here $Fr = 0.1$ (solid line) and $Fr = 0.7$ (broken line) are rescaled by $Fr^2$, since the depth of the fluid is proportional to $Fr^2$. For small Ro($=5, 10$), the amplitude of high-frequency components of an unsteady rotational flow decreases for larger Fr. This is because the vortex interaction is suppressed due to the small deformation radius for large Fr and small Ro. On the other hand, for large Ro($=50, 100$), there is no significant decrease for larger Fr, because of the large deformation radius for large Ro. For Ro $\leq 10$ the inertial cutoff frequency is so high that the decrease of the high-frequency motion crucially affects spontaneous gravity wave radiation. This is why the breakdown of the power law of Fr is observed. For Ro $\geq 30$, the inertial cutoff frequency is so low and the decrease of deformation radius is not so significant that gravity wave flux depends on the power law of Fr.
5. Summary and discussion

In the present study, we numerically investigate spontaneous gravity wave radiation, which is generated continuously, from an unsteady jet flow in an $f$-plane shallow water system in a wide parameter range. Although Ford (1994) investigated spontaneous gravity wave radiation from an initial unstable vorticity strip in a similar system and obtained the power law of Fr for the case of small Ro, his study treated only initial unsteady rotational flow using a nonforcing system. Instead of this numerical experimental setting, we keep an unsteady rotational flow that changes periodically with time by adding forcing to maintain the zonal jet flow. In addition, we investigate the Ro and Fr dependence of gravity wave flux.

For large Ro($\geq 30$), the effect of the earth’s rotation is so small that gravity wave flux depends on the power law of Fr, which is consistent with the theory of aeroacoustic sound wave radiation (the Lighthill theory). For smaller Ro, on the other hand, since the effect of the earth’s rotation is important, numerical results of gravity wave radiation differ from those of the Lighthill theory as follows:

Case 1. Ro $\leq 5$ for all Fr, and gravity wave flux decreases suddenly.
Case 2. Ro = 10 and Fr $\leq 0.5$, and gravity wave flux has a local maximum.
Case 3. Ro $< 30$, and gravity wave flux does not depend on the power law of Fr.

For all cases, while the frequency of unsteady jet flows is almost the same for different Ro, the inertial cutoff frequency, $f/2\pi$, of gravity waves increases with small Ro, since small Ro corresponds to high $f$. In addition, gravity wave sources related to unsteady jet flows are almost the same for different Ro. On the other hand, gravity wave sources related to $f$ increase for smaller Ro.

For case 1, since the inertial cutoff frequency exceeds the peak frequency of an unsteady jet flow, gravity wave flux decreases suddenly.

For case 2, while gravity wave sources related to $f$ are larger than those related to vortex motion, the inertial cutoff frequency does not affect gravity wave radiation. In this case, the effect of the earth’s rotation intensifies gravity wave radiation.

For case 3, both the effects of small deformation radius and high inertial cutoff frequency are important for spontaneous gravity wave radiation. Since the deformation radius is small for large Fr and small Ro, the vortex interaction is suppressed. In addition, for small Ro the inertial cutoff frequency is so high that the decrease in unsteady rotational flow leads to the breakdown of the power law of Fr. In contrast, for Ro $\geq 30$ the decrease in unsteady rotational flow is negligible.

![Fig. 12. The frequency spectrum of the zonally averaged source integrated in the jet region for (a) Ro = 100, (b) Ro = 50, (c) Ro = 10, and (d) Ro = 5. The solid line is for Fr = 0.1, while the broken line is for Fr = 0.7. The spectra are rescaled for each Fr, considering the dependence on Fr of the gravity wave source ($S' = S \times Fr^2$).](image-url)
because of the large deformation radius and low inertial cutoff frequency.

For $Ro = 1$, unsteady rotational flow is no longer observed for $Fr \geq 0.5$ because the deformation radius is too small. The decrease in unsteady rotational flow for large $Fr$ also directly leads to the decrease of $\omega$ in (29).

Finally, we can conclude that these results have some implications for the real atmosphere. First, gravity waves could be radiated from rotational flows in the real atmosphere, even in the parameter range in which rotational balanced flows are thought to be dominant in the nonlinear phase of instability (Sugimoto et al. 2007c). Second, in phenomena having large Ro, such as tornados and hurricanes, it may be that the power law of Fr for gravity wave amplitude can be observed. In contrast, in phenomena having small Ro, such as large-scale cyclones, there is a breakdown of the power law of Fr for spontaneous gravity wave radiation. Third, in some special situations, gravity wave amplitude could be intensified by the effect of the earth’s rotation. Since an $f$-plane shallow water system is introduced on the assumption of strong stratification, this system could be applied to the phenomena in the middle atmosphere. The results could also be useful in understanding spontaneous gravity waves radiation from polar night jets in the middle atmosphere. However, an $f$-plane shallow water system has external gravity waves only, which propagate in the horizontal direction. A two-layer model having internal gravity waves or a three-dimensional model, in which gravity waves propagate in the vertical direction, will be one of the next steps in studying the phenomena of polar night jets. In the present study, the parameter regimes explored are only for Ro greater than one. The more relevant parameter regime for synoptic-scale flows would be that where the Rossby number is small (Saujani and Shepherd 2002).

In the future, more studies from various viewpoints will be needed for a comprehensive understanding of gravity wave radiation from unsteady rotational flows.

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