Dissipation of Synoptic-Scale Flow by Small-Scale Turbulence

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ABSTRACT

Although it is now accepted that imbalance in the atmosphere and ocean is generic, the feedback of the unbalanced motion on the balanced flow has not received much attention. In this work the parameterization problem is examined in the context of rotating stratified turbulence, that is, with a nonhydrostatic Boussinesq model. Using the normal modes as a first approximation to the balanced and unbalanced flow, the growth of ageostrophic perturbations to the quasigeostrophic flow and the associated feedback are studied. For weak stratification, there are analogies with the three-dimensionalization of decaying 2D turbulence: the growth rate of the ageostrophic perturbation follows a linear estimate, geostrophic energy is extracted from the base flow, and the associated damping on the geostrophic base flow (the “eddy viscosity”) is peaked at large horizontal scales. For strong stratification, the transfer spectra and eddy viscosities maintain this structure if there is synoptic-scale motion and the buoyancy scale is adequately resolved. This has been confirmed for global Rossby and Froude numbers of $O(0.1)$. Implications for atmospheric and oceanic modeling are discussed.

1. Introduction

In computational fluid dynamics, models with limited spatial resolution are employed. For this reason, the parameterization of unresolved motion has attracted much attention. In the popular large-eddy simulation (LES) approach (e.g., Fröhlich and Rodi 2002; Mason 1994), one obtains an accurate simulation with a coarse grid by modeling interactions between resolved and unresolved modes. As an alternative to subgrid-scale modeling, numerical data from high-resolution simulations can be analyzed; the resulting eddy viscosity, which represents the effects of unresolved interactions, is then appended to the kinematic viscosity (Domaszela et al. 1987, 1993). LES is a standard technique in the modeling of industrial flows. It also lies behind turbulence schemes used in atmospheric and oceanic models (e.g., the well-known Smagorinsky model).

Nevertheless, the problem is more intricate in geophysical fluid dynamics. Here we desire information beyond the total contribution of unresolved, small-scale processes, namely, the separate contributions from different modes. For practical and theoretical reasons, it may be desirable to partition the “feedback” into distinct contributions from, say, gravity waves and vortical modes, or more generally into interactions between balanced (slow) and unbalanced (fast) modes.

The feedback of the unbalanced modes onto the balanced ones is a quantity of great interest. By definition this quantity is zero in a balance model. Even so, this feedback is crucial in real geophysical flows. It has been suggested that ageostrophic circulations play a role in, for example, frontogenesis and troposphere folding (cf. O’Sullivan and Dunkerton 1995).

There have been several studies of this problem. For example, Dewar and Killworth (1995) showed analytically that the effect of inertia–gravity waves vanishes to leading order in the Rossby number and confirmed this prediction numerically with reduced-gravity layer models. For rotating shallow water, minimal energy transfer between the quasigeostrophic flow and the inertia–gravity waves was observed in the numerical simulations of Farge and Sadourny (1989). In the study whose objectives are closest to our own, Errico (1982) argued, using a low-order version of a two-layer, primitive equation model, that the energy cascade to small scales and the rate of energy dissipation may be affected by weak ageostrophic processes. Less is known about bal-
anced–unbalanced interactions for larger Rossby and Froude numbers. If we accept that imbalance in the atmosphere and ocean is generic and nonnegligible, quantifying interactions between (nominally) balanced and unbalanced modes at realistic Rossby and Froude numbers should be a useful line of research.

In a previous publication of ours this problem was addressed in the context of the three-dimensionalization of decaying 2D turbulence (Ngan et al. 2005, hereafter P2). We showed that the 3D modes (the perturbation) extract energy from the 2D modes (the base flow), which is then cascaded from large scales to small scales. The spectral eddy viscosity, which represents the effects of 3D modes on 2D modes, (i) peaks strongly at low wavenumbers, (ii) approaches zero above a crossover wavenumber, and (iii) decreases in magnitude as the aspect ratio of the domain is decreased. Although these results were obtained for unstratified, nonrotating flow, they may hold more generally. On synoptic scales the 2D modes may be analogous to the geostrophic motion and 3D modes to the ageostrophic motion. Thus the effects of gravity waves could be straightforward to parameterize: the accuracy of the parameterization would not hinge on the modeling of small-scale dynamics, about which uncertainty is greatest.

Using a numerical model of the nonhydrostatic Boussinesq equations, we examine the growth of ageostrophic perturbations to a quasigeostrophic base flow and quantify the interactions between base flow and perturbation for global Rossby numbers of $O(0.1)$. The quasigeostrophic base flow may be viewed as a proxy for balanced motion, and the ageostrophic motion as a proxy for unbalanced motion. Obviously, more sophisticated definitions of balance exist, but this is a reasonable starting point for synoptic-scale flow: if the Rossby and Froude numbers are small, then only a small fraction of the motion will be associated with a higher-order balance. Following P2, transfer spectra and eddy viscosities are calculated, and the spectral structure of balanced–unbalanced interactions is considered in detail.

Geostrophic and ageostrophic modes are defined using normal modes of the nonhydrostatic Boussinesq equations. This facilitates connections with rotating stratified turbulence phenomenology. Scale analysis of the nonhydrostatic Boussinesq equations (Lilly 1983) implies that geostrophic modes decouple from ageostrophic modes as $Ro, Fr \to 0$, in agreement with quasigeostrophic theory. It has been shown that the energy transfers for $Ro, Fr \ll 1$ can be understood by analyzing the dynamics of the triads (Bartello 1995), geostrophic–geostrophic–geostrophic (GGG), geostrophic–ageostrophic–ageostrophic (GAG), and ageostrophic–ageostrophic–ageostrophic (AAG). GGG triads yield quasiageostrophic dynamics; AAG triads are responsible for nonlinear geostrophic adjustment in the limit $Ro, Fr \to 0$, that is, the downscale cascade of ageostrophic energy, the geostrophic mode acting as a catalyst on account of potential vorticity conservation; GAG triads, while nonresonant, effect geostrophic–ageostrophic energy transfer for finite $Ro$ and $Fr$.

Many studies of balance have employed normal modes (cf. Errico 1981; Leith 1980). This suggests connections with recent work on “spontaneous imbalance.” Following the pioneering work of Warn (1986) and Warn and Menard (1986), it has become clear that unbalanced motion may be generated through the action of the slow, balanced flow: balance is intrinsically “fuzzy.” In principle, spontaneous imbalance may occur even for unbalanced motion of arbitrarily small amplitude (Ford et al. 2000; Vanneste and Yavneh 2004), by contrast with the classic Rossby adjustment (or “dam break”) problem (e.g., Gill 1982), in which the adjustment occurs on the fast time scale and depends crucially on the (unbalanced) initial conditions. Indeed the phenomenon of spontaneous imbalance is essentially independent of the definition of balance that one chooses to adopt [Vanneste and Yavneh (2004) consider a special flow for which balance may be defined exactly].

We examine the extent to which geostrophic–ageostrophic interactions can be attributed to a specific mechanism of spontaneous imbalance. The three-dimensionalization of decaying 2D turbulence occurs via random straining of small-aspect-ratio 3D perturbations by the 2D base flow (specifically through a time-dependent, hyperbolic instability; see (Ngan et al. 2004, hereafter P1, and appendix B for details); if an analogous mechanism applied to rotating stratified turbulence, this would represent a route by which energy could be extracted from the balanced modes and transferred to unbalanced modes, whereupon a forward cascade to dissipation scales may occur (cf. Straub 2003). In other words, the unbalanced motion may induce an effective dissipation on the balanced flow. Since large-scale horizontal random straining is ubiquitous in the upper troposphere and lower stratosphere (cf. Shepherd et al. 2000), this mechanism may be of wide applicability. The generation of an essentially 3D response to a quasi-2D flow is reminiscent of the vertical tracer cascade in stratospheric dynamics (Haynes and Anglade 1997).

We argue that the behavior is analogous (in some important respects) to that of unstratified turbulence if the stratification is (locally) weak and the global Rossby number comparable to or smaller than the
global Froude number; in particular, the growth of the ageostrophic energy follows a linear estimate and the damping of the (quasi-) geostrophic base flow by ageostrophic motion occurs preferentially at large horizontal scales. Even for global Froude numbers as small as \(O(0.1)\), the geostrophic–ageostrophic transfer is maximized at large horizontal scales.

Although the dynamics of a triply periodic Boussinesq model may be broadly representative of those of the real atmosphere and ocean, our results cannot be interpreted literally. Phase information and temporal dependence are discarded in the calculation of the eddy viscosity and transfer spectra. Nevertheless, the eddy viscosity may still delineate constraints that a successful parameterization should respect. We emphasize, however, that our eddy viscosity is not the standard subgrid eddy viscosity, for it represents the effects of imbalance: the dissipation described in this paper arises from neither molecular nor eddy processes. “Wave drag” may be a more appropriate term, but as the risk of confusion would be even greater, “eddy viscosity” is retained.

The organization of the paper follows P2. After reviewing the numerical formulation and procedure (section 2), numerical results for weak stratification (section 3) and strong stratification (section 4) are considered in turn, transfer spectra and eddy viscosities being highlighted. We argue that the results are analogous to those for unstratified turbulence if the perturbation grows on vertical scales smaller than the buoyancy (or overturning) scale and the base flow has horizontal length scale greater than the deformation radius; we underscore the role of the buoyancy scale in determining the resolution dependence of the results. Implications for atmospheric and oceanic modeling are discussed in section 5. Background material on the definition of the basic state (appendix A) and the time-dependent, hyperbolic instability (appendix B) is deferred to the appendixes.

2. Numerical formulation and procedure

a. Model

We consider the nonhydrostatic Boussinesq equations for rotating stratified flow:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + 2 \Omega \times \mathbf{v} = -\frac{1}{\rho_0} \nabla p + b \mathbf{z} + \mathcal{D}(\mathbf{v})
\]

\[
\frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b = -N^2 w + \mathcal{D}(b)
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

where \(b := g \theta'/\theta_0\) is the (perturbation) buoyancy, \(\theta = \theta_0((d\theta/dz)z + \theta')\) is the potential temperature, \(\Omega = \Omega \mathbf{z}\) is the angular velocity of the rotating frame, \(N = [(g/\theta_0)(d\theta/dz)]^{1/2}\) is the constant Brunt–Väisälä frequency, \(\mathbf{v}\) is the 3D velocity with vertical component \(w\), and \(\mathcal{D}\) denotes the diffusion operator. Our numerical model is triply periodic, cylindrically truncated, pseudospectral, and dealiased with the two-thirds rule; a weak Robert filter (Asselin 1972) is applied to suppress the computational mode. The grid is isotropic with \(N_h^2 \times N_e\) collocation points and dimensions \(L_0 = 2\pi\) and \(H_0 = 2\pi N_v/N_h\). The domain may be either isotropic or anisotropic. For an isotropic domain, the aspect ratio, \(e_a := N_v/N_h\), is unity; for an anisotropic, thin domain, \(e_a < 1\).

We use \(\nabla^8\) hyperviscosity for \(\mathcal{D}\). The hyperviscosity coefficient is chosen so that the dissipation at the truncation scale stays constant (\(\nu = \nu_0/N_h^8\), where \(\nu_0 = 2.62 \times 10^4\)). The time step is chosen so that buoyancy oscillations are explicitly resolved for all values of \(N\): the time step is at least 200 times smaller than the buoyancy time scale. Here \(N_h\) is chosen so that the buoyancy scale (see section 2d) is resolved.

b. Procedure

The procedure is a variant of that employed in P1 and P2. For each \((N, f)\) pair, a basic state is generated from random initial conditions by setting the amplitude of the wave modes to zero (cf. Errico 1984). See appendix A for details. At \(t = 0\), the geostrophic basic state is perturbed with a random, small-amplitude perturbation. The total energy of the basic state is identical for all runs, irrespective of \(N\) and \(f\). The kinetic energy of the perturbation is a specified fraction of the basic state’s. Figure 1 summarizes the procedure.

The base-flow energy is chosen so that the (initial) eddy turnover time, defined by the rms vertical vorticity, is \(O(1)\). Likewise, the (initial) velocity scale, defined by the rms horizontal velocity, satisfies \(U \sim 0.6\).\(^1\) For the runs described herein, \(f = 1\).

The Rossby and (vertical) Froude numbers are defined as \(\text{Ro} := U/fL\) and \(\text{Fr} := U/NH\), respectively. Some freedom exists in the choice of \(H\) and \(L\): a small-scale, “micro” (or local) definition may be adopted based on the vorticity; alternatively, the large-scale, “macro” (or global) definitions, \(L \sim L_0, H \sim H_p\), may be used. Neither is appropriate in all cases: whereas the former is suited to the growth of small-scale perturbations, the latter best describes the effect of small-scale

\(^1\) For convenience we use the rms total velocity rather than the rms horizontal velocity. The latter depends weakly on Fr.
perturbations on the large-scale base flow (the geostrophic energy spectra are fairly steep; initially $k^{-4}$ or steeper). Given the concerns of this paper, and its widespread use in atmospheric science, the macrodefinitions

$$\text{Ro} := \frac{U}{fL_0}, \quad \text{Fr} := \frac{U}{NH_0}, \quad (2)$$

seem natural. This yields a Rossby number of $O(0.1)$, which is representative of synoptic-scale, tropospheric flow, though it may be an overestimate.

Small errors are incurred in approximating the Rossby and Froude numbers with nominal, time-independent values based on $H_0$ and $L_0$. In our simulations, the Rossby and Froude numbers typically vary over an order of magnitude. It is also possible to estimate $H$ and $L$ with the first moments, $H_1$ and $L_1$, of the geostrophic energy spectrum. In an isotropic domain, $L_1$ agrees with $L_0$ to within a factor of 2.

c. Normal modes

The time-dependent base flow and perturbation, that is, the nominal balanced and unbalanced modes, are defined using the linear normal modes of the nonhydrostatic Boussinesq equations (Bartello 1995):

$$A_k^{(0)} = \frac{Nk h \hat{\xi}_k + ifk\tau_k}{\sigma_k k},$$

$$A_k^{(a)} = \frac{\sigma_k k D_k + ik\tau_k \hat{\xi}_k + Nk \tau_k}{\sqrt{2} \sigma_k k}, \quad (3)$$

where the scaled variables $\hat{\xi}_k = i(k_x u_k - k_y v_k)$, $D_k = k/k_z(i k_x u_k + i k_y v_k)$, and $\tau_k = k/h_0/N$. The modal frequency $\sigma_k = \sqrt{f^2 k_z^2 + N^2 k_z^2}/k$, and it is assumed that $k_x$, $k_z \neq 0$. Here $A_k^{(0)}$ denotes vortical modes, while $A_k^{(a)}$ denotes gravity waves. We shall refer to the former as “geostrophic” and the latter as “ageostrophic”; in the initial spinup $A_k^{(z)} = 0$ (see appendix A).

The ageostrophic and geostrophic energies are given by

$$E_{\text{ageo}} := \sum_k \frac{1}{k^2} [A_k^{(z)} A_k^{(z)*} + A_k^{(a)} A_k^{(a)*}].$$

$$E_{\text{geo}} := \sum_k \frac{1}{k^2} A_k^{(0)} A_k^{(0)*}, \quad (4)$$

where the asterisk denotes complex conjugation.

Some of the calculations require geostrophic or ageostrophic velocities. They are obtained by projection. For example, in the former case,

$$(\xi_k, D_k)_G = \left[ N k_h A_k^{(0)}, 0 \right], \quad (5)$$

from which the geostrophic velocities ($u_g$, $v_g$) follow; $w_g = 0$.

Note that while the normal modes are exact solutions only for linear dynamics, that is, for $\text{Ro}$, $\text{Fr} \to 0$, the $[A_k^{(0)}, A_k^{(z)}, A_k^{(a)}]$ basis can be used generally, for it is complete. The normal-mode basis has been useful in the study of unbalanced turbulence, where $\text{Ro}$ and $\text{Fr}$ might not be small (e.g., Bartello et al. 1996; Bartello 1995).

At finite $\text{Ro}$ and $\text{Fr}$, the geostrophic solution$^3$ does not remain an exact solution of the Boussinesq equations. Departures from exact quasigeostrophy enable ageostrophic perturbations to grow. For the Rossby and Froude numbers described herein, the evolution is insensitive to the structure of the (external) perturbation; runs with small temperature perturbations are nearly indistinguishable from runs with small vorticity perturbations, or even with no perturbation at all.

Although the normal-mode basis is formally valid for all $\text{Ro}$ and $\text{Fr}$, the use of linear normal modes is debatable for larger values. In this case geostrophic–ageostrophic interactions might not be representative of balanced–unbalanced interactions. This is addressed in section 3.

$^2$ See Bartello (1995) for details on the treatment of the special cases $k_x$, $k_z = 0$.

$^3$ Hereafter, to conform with conventional usage, we drop the qualifier.
Using the normal-mode basis, the velocity field is decomposed into a geostrophic base flow, \( \mathbf{v}_g \), and an ageostrophic perturbation, \( \mathbf{v}_a \). The full velocity \( \mathbf{v} := \mathbf{v}_g + \mathbf{v}_a \). Diagnostics will be defined as needed.

d. Key length scales

In analyzing the results, reference will be made to two important length scales: the buoyancy scale and the deformation radius.

At the buoyancy scale, \( H_b \), the vertical advective time scale matches the buoyancy time scale, \( N^{-1} \). With the definition

\[
H_b := \frac{U}{N},
\]

Fr > 1 for \( H < H_b \); in other words, the stratification is relatively weak. Recently it has been shown that \( H_b \) does indeed define an overturning scale for stably stratified turbulence (Waite and Bartello 2006). If

\[
H_0 < H_b,
\]

we say there is weak stratification.

Other definitions exist. For example, it is possible to use the characteristic vertical velocity, \( W \) (Hopfinger 1987), though the previous definition makes sense when characterizing the transition from quasi-horizontal motion to motion with strong buoyancy fluctuations. In phenomenological theories of stratified turbulence (e.g., Lumley 1964), much attention has been devoted to the so-called Osmidov scale, \( H_{\text{os}} := (\varepsilon/N^3)^{1/2} \), where \( \varepsilon \) is the turbulent dissipation rate.

The deformation radius, \( L_d \), characterizes the importance of rotation. It may be defined as

\[
L_d := \frac{N H_0}{f},
\]

where \( H_0 \) is used for the vertical length scale (Gill 1982). For horizontal length scales that are large compared to \( L_d \), that is, \( L_d/L < 1 \), rotation dominates, and, according to the Taylor–Proudman theorem, there is quasi-2D motion for \( \text{Ro} \ll 1 \). The flow is nearly barotropic, but baroclinic effects are retained. If

\[
\frac{L_d}{L_0} < 1,
\]

we say there is synoptic flow. This condition is equivalent to \( \varepsilon_d N f < 1 \) or \( \text{Ro}/\text{Fr} < 1 \). In the “synoptic limit” \( L_d/L_0 \to 0 \).

The structure of the base flow changes with \( L_d/L_0 \). As \( L_d/L_0 \) is increased (at fixed, small Ro) there is a transition from “tubes” to “pancakes,” in accord with rotating stratified turbulence phenomenology (e.g., Riley and Lelong 2000). This can be seen in Fig. 2, which plots vertical vorticity isosurfaces at \( t \sim 0 \). For synoptic flow and \( L_d/L_0 < 1 \) (Figs. 2a,b), there are coherent vortex tubes; for subsynoptic flow and \( L_d/L_0 > 1 \) (Fig. 2d), there are thin “pancakes.” For \( L_d/L_0 = 1 \) (Fig. 2c), vortex tubes are present but less coherent than before.

Intuitively, the numerical results are expected to be analogous to those in the unstratified case (cf. P2) if there is weak stratification and quasi-2D motion. Connections with three-dimensionalization and the time-dependent, hyperbolic instability are discussed in appendix B. In sections 3 and 4 we investigate the effects of varying the buoyancy scale and the deformation radius: we progress from a regime in which both (7) and (9) are satisfied, through to one in which neither holds.

3. Weak stratification, synoptic flow (\( N f/\text{Fr} < 1 \))

In this section we examine weak stratification, \( H_0 \sim H_b \), and synoptic flow, \( L_d/L_0 < 1 \). The base flow is dominated by “tubes” (cf. Figs. 2a,b). Table 1 lists relevant parameters and length scales for the different runs; resolution checks have been excluded for brevity. Although \( N f/\text{Fr} \sim 100 \) in the atmosphere and ocean, \( N f/\text{Fr} < 1 \) in this section. This allows the qualitative arguments of section 2d to be examined in a regime where the unstratified results should extend. Weak stratification might, however, be relevant to the atmospheric boundary layer or the oceanic mixed layer.

a. Ageostrophic energy

We begin by analyzing the growth of \( E_{\text{ageo}} \) (4). Figure 3 plots \( E_{\text{ageo}} \) against time for \( fr \approx 1 \). In all cases roughly exponential growth is established before \( E_{\text{ageo}} \) saturates and decays for \( t \approx 10 \). This is analogous to P1, in which there is exponential growth of the 3D perturbation.

Although we are more interested in the long-time interactions between geostrophic and ageostrophic modes than in the short-time growth of the ageostrophic energy, it is instructive to pursue the analogy. In P1, the growth rate of the 3D energy, \( E_{3D} \), was compared to the linear growth estimates, \( \Gamma_{\text{lin}} \) and \( \Gamma_0 \) (see appendix B for definitions). More specifically, the actual growth rate, \( \bar{\gamma} \), agrees well with the domain average of \( \Gamma_0 \), where the overbar denotes a time average.

\[\text{In fact, we plot the geostrophic projection of the total vorticity. However, the difference is immaterial at } t \sim 0, \text{ because the ageostrophic component starts small. At later times the picture is more complicated, but broadly similar.}\]
and the angle brackets denote a domain average. (The agreement is not exact, but given the averaging and the turbulent nature of the flow, agreement within 50% is surprisingly good; see P1, section IV.B for discussion.)

In the present case, $E_{ageo}$ would grow at a rate $2\overline{(V_0)}$ if the linear growth rate estimates applied. In Fig. 3b, $\exp(2\overline{(V_0)}t)$ is plotted for the three largest stratifications; for clarity Fr = 1000 is omitted and each curve is shifted so as to be adjacent to the corresponding $E_{ageo}$ curve. There is very good agreement up to Fr = 1: $2\overline{(V_0)} \sim 1.2$ and $\gamma \sim 1.4$ for Fr = 100. Here $2\overline{(V_0)}$ underpredicts the actual growth for Fr = 1, the strongest stratification (though $\gamma$ remains bracketed by $2\overline{(V_0)}$ and $2\overline{(T)}$, as in the unstratified case). Evidently the perturbation growth may differ for stronger stratifications; this is discussed in section 4a(1). Recall that Fr = 1 is a borderline case, lying between weak and strong stratification, and that $dH_b/dt \leq 0$ for decaying flow, meaning that not all modes remain within the “weakly stratified regime.”

For $t \in [1, 5]$, in which there is exponential growth of the perturbation, $\gamma$ is approximately independent of Fr.

Fig. 2. Vertical vorticity isosurfaces for the geostrophic modes at $t = 0$. These fields correspond to the initial conditions of the runs analyzed in Figs. 3 and 15; Ro = 0.1. The isosurface values are defined by the rms vertical vorticity. For clarity, vertical slices have been extracted from the isosurfaces. The panels correspond to different values of the deformation radius, $L_d$: (a) $L_d/L_0 = 0.01$, (b) $L_d/L_0 = 0.1$, (c) $L_d/L_0 = 1$, (d) $L_d/L_0 = 10$. 

MARCH 2008 NGAN ET AL. 771
Instability mechanisms that are not strain dominated may show a stronger dependence on the stratification. For example, the so-called ageostrophic, anticyclonic instability shuts off for weak stratification (McWilliams et al. 2004; Molemaker et al. 2005).

On very short time scales, $E_{\text{ageo}}$ grows much faster than $w_{\text{rms}}^2$ for $Fr/L \gg 1$ and $Fr/L \ll 1$. Note the change in slope at $t \sim 1$. This rapid adjustment is likely related to the fact that $w$ for geostrophic flow [cf. (5)]. Figure 4 compares time series of $E_{\text{ageo}}$ and $w_{\text{rms}}^2$, with $w_{\text{rms}}$ being the rms vertical velocity. Like $E_{\text{ageo}}$, $w_{\text{rms}}^2$ undergoes a rapid adjustment for $t \lesssim 1$. A similar adjustment may also be seen in Bartello (1995), where the adjustment arises on account of initial conditions with $b = 0$ rather than $w = 0$. Since the adjustment occurs on the fast, inertial time scale, $T_i \sim 1$, rather than the slow, advective time scale, $T_{\text{adv}} \sim 10$, it cannot be attributed to spontaneous imbalance and the time-dependent, hyperbolic instability.

The initial adjustment can be mitigated with a more judicious choice of basic state, for example, a higher-order balance with $w \neq 0$. This is demonstrated explicitly in section 4a(1).

The ageostrophic energy $E_{\text{ageo}}$ is a coarse-grained diagnostic. To characterize the spectral signature of the growth another diagnostic is needed. We compute the spectral growth rate, $\sigma_{E_{\text{ageo}}}$, Following P1,

$$\sigma_{E_{\text{ageo}}}(k_h, k_z; t, t_0) = \frac{1}{t - t_0} \log \left[ \frac{E_{\text{ageo}}(k_h, k_z; t)}{E_{\text{ageo}}(k_h, k_z; t_0)} \right],$$  \hspace{1cm} (10)

where $t_0$ is a reference time. The results are sensitive to $t$ and $t_0$. We choose $t_0 = 0.8$, which roughly corresponds to the onset of exponential scaling for $Fr = 1$ and $Fr = 10$. Here $t$ is chosen to lie inside the exponential regime.

Figure 5 shows $\sigma_{E_{\text{ageo}}}$, snapshots taken after the onset of exponential growth but before the saturation of the ageostrophic perturbation. The key point is that $\sigma_{E_{\text{ageo}}}$ is maximized near $k_h = 1$ and decreases below the line $k_h = k_z$, just as in the three-dimensionalization of decaying 2D turbulence (cf. P1, Fig. 11): the growth is preferentially determined by small-aspect-ratio modes with small $k_h$ and large $|k_z|$. This may be clearly seen for $Fr = 10$ (Fig. 5a). Maximization of the growth rate near $k_h = 0$ has also been observed in stratified mixing layers (Caulfield and Peltier 2000). At later times the structure becomes diffuse, but growth for $k_h > k_0$ continues to be favored. For $Fr = 1$ (Fig. 5b) the largest growth rates remain localized around $k_h = 1$.

### Table 1. Parameters for weak stratification (section 3).

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<th>$Ro$</th>
<th>$Fr$</th>
<th>$N$</th>
<th>$f$</th>
<th>$N_p$</th>
<th>$N_s$</th>
<th>$L_d/L_0$</th>
<th>$H_b$</th>
<th>$\Delta z$</th>
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![Fig. 3. Ageostrophic energy $E_{\text{ageo}}$ vs $t$ for weak stratification ($Ro = 0.1$, $Fr \geq 1$). The 2($T_0$) curves are obtained by averaging data over the time interval $t \in [1, 5]$; for clarity only the $Fr = 1$, 10, and 100 curves are shown: (a) $t \leq 60$, (b) $t \leq 8$; $L_d/L_0 < 1$ in all cases. Note the exponential scaling prior to saturation.](image-url)
In both panels there are large growth rates in the vicinity of the vertical truncation scale. This arises from the initial decay of the kinetic energy at large $k_z$. Growth is initiated near $k_h = 1$, where there is weak horizontal dissipation, before spreading to larger $k_n$ (not shown). See section 4b(1).

b. Transfer spectra

1) Definitions

We now consider geostrophic–ageostrophic energy transfers. The transfer functions are given by

$$ T_G(k_h) := \Re \sum_{|k| = k_h} \mathcal{N}_G \mathcal{E}_G(k_h) $$

$$ T_A(k_h) := \Re \sum_{|k| = k_h} \mathcal{N}_A \mathcal{E}_A(k_h) $$

where the former is associated with the advection of velocity and the latter with advection of buoyancy. Here $\Re$ denotes the real part. Note that the sum is taken over a cylinder in spectral space (i.e., there is an implicit sum over $k_z$).

It is convenient to reexpress these transfer functions in terms of the normal-mode variables. This is done by projecting the state variables $(\alpha, b)$ and the nonlinear terms $(\mathcal{N}_G, \mathcal{N}_A)$ onto the normal-mode variables. Since the projection is linear and the basis orthonormal and complete, the total transfer is unchanged. Defining $T(k_h) := T_G(k_h) + T_A(k_h)$,

$$ T(k_h) = \Re \sum_{|k| = k_h} \left[ A_0^0 \mathcal{N}_A^{(0)} + A_k^{(+)} \mathcal{N}_A^{(+)} + A_k^{(-)} \mathcal{N}_A^{(-)} \right], $$

where $[\mathcal{N}_A^{(0)}, \mathcal{N}_A^{(+)}, \mathcal{N}_A^{(-)}]$ denotes the projected nonlinear terms. The geostrophic and ageostrophic contributions to the transfer are given by

$$ T_G(k_h) = \Re \sum_{|k| = k_h} A_0^0 \mathcal{N}_A^{(0)}, $$

$$ T_A(k_h) = \Re \sum_{|k| = k_h} [A_k^{(+)} \mathcal{N}_A^{(+)} + A_k^{(-)} \mathcal{N}_A^{(-)}]. $$

Fig. 5. Illustration of the effect of the initial vertical velocity. Comparison between time series of the ageostrophic energy, $E_{ageo}$ and square of the rms vertical velocity, $w_{rms}^2$. Here, $Fr = 10$.

Fig. 4. Illustration of the effect of the initial vertical velocity.

In both panels there are large growth rates in the vicinity of the vertical truncation scale. This arises from the initial decay of the kinetic energy at large $k_z$. Growth is initiated near $k_h = 1$, where there is weak horizontal dissipation, before spreading to larger $k_n$ (not shown). See section 4b(1).

MARCH 2008 NGAN ET AL. 773
The nonlinear terms, which represent convolutions, may be further decomposed. Clearly,

$$N_{Ak} / H^2 = N_{AG} / H^2 + N_{AA} / H^2,$$

where $N_{GG}$ is the purely geostrophic convolution, $N_{AA}$ the purely ageostrophic convolution, and $N_{AG}$ the mixed nonlinear term. The hat denotes a projection onto geostrophic modes, and $N_{GG}$ and $N_{AA}$ can be obtained by filtering $v$ and $b$. Substituting

$$T_{A_0} / H^2 = T_{G} / H^2 + T_{AA} / H^2 + T_{AG} / H^2,$$

and

$$T_{GG-k} / H^2 = \Re \sum_{|k|=k_h} A_{k}^{(0)} \hat{N}_{GG}^*,$$

$$T_{AA-k} / H^2 = \Re \sum_{|k|=k_h} A_{k}^{(0)} \hat{N}_{AA}^*$$

with the cross term

$$T_{GA-k} / H^2 = T_{G} - T_{GG-k} - T_{AA-k}$$

(12)

Here $T_{GG-k}$ is associated with the GGG triads, $T_{AA-k}$ with the AAG triads, and $T_{GA-k}$ with the GAG triads; $T_{AA-k}$ and $T_{GA-k}$ cannot be represented with quasigeostrophic processes. There are analogous expressions for $T_{GG-k}$, $T_{GA-k}$, and $T_{AA-k}$.

By contrast with section 3a, we focus on the long-time dynamics, which should be independent of the initial conditions. The results are temporally averaged over the interval $20 \leq t \leq 60$. Similar results have been obtained with other time intervals (e.g., $[10, 20]$ or $[50, 60]$).

2) SPECTRAL STRUCTURE

Figure 6a shows $T_{GA-k}$, $T_{AA-k}$, and $T_{GG-k}$ for $Fr = 100$. Since log-linear axes are used, the $k_h = 0$ transfer cannot be shown; however, it is at least several orders of magnitude smaller than the $k_h = 1$ transfer. To maintain area preservation the transfer functions are scaled by $k_f$.

The expression $T_{GG-k}$, which describes quasigeostrophic dynamics, is positive at $k_h = 1$ and approximately zero for large $k_h$; this is consistent with an inverse energy cascade. Detailed balance is satisfied: $\Sigma_k T_{GG-k} \approx 0$. At early times, before the ageostrophic perturbation has experienced much growth, $T_{G} \approx T_{GG-k}$ (not shown). At the late times corresponding to Fig. 6, the nonlinear transfer by the GAG and AAG triads is larger at small $k_h$.

In resonant interaction theory, the GAG triads are solely responsible for geostrophic–ageostrophic transfers (Bartello 1995). For $Fr \rightarrow 0$ geostrophic modes catalyze transfer between ageostrophic modes in AAG triads. But, in the present case, $Fr$ is large and the AAG triads can make an important contribution to the geostrophic–ageostrophic transfer. Thus, the net transfer is given by

$$T_{net} := T_{GA-k} + T_{AA-k}$$

(13)

5 The sum is many orders of magnitude smaller than the peak, but detailed balance is not satisfied exactly on account of roundoff errors.
In the unstratified case, by contrast, transfer between the 2D and 3D modes can be effected only by 3D–3D–2D triads, namely by

\[ T_{3D3D} := -\Re \sum_{k_h} U_h (\hat{u} \cdot \nabla \hat{u}) (k_h), \]

where \( U \) is the 2D component, \( u \) the 3D component, and the hat denotes the 2D projection.\(^6\) Assuming \( v_g \sim U \) and \( v_a \sim u \), the structure of \( T_{AA-G} \) should then resemble that of \( T_{3D3D-2D} \).

Figure 6a confirms this for weak stratification: \( T_{AA-G} \) has a strong negative peak at small \( k_h \). Just as 3D perturbations represent a strong damping on 2D decaying turbulence, ageostrophic perturbations represent a damping on geostrophic turbulence: there is extraction of base-flow energy at small \( k_h \). Note that this effective damping should not be confused with explicit dissipation.

As expected, \( T_{GA-G} \) is nonzero: the transfer between the base flow and the perturbation cannot be ascribed solely to AAG triads. Nevertheless, \( T_{GA-G} \) is smaller in magnitude than \( T_{AA-G} \). This is reassuring inasmuch as, strictly speaking, \( Fr \ll 1 \) in order for the normal-mode basis to be applicable.

The analogy between geostrophic motion and 2D flow, on the one hand, and ageostrophic motion and 3D flow, on the other, is reinforced by Fig. 6b, which plots \( T_{3D3D-2D} \) and \( T_{net} \). There is good agreement between them for \( Fr = 100 \). The deviations between the curves are larger, at a given value of \( k_h \), for smaller \( Fr \) (not shown). Note that \( \sum_{k_h} T_{net} < 0 \).

Extraction of energy continues after the saturation of the ageostrophic perturbation. Even though the perturbation energy decays for \( t \approx 10 \) (cf. Fig. 3), energy transfers between geostrophic and ageostrophic modes persist. This is consistent with the 2D–3D transfers described in P2. The maximum amplitude of \( T_{net} \) exceeds that of the quasigeostrophic GGG transfer.

Time averaging obscures fluctuations on a given time slice. For example, \( T_{net} \) can be positive at small \( k_h \).

While such contributions are important in subgrid-scale modeling (Mason 1994; Fröhlich and Rodi 2002), they lie beyond the scope of this study.

3) FR DEPENDENCE

Figure 7a shows \( T_{net} \) for the runs analyzed in section 3a. The structure remains more or less unchanged, though the magnitude of the transfer (i.e., extraction) is noticeably smaller for \( Fr = 1 \). Recall that \( Fr = 1 \) lies at the boundary between weak and strong stratification. Here \( T_{net} \) resembles \( T_G \), which we omit for brevity.

Figure 7b, which shows \( T_A \), indicates that energy extracted from the geostrophic flow by the ageostrophic perturbation cascades toward small scales. This is analogous to the forward cascade of 3D energy in P2. The magnitude of the ageostrophic transfer increases for weaker stratification. The absence of a peak at low \( k_h \) implies that the injection of geostrophic energy is balanced by the loss of energy toward small scales.

The extraction at small \( k_h \) is not an artifact of the initial conditions. Although the basic state is dominated

\(^6\) In P2 this transfer was labeled as \( T_{3D-2D} \).
by tubes, only a single (cycloic) tube survives in the geostrophic base flow at later times; however, it becomes small and amorphous for stronger stratification. Moreover, the extraction occurs over a range of horizontal scales ($\tau_{\text{net}} < 0$ for $k_h \approx 10$).

4) ROBUSTNESS

In these runs the global Rossby number is $O(0.1)$; a smaller Rossby number has been obtained from a less-energetic basic state. With $Ro = 0.01$, the transfer spectra closely resemble those depicted in Fig. 7: the extraction of energy from the large-scale base flow is robust. Because the horizontal momentum equation is dominated by inertial oscillations for $Ro \ll 1$, the geostrophic base flow has a tubelike structure because $L_d/L_0 < 1 \approx 1$ (cf. Figs. 2a,b). We fix $N$ and $f$ and vary $\varepsilon_a$; equivalently, $Ro$ is fixed and $Fr$ varied. See Table 2. We choose $N/f = 10$ because it is less computationally demanding than the more realistic $N/f = 100$.

1) AGEOSTROPHIC ENERGY

Time series of $E_{\text{ageo}}$ for $L_d/L_0 \in [0.21, 3.33]$ are plotted in Fig. 8. There is rapid adjustment followed by saturation and slow decay. The flow stays nearly geostrophic: for large $t$, $E_{\text{ageo}} \approx 0.1$ for $L_d/L_0 = 3.33$ (the ratio is even smaller in the synoptic limit). The growth and saturation of the ageostrophic perturbation agrees with low-resolution, hydrostatic Boussinesq simulations (Errico 1984), and the notion that generation of inertia–gravity waves by balanced flow is ubiquitous and inescapable (cf. Warn 1986; Warn and Mendard 1986). The “saturation level” decreases for smaller $\varepsilon_a$ (or $H_0$); this is analogous to the scaling estimate $u \sim \varepsilon_a U$ of P2.

Although the initial adjustment occurs on the $O(1)$ inertial (or buoyancy) time scale, there is an important difference with respect to Fig. 3: it is not followed by a transition toward a regime with clean exponential scaling. The expanded view of the initial growth confirms this (Fig. 8b). However, there is a hint of exponential scaling for synoptic flow (e.g., $L_d/L_0 = 0.21$).

In section 3a, we discussed the influence of the initialization on the rapid adjustment for $t \leq 1$. The rapid adjustment may arise from the use of a basic state in which $w = 0$. As is well known, however, the vertical velocity is implicit in quasigeostrophic theory even

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In section 3 there is weak stratification and $N/f < 1$. This is a sufficient condition for synoptic flow in an isotropic domain; that is, $\varepsilon_a N/f < 1$. With strong stratification, however, synoptic motion is not guaranteed. Here $L_d/L_0 < 1$ only if $\varepsilon_a$ is sufficiently small.

4. Strong stratification, synoptic and subsynoptic flow ($N/f \geq 1$)

In section 3 there is weak stratification and $N/f < 1$. This is a sufficient condition for synoptic flow in an anisotropic domain, synoptic flow

Anisotropic, strongly stratified flow represents the natural extension of weak stratification (section 3): for $\varepsilon_a \ll 1$, the geostrophic base flow has a tubelike structure because $L_d/L_0 < 1 \approx 1$ (cf. Figs. 2a,b). We fix $N$ and $f$ and vary $\varepsilon_a$; equivalently, $Ro$ is fixed and $Fr$ varied. See Table 2. We choose $N/f = 10$ because it is less computationally demanding than the more realistic $N/f = 100$.

1) AGEOSTROPHIC ENERGY

Time series of $E_{\text{ageo}}$ for $L_d/L_0 \in [0.21, 3.33]$ are plotted in Fig. 8. There is rapid adjustment followed by saturation and slow decay. The flow stays nearly geostrophic: for large $t$, $E_{\text{ageo}} \approx 0.1$ for $L_d/L_0 = 3.33$ (the ratio is even smaller in the synoptic limit). The growth and saturation of the ageostrophic perturbation agrees with low-resolution, hydrostatic Boussinesq simulations (Errico 1984), and the notion that generation of inertia–gravity waves by balanced flow is ubiquitous and inescapable (cf. Warn 1986; Warn and Mendard 1986). The “saturation level” decreases for smaller $\varepsilon_a$ (or $H_0$); this is analogous to the scaling estimate $u \sim \varepsilon_a U$ of P2.

Although the initial adjustment occurs on the $O(1)$ inertial (or buoyancy) time scale, there is an important difference with respect to Fig. 3: it is not followed by a transition toward a regime with clean exponential scaling. The expanded view of the initial growth confirms this (Fig. 8b). However, there is a hint of exponential scaling for synoptic flow (e.g., $L_d/L_0 = 0.21$).

In section 3a, we discussed the influence of the initialization on the rapid adjustment for $t \leq 1$. The rapid adjustment may arise from the use of a basic state in which $w = 0$. As is well known, however, the vertical velocity is implicit in quasigeostrophic theory even
though it does not appear explicitly in the potential vorticity equation. Thus for approximately quasigeostrophic flows, the (ageostrophic) vertical velocity may be estimated with the equation (Hoskins et al. 1978):

$$N^2 \nabla_h w + f^2 \frac{\partial^2 w}{\partial z^2} = \nabla \cdot Q,$$

$$Q = \left( \frac{\partial y_g}{\partial x} \nabla \theta, \frac{\partial y_g}{\partial y} \nabla \theta \right).$$ (16)

Initial conditions defined in this way correspond to a more accurate balance (cf. appendix A).

Figure 9a shows the effect of defining $w(t = 0)$ with the $\omega$ equation. To facilitate comparison the vertical axis is identical to that of Fig. 8b. The rapid adjustment is largely suppressed, as may be expected from the discussion of section 3a; however, it is not suppressed completely. Indeed, there is quasi-exponential scaling.

Because $E_{ageo}(t = 0)$ is nonnegligible when using the $\omega$ equation, there are systematic differences when the

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**Fig. 8.** (a) Ageostrophic energy $E_{ageo}$ vs $t$ for strong stratification in an anisotropic domain. (b) An expanded view is shown. Note the rapid adjustment. Here $N/f = 10$.

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**Fig. 9.** Evolution of $E_{ageo}$ from a basic state defined with the $\omega$ equation (strong stratification, anisotropic domain). (a) Time series corresponding to Fig. 8b. The $2(\Gamma_0)$ curve for $L_d/L_0 = 0.33$ is plotted for reference. (b) Comparison with the initial evolution defined using the regular procedure, that is, geostrophic modes with $w = 0$. Here, $L_d/L_0 = 1.33$. 
time evolution is compared to Fig. 8a. These systematic differences do not vanish at large $t$: Fig. 9b shows this for $L_d/L_0 = 1.33$. Crucially, however, the decay rates show much better agreement: memory of the initial conditions is lost and we may expect energy transfer rates to be comparable. There is similar behavior for other values of $L_d/L_0$.

The $\omega$ equation could be used for all calculations. However, the same could be said about any higher-order balance, and moreover the differences are not very important on long time scales. The key point is that, in accord with present understanding of spontaneous imbalance (Vanneste and Yavneh 2004), growth of the unbalanced motion persists.

Even with the modified initial conditions, the growth of $E_{geo}$ occurs more rapidly than predicted by $\langle T_B \rangle$. There are several possible explanations for this discrepancy. First, even with the $\omega$ equation the rapid adjustment is not entirely eliminated; consequently, the subsequent growth may be affected (i.e., masked). Second, strong stratification yields markedly nonstationary flow: for $L_d/L_0 = 0.33$, $\langle T_B \rangle$ increases by almost two orders of magnitude from $t = 0.2$ to $t = 0.6$. Averaging $\langle T_B \rangle$ in space and time cannot be justified as before. A final reason relates to the vertical resolution. From Table 2, $H_b/\Delta z \sim 1 - 3$, meaning that the buoyancy scale is only marginally resolved. This may be significant if, by analogy with the three-dimensionalization of decaying 2D turbulence, growth of the ageostrophic perturbation were influenced by the time-dependent hyperbolic instability. As explained in section b of appendix B, pressureless random straining occurs for $k_z > k_b$ [cf. (A15)]; therefore adequate resolution of the buoyancy scale may be crucial. The influence of $H_b$ is considered in section 4b.

A number of authors have described the growth of gravity waves in flows for which conventional mechanisms, like stratified shear instability, cannot be invoked. Charron and Brunet (1999) speculated that resonant triad interactions could be responsible for the growth of a gravity wave mode diagnosed from the SKYHI general circulation model; in their numerical simulations of a baroclinic life cycle, O’Sullivan and Dunkerton (1995) observed the generation of inertia–gravity waves via processes engendered by geostrophic adjustment. Geostrophic adjustment, that is, removal of wave energy, seems distinct from spontaneous imbalance, that is, a mechanism for the growth of the wave energy; however, generation of the ageostrophic perturbation may continue while there is a forward cascade of ageostrophic energy [cf. section 4a(2)]. Below it will be shown that the AAG triad, which is responsible for nonlinear geostrophic adjustment in the Ro, Fr $\to 0$ limit, plays an important role in the extraction of large-scale geostrophic energy.

2) TRANSFER SPECTRA

The transfer spectra, Fig. 10, generally resemble those for weak stratification (section 3b). There is extraction of geostrophic energy and a forward cascade of ageostrophic energy. With respect to the former, the (negative) low-$k_h$ peak in $T_{net}$ persists and is separated from the dissipation range by an inertial range (Fig. 10a); with respect to the latter, $T_A$ increases sharply toward small scales, though the baseline is not as flat as before (Fig. 10b). The purely geostrophic transfer,
is independent of \( L_d/L_0 \) for \( L_d/L_0 > 0.21 \) (not shown). By contrast with the results for weak stratification (Fig. 6), GAG triads make a greater contribution to \( T_{\text{net}} \). There is qualitatively similar behavior when the average is defined using the integral time, \( \tau \) (not shown).

The ratio of the \( T_{\text{net}} \) peak to the \( T_{\text{GG}} \) peak remains \( O(1) \) or greater. Given that Ro and Fr are small and \( E_{\text{geo}} \gg E_{\text{ageo}} \), this may seem counterintuitive. As emphasized, however, by Errico (1982), correlations between the geostrophic and ageostrophic motion may render products involving geostrophic and ageostrophic variables comparable to products involving the geostrophic motion alone, at least in a time-averaged sense.

The influence of \( L_d \) cannot be easily discerned from Fig. 10. The trend with respect to \( L_d/L_0 \) is quantified in Fig. 11. The net geostrophic damping, \( D_g \), decreases with \( L_d/L_0 \) only down to \( L_d/L_0 \sim 0.5 \) (Fig. 11a). This is consistent with the maximization of the transfer around \( L_d/L_0 \sim 0.67 \) (cf. Fig. 10a). The trend is nonmonotonic: \( D_g \) increases for \( L_d/L_0 \sim 0.5 \).

This behavior is the product of two competing effects. One has been described already, the well-known decoupling between geostrophic and ageostrophic modes for strongly stratified flows (Lilly 1983). The other pertains to synoptic flow. As \( L_d/L_0 \) decreases, so does the aspect ratio of the domain; but from P2, the damping should weaken in a thinner domain, for it is more two-dimensional. Putting these trends together, there is a critical value, \( L_d/L_0 \sim 1 \), at which \( D_g \) is maximized; \( D_g \) reflects the transition from subsynoptic to synoptic flow.

This is confirmed by Fig. 11b, which plots \( E_{\text{geo}}(k_z = 0)/E_{\text{geo}} \) versus \( L_d/L_0 \). This diagnostic characterizes the two-dimensionality of the geostrophic basic flow. There is a kink around \( L_d/L_0 = 3 \). The two diagnostics in Fig. 11 yield slightly different estimates of the transition from subsynoptic to synoptic flow.

This transition cannot be attributed to the thinness of the domain. Even for our thinnest domain, \( H_b \) (cf. Table 2). The buoyancy scale is comparable to the height of the computational box for

\[
\varepsilon_c \sim \frac{U}{N^2},
\]

implying the crossover occurs at \( \varepsilon_c \sim 0.01 \). From Table 2, \( \varepsilon_u > 0.03 \).

Previous results on the noninteraction between gravity waves and balanced flow (e.g., Dewar and Killworth 1995) may appear to be at odds with Figs. 10 and 11. However, there is no contradiction, for those results were obtained with discrete, reduced-gravity layer models. Moreover, the net energy transfer from geostrophic to ageostrophic modes is quite weak: at \( L_d/L_0 \sim 1 \), the net damping induced by \( D_g \) over \( t \in [20, 60] \) is about 10% of \( E_{\text{geo}} \).

These results vindicate the choice \( H \sim H_0, \ L \sim L_0 \) used to define Ro and Fr. A “micro-Froude” number defined by the first moment of the vertical energy spectrum would not capture this behavior, for it is approxi-
mately independent of $e_g$ (not shown). Since the large-scale structure of the flow obviously plays an important role, $R_o$ and $F_r$ should be defined accordingly.

We note also that, while the results described above were obtained with a geostrophic basic state, they appear insensitive to the initial vertical velocity. Calculations with the $\omega$ equation yield similar long-time statistics; that is, Fig. 10 is qualitatively unchanged, though the magnitude of the transfer is considerably smaller in the synoptic limit (not shown). The $v_{\text{eddy}}$ magnitudes (see Fig. 13 below) show much less sensitivity to the base state.

**3) EDDY VISCOSITY**

In subgrid-scale modeling the eddy viscosity models the effect of unresolved scales on resolved scales (e.g., Domaradzki et al. 1987, 1993). Calculation of the eddy viscosity is a standard problem in turbulence modeling. For an application to the Boussinesq equations see Bartello et al. (1996); for a discussion of the GCM problem see Koshyk and Boer (1995). Here we consider a slightly modified procedure in which ageostrophic modes rather than small-scale ones are filtered (see P2 for background). In this way the effects of ageostrophic motion (i.e., gravity waves) can be modeled. This procedure can also be applied to other definitions of balance, provided they can be formulated unambiguously.

We recall the procedure for unstratified, nonrotating flow. Letting $U$ and $u$ denote, respectively, the 2D base flow and 3D perturbation, we obtain a closed set of nonlinear equations:

$$\frac{Du}{Dt} = -\nabla P - \nu \nabla^2 U - (u \cdot \nabla U)' + (u \cdot \nabla u)',$$

$$\frac{D}{Dt} = -\nabla P - \nu \nabla^2 U - u \cdot \nabla \nabla U - u \cdot \nabla^2 u,$$

where the hat denotes a projection onto the 2D base flow and the prime a projection onto the perturbation; $D/dt = \partial/dt + U \cdot \nabla$. These equations amount to a Reynolds stress decomposition. In spectral space the energy equation for the base flow is

$$\frac{D}{Dt} E_{\text{base}}(k_h) = - \sum_{|k|=k_h} \left[ - ikPU + \nu k^2 U U^* \right]$$

$$- U(f(u \cdot \nabla u)^2(k) - U(f(u \cdot \nabla U)^2(k),$$

where the base-flow energy $E_{\text{base}}(k_h) = \sqrt{2} \sum_{|k|=k_h} U_i U_i^*(k).$

From this equation an eddy viscosity can be derived.

The eddy viscosity represents perturbation terms in the base-flow momentum equation as $v_{\text{eddy}} \nabla^2 U$. Thus

$$v_{\text{eddy}} := - \frac{\mathcal{N}_i}{k_h^2 U_i U_i^*},$$

where $\mathcal{N}_i$ is the $i$th perturbation term in (19). Defining $v_{\text{eddy}}$ with the perturbation advection terms,

$$v_{\text{eddy}}(k_h) = - \sum_{|k|=k_h} - \frac{U_i(u \cdot \nabla u + u \cdot \nabla U)^2(k)}{k_h^2 |U|^2(k_h)},$$

(20)

[Note that there is a misprint in the corresponding equation of P2, Eq. (14).] Obviously $v_{\text{eddy}}$ recasts information that is already present in the transfer spectra; however, it has an attractive physical interpretation and can be more conveniently employed in a parameterization. Furthermore $v_{\text{eddy}}$ is normalized with respect to $U$. This ensures that the eddy viscosity reflects interactions between the base flow and the perturbation.

For rotating stratified flow there is a slight change of notation. Here we replace $U$ and $u$ with $v_g$ and $v_a$, respectively, yielding

$$v_{\text{eddy}}(k_h) = - \sum_{|k|=k_h} \frac{A_k^0 (\lambda_{AA} \lambda_{AG}^a(k))}{k_h^2 |v_g|^2(k_h)},$$

(21)

where the hat now denotes a projection onto the geostrophic modes. This is the definition that will be used for all the calculations described below. Here, $v_{\text{eddy}}$ represents perturbation terms in the geostrophic momentum equation as $v_{\text{eddy}} \nabla^2 v_g$. Both AAG and GAG triads contribute to $v_{\text{eddy}}$, as indicated by the two terms in the numerator. Again, note that the terminology is somewhat misleading: $v_{\text{eddy}}$ need not represent the effect of eddies.

Figure 12 plots eddy viscosities for $L_d/L_0 = 3.33$ and $L_d/L_0 = 0.21$. We show the individual contributions from the GAG and AAG triads, as well as $v_{\text{eddy}}$. The magnitudes of the GAG and AAG contributions are comparable for the subsynoptic flow ($L_d/L_0 = 3.33$), but $v_{\text{eddy}, \text{AAG}}$, which is mostly positive, dominates for synoptic flow ($L_d/L_0 = 0.21$).

According to resonant interaction theory, only GAG triads effect geostrophic–ageostrophic transfer for $F_r \ll 1$. The large AAG contributions in Fig. 12 may arise from a preference for interactions at small vertical scales, where $F_r$ might not be small.

In section 4a(2), the extraction of geostrophic energy was described; here there is damping of large-scale geostrophic modes. The negative, low-$k_h$ peak in $\tau_{\text{net}}$ (Fig. 10) is now manifested as a positive, low-$k_h$ peak in $v_{\text{eddy}}$. 
This is analogous to unstratified turbulence (P2, section VI) and weak stratification (omitted for brevity from section 3). The contributions from low $k_h$ dominate $\nu_{\text{eddy}}$, even with strong stratification. The importance of $\nu_{\text{eddy}}$ can be assessed by comparing it to the effective viscosity, $\nu_{\text{eff}}$. Letting $N_i U / \nu_{\text{eff}}$, $|k_h|^2 / \nu_{\text{eff}}$, and $|k_h|^2 / \nu_{\text{eddy}}$, if the dominant contribution to the numerator comes from $k_z \sim 0$, $\nu_{\text{eff}} \sim k_h$.

In Fig. 12, the dissipation due to $\nu_{\text{eddy}}$ is several orders of magnitude larger than hyperviscosity at small $k_h$; that is, $\nu_{\text{eddy}} \gg \nu_{\text{eff}}$. (It is, however, small in absolute terms: it occurs on a time scale that is about an order of magnitude greater than that associated with the decay of the total geostrophic energy, which is dominated by small-scale contributions). At small scales the roles of $\nu_{\text{eddy}}$ and $\nu_{\text{eff}}$ are reversed, while at intermediate scales $|\nu_{\text{eddy}}| \sim \nu_{\text{eff}}$. For homogeneous turbulence the subgrid-scale eddy viscosity has a completely different structure: there is a “cusp” at the truncation scale (Kraichnan 1976; Chollet and Lesieur 1981): $\nu_{\text{eddy}}$ increases at small scales.

In P2 the large, low-$k_h$ peak is coupled with smaller values for high-$k_h$. Here $\nu_{\text{eddy}}$ can be modeled as

$$\nu_{\text{eddy}} = \begin{cases} f(k_h), & k_h < k_{\text{cross}}, \\ 0, & k_h > k_{\text{cross}}, \end{cases} \quad (23)$$

where $f(x)$ is a monotonically decreasing function and $k_{\text{cross}}$ is a crossover wavenumber. This is a good approximation for synoptic flow, though there is slight undershoot at large $k_h$ (Fig. 12b); that is, $\nu_{\text{eddy}} < 0$. For subsonic flow, (23) does not apply as well: there is broadband structure and the baseline deviates more strongly from zero (Fig. 12a). Again, the effects of the ageostrophic perturbation on the geostrophic base flow do not average out.

To investigate the influence of $L_d/L_0$, consider the eddy viscosities of Fig. 13. The peak magnitude of $\nu_{\text{eddy}}$ is maximized for $L_d/L_0 = 0.67$ (Fig. 13a), in agreement with Fig. 11. The structure of $\nu_{\text{eddy}}$ also changes. This can be seen more clearly in Fig. 13b, which shows $\nu_{\text{eddy}}^\ast$, the eddy viscosity normalized by its peak value. For smaller $L_d/L_0$ there is less undershoot, though the change is most pronounced for the smallest value, $L_d/L_0 = 0.21$. In the synoptic limit geostrophic–ageostrophic interactions resemble 2D–3D interactions in unstratified turbulence (cf. P2, Fig. 15).

If the base flow is not 2D, we might expect $\nu_{\text{eddy}}$ to be qualitatively different. For $L_d/L_0 \approx 1$, a more broadband structure would ensue if geostrophic–ageostrophic interactions did not remain confined to small $k_h$ (or small $\varepsilon_s$). Indeed, growth via random straining (section b of appendix B) is favored in a thin domain because there are fewer small-$k_z$ modes that depart from quasi-2D motion and modify interactions with small-scale modes below $H_b$.

Figure 14 depicts the resolution dependence for the subsynoptic value, $L_d/L_0 = 3.33$ (Fig. 14a), and the
synoptic value, $L_d/L_0 = 0.33$ (Fig. 14b). Because the grid is isotropic, the horizontal and vertical resolutions increase in step. There is significant undershoot for the coarsest resolutions—the low-$k_h$ peak disappears in Fig. 14a. This phenomenon is robust and has been observed for other parameters. The structure is qualitatively unchanged when the time averages are defined using the integral time, $\tau$ (not shown).

We speculate that the growth of the perturbation cannot be properly modeled unless $H_b$ is adequately resolved; that is, $\Delta z \approx H_b$. As the resolution increases, $\nu_{\text{eddy}}$ loses some of its broadband structure: for $L_d/L_0 = 3.33$, the low-$k_h$ peak begins to emerge at $N_h \approx 150$ (Fig. 14a), from which $H_b$ is marginally resolved (cf. Table 2). Even for synoptic flow (Fig. 14b), convergence is difficult to ascertain in an anisotropic domain: the ensemble variation for $N_h = 240$ (not shown) is comparable to the variation between the curves. The inconclusive convergence may be a by-product of the limited number of vertical modes employed in these anisotropic simulations.

The sensitivity to the buoyancy scale may be attrib-
uted to geostrophic–ageostrophic interactions [i.e., the numerator in (21)] or the geostrophic energy spectrum [i.e., the denominator]. Waite and Bartello (2006) have shown that energy spectra exhibit a bumpy structure at large scales if $k_h$ is not adequately resolved. In section 4b(2) we explicitly examine the influence of contributions from large $k_z$ to geostrophic–ageostrophic interactions.

For strong stratification and synoptic flow, the dominant contribution to $\nu_{\text{eddy}}$ comes from the smallest $k_h$, in agreement with (23). This supports the idea that $\nu_{\text{eddy}}$ may be amenable to parameterization.

b. Isotropic domain, subsynoptic flow

Strong stratification in an isotropic domain is a case without obvious geophysical relevance. Yet this case is worth considering anyway to elucidate the influence of synoptic flow. Table 3 lists parameters. For $L_d/L_0 > 1$ the geostrophic base flow consists of pancakes rather than tubes (see Fig. 2d) and quasi two-dimensionality cannot be assumed.

1) Ageostrophic energy

The growth of $E_{\text{ageo}}$ (Fig. 15a) follows the same pattern established by anisotropic, subsynoptic flow. Rapid initial adjustment is followed by a brief transitional regime before the perturbation saturates. The Fr = 1 curve, taken from the results for weak stratification, is shown for comparison, to underscore the rapidity of the growth.

As before, the rapid adjustment may be partly attributed to the initial conditions. Using the $w$ equation (16), to define $w(t = 0)$ greatly reduces the magnitude of the adjustment (by about five orders of magnitude; not shown), but the long-time behavior is qualitatively unchanged ($E_{\text{ageo}}$ differs by $\sim 10\%$ for Fr = 0.1 and 0.01).

The accuracy of the linear growth estimates is poor. For Fr = 0.1, the growth rate is over 8 times larger than $\left(\Gamma/\nu\right)_0$; for Fr = 0.02 and Fr = 0.01, the agreement breaks down completely as $\left(\Gamma/\nu\right)_0 \sim 0$ (more precisely, $\left(\Gamma/\nu\right)_0 < 0$).

Nevertheless, the random straining mechanism may still be relevant. We hypothesize that vertical modes

<table>
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<th>$f$</th>
<th>$N_h$</th>
<th>$N_v$</th>
<th>$L_d/L_0$</th>
<th>$H_b$</th>
<th>$\Delta z$</th>
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<td>1</td>
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Fig. 15. Ageostrophic energy $E_{\text{ageo}}$ vs $t$ for strong stratification, isotropic domain; (a) Fr = 0.1. For comparison, the Fr = 1 curve is also shown. (b) Subsampled results for Fr = 0.1. Only modes satisfying $k_z \geq k_r$ are retained.
below $H_b$, that is, $k_z > k_h$, may experience exponential growth. To investigate this we subsample the data:

$$E_{ageo}(k_z) = \sum_{k_z = k_h}^{K_1} \left[ A_k^+ A_k^+ \right] (k) + A_k^- A_k^- (k),$$

where $k_z$ denotes the vertical wavenumber above which ageostrophic modes are retained. Thus there is no filtering for $k_z \leq k_h$.

Figure 15b shows the effect of subsampling for $Fr = 0.1$. The maximum value of $E_{ageo}$ decreases as large vertical scales are filtered out, that is, as $k_z$ increases; here $k_h \approx 4$ if the long-time value $U \sim 0.3$ is used. Furthermore, the rapid adjustment appears to be followed by quasi-exponential growth for $t \approx 1$. The quasi-exponential scaling is cleanest for $k_z \geq k_h$. Qualitatively, the change is greater than that yielded by the $\omega$ equation. Although the growth rate does not match $2(1/O)$, which is considerably smaller, a modified linear growth rate estimate could be obtained by taking the vertical structure of $v_g$ into account. Subsampling data generated from initial conditions defined by the $\omega$ equation yields qualitatively similar results (not shown).

In spectral space the growth rate tends to be maximized near the vertical truncation scale, irrespective of $Fr$. Here $\bar{E}_a (k_z; t = 3, t_0 = 0.39)$ increases toward large $k_z$ (Fig. 16a), as with weak stratification (cf. Fig. 5). By contrast, $\bar{E}_a (k_h; t = 3, t_0 = 0.39)$ does not always peak at the smallest horizontal scales: for $Fr = 0.1$ the growth rate is maximized at large horizontal scales, that is, small $k_h$. The choice of $t$ is arbitrary; similar trends are obtained with smaller $t$.

This behavior is consistent with the anisotropic results (cf. Fig. 5). In the synoptic limit, the structure of geostrophic–ageostrophic interactions resembles that for unstratified turbulence and the growth of the ageostrophic perturbation is favored at large $k_z$ and small $k_h$. This agrees with the behavior of decaying 2D turbulence (cf. P2 and appendix Ba). This is also consistent with properties of inertia–gravity waves in the upper troposphere and lower stratosphere, which have large horizontal scales ($\sim 1000–2000$ km) and small vertical scales ($\sim 1–4$ km; cf. O’Sullivan and Dunkerton 1995 and references therein). The random straining mechanism could be relevant to the initial growth of the ageostrophic perturbation on scales $k_z \geq k_h$, even for strong stratification. Nevertheless, the evidence is circumstantial: the (rough) exponential scaling in Fig. 15b does not extend over a wide range and the structure of $\bar{E}_a (k_h, k_z)$ is sensitive to $t$.

2) TRANSFER SPECTRA AND EDDY VISCOSITIES

The anisotropic simulations of sections 4a(2) and 4a(3) suffer from a reduced number of vertical modes for $e_v \ll 1$. Although, in principle, anisotropic and isotropic simulations are equivalent—the same values of $Ro$ and $Fr$ can be examined with either configuration—it is useful to verify the robustness of the previous results. Figure 17 confirms that, for subsynoptic flow, there continues to be extraction of geostrophic energy at small $k_h$ (Fig. 17a), while the undershoot of $\nu_{edd}$ persists (Fig. 17b). Again, defining the basic state with the $\omega$ equation does not alter these conclusions.
More interestingly, the resolution dependence of \( \nu_{\text{eddy}} \) can be examined more carefully with an isotropic domain. Here \( \nu_{\text{eddy}} \) appears to converge for marginally synoptic flow, \( L_d/L_0 = 1 \) (Fig. 18a): the persistent undershoot in Fig. 14 could be an artifact of the limited number of vertical modes. By contrast, the convergence is slower for subsynoptic flow, \( L_d/L_0 = 10 \) (Fig. 18b). The structure of \( \nu_{\text{eddy}} \) is different in the two cases: whereas increasing the resolution primarily serves to extend and displace the baseline in the former case, the structure of \( \nu_{\text{eddy}} \) changes in the latter. Although there is minimal undershoot when \( H_b \) is resolved, as in the synoptic case (Fig. 18a), it is unclear whether the undershoot will also disappear for subsynoptic flow when \( H_b \) is adequately resolved (cf. Fig. 18b and Table 3).

There are two explanations for the resolution dependence of \( \nu_{\text{eddy}} \). The geostrophic energy spectrum or the geostrophic–ageostrophic interactions could change qualitatively as the resolution increases; alternatively, small-scale vertical modes below \( k_b \) should make a significant contribution to geostrophic–ageostrophic interactions if the random straining mechanism is relevant. To examine the contribution of small-scale modes with \( k_z > k_b \), we subsample the data. Redefining the transfer spectra as

Fig. 17. Transfer spectra and eddy viscosities (strong stratification, isotropic domain). (a) Net geostrophic transfer, \( \mathcal{T}_{\text{net}} \); (b) total eddy viscosity, \( \nu_{\text{eddy}} \).

Fig. 18. Resolution dependence of \( \nu_{\text{eddy}} \) (strong stratification, isotropic domain); (a) \( \text{Fr} = 0.1 \), (b) \( \text{Fr} = 0.01 \).
creases for flow, we might expect were responsible for dissipation of the synoptic base flow. If the growth of small-scale ageostrophic modes (23b), which implies a damping on the large-scale base flow. Random straining in the weakly stratified range (cf. Fig. 17b), the growth rate of the ageostrophic perturbation may return energy to the base flow. At early times, \( \nu_{\text{eddy}} (k_h; k_z) > 0 \) for \( k_h \approx 10 \) and the structure of \( \nu_{\text{eddy}} (k_h; k_z) \) does not change significantly with \( k_z \); at late times, the perturbation may return energy to the base flow so that \( \nu_{\text{eddy}} (k_h; k_z) < 0 \) for \( k_z > k_h \).

\[
\nu_{\text{eddy}} vs. \text{kh (L_d/L_0=1, subsampled)}
\]

\[
\nu_{\text{eddy}} vs. \text{kh (L_d/L_0=10, subsampled)}
\]

where \( \tilde{A} \) denotes subsampled ageostrophic modes, that is, \( A^z_k = 0 \) for \( k_z > k_s \), we can examine the influence of the cutoff scale, \( k_z \).

Figure 19 shows \( \nu_{\text{eddy}} (k_h; k_z) \) for \( \text{Fr} = 0.1 \) and \( \text{Fr} = 0.01 \). In the marginally synoptic case (Fig. 19a), \( \nu_{\text{eddy}} (k_h; k_z) \) converges for \( k_z \approx 10 \). This agrees with the estimate \( k_b = 4 \) and explains the rapid convergence of \( \nu_{\text{eddy}} \) with resolution (Fig. 18a). Because of the way we have chosen to define our numerical viscosity, higher resolution implies a smaller viscous scale and a wider weakly stratified range. In the subsynoptic case (Fig. 19b), convergence is not achieved: \( \nu_{\text{eddy}} (k_h; k_z) \) increases for \( k_z > k_b \approx 40 \). This explains why the structure of \( \nu_{\text{eddy}} \) changes with resolution in Fig. 18b: the “rectification” of \( \nu_{\text{eddy}} \) at small \( k_b \), that is, the decreased undershoot of the baseline at higher resolution, follows from the increase in \( \nu_{\text{eddy}} (k_h; k_z) \). In this way agreement with (23) is improved.

These results indicate that small-scale modes with \( k_z > k_b \) may exert an important influence on geostrophic–ageostrophic interactions. They are also consistent with growth of the ageostrophic perturbation via random straining in the weakly stratified range (cf. Fig. 15b), which implies a damping on the large-scale base flow. If the growth of small-scale ageostrophic modes were responsible for dissipation of the synoptic base flow, we might expect \( \nu_{\text{eddy}} (k_h; k_z) > 0 \) for \( k_z > k_b \). This is not observed in Fig. 19 because of decay of the ageostrophic energy. At early times, \( \nu_{\text{eddy}} (k_h; k_z) > 0 \) for \( k_h \approx 10 \) and the structure of \( \nu_{\text{eddy}} (k_h; k_z) \) does not change significantly with \( k_z \); at late times, the perturbation may return energy to the base flow so that \( \nu_{\text{eddy}} (k_h; k_z) < 0 \) for \( k_z > k_b \).

5. Summary and discussion

This paper has examined the extent to which our understanding of the three-dimensionalization of decaying 2D turbulence can be used to analyze and interpret the growth and feedback of ageostrophic perturbations in rotating stratified turbulence. For weak stratification, the growth rate of the ageostrophic perturbation is well predicted by a linear estimate (Fig. 3), the ageostrophic perturbation extracts energy from the geostrophic base flow (Fig. 6), and the eddy viscosity, \( \nu_{\text{eddy}} \), assumes a “characteristic structure” in which there is a preference for interactions at large horizontal scales. For strong stratification, applicability is more subtle. The growth rate departs from the linear estimate (Fig. 15) and \( \nu_{\text{eddy}} \) develops broadband structure for subsynoptic flow (Fig. 17b). But with a small aspect ratio, \( \varepsilon_u \), there is synoptic-scale motion, \( NH_0/fL_0 < 1 \), and \( \nu_{\text{eddy}} \) reverts to its previous form in the limit of high resolution (Fig. 14), that is, for wavenumbers \( k_z \gg k_b \), where \( k_b \) is the buoyancy wavenumber.

These results suggest that the time-dependent, hyperbolic instability responsible for the three-dimensionalization of decaying 2D turbulence (P1, P2) may apply to rotating, stratified turbulence, at least in some important respects. The mechanism generalizes (appendix Bb) if (i) the buoyancy scale, \( H_b \), is large compared to the vertical length scale (\( H_b > H_0 \)) and (ii) the
deformation radius is small compared to the horizontal length scale \( (L_\theta > L_d) \). In addition to the evidence adduced above, this claim is also supported by the anisotropy of geostrophic–ageostrophic interactions. According to appendix B, the perturbation and base flow should interact most strongly at large horizontal scales (small \( k_h \)) and small vertical scales (large \( k_z \)). For weak stratification, spectral growth rates, \( \tilde{\sigma}_{E_z} \) confirm that the ageostrophic perturbation is maximized for small \( k_h/k_z \) (Fig. 5). For strong stratification, the \( \sigma_{E_u} \) increase toward large \( k_z \) (Fig. 16a); if there is synoptic flow, they also increase toward small \( k_h \) (Fig. 16b).

We have considered parameters that are representative of large-scale tropospheric flow. The global Rossby number \( \text{Ro} = O(0.1) \) with global Froude numbers ranging from synoptic values (\( Fr \leq 0.1, L_d/L_\theta < 1 \)) to subsynoptic values (\( Fr > 0.1, L_d/L_\theta > 1 \)). Although we have tried to interpret the results for weak and strong stratification in terms of the same mechanism, there are some important differences. Interactions between the large-scale geostrophic flow and the small-scale ageostrophic perturbation depend on \( L_d \) and \( H_b \). With respect to \( L_d \), there are large-scale buoyancy oscillations for subsynoptic flow, implying that the geostrophic base flow cannot be modeled with horizontal random strain; with respect to \( H_b \), small-scale perturbation modes cannot be captured if it is not adequately resolved. The eddy viscosity curves show persistent “undershoot” at small \( k_h \), that is, \( \nu_{\text{eddy}} < 0 \) (Figs. 14 and 18) for subsynoptic flow with marginal resolution of \( H_b \). Calculations with subsampled data indicate that the “rectification” of the \( \nu_{\text{eddy}} \) at higher resolution can indeed be partly ascribed to small-scale vertical modes with \( k_z > k_h \) (Fig. 19).

Similarly, quasi-exponential growth of the ageostrophic energy, \( E_{\text{ageo}} \), is obtained if the data are subsampled to restrict attention to large \( k_z \) (Fig. 15b). The idea that the small-scale flow can exert a strong feedback on the largest scales also appears in other contexts; this is the defining property of the alpha effect in magnetohydrodynamics (e.g., Gilbert 2003).

A subtle point concerns the balanced basic state. Using the geostrophic modes yields quasigeostrophic dynamics but a vanishing vertical velocity (the latter does not appear explicitly in quasigeostrophic theory). Consequently a rapid adjustment on inertial time scales ensues (Fig. 4). This adjustment can be mitigated by defining the basic state with the quasigeostrophic \( \omega \) equation (Fig. 9a). Robustness tests indicate that long-time statistics are qualitatively insensitive to the \( \omega \) equation (e.g., Fig. 9b). Of course, any definition of balanced flow may be questioned; nevertheless, the one adopted here is a reasonable starting point for tropospheric flow.

From a theoretical perspective, random straining by the geostrophic base flow may be a generic mechanism for the generation of imbalance via a nominally balanced flow (i.e., spontaneous imbalance). This mechanism could be important in the middle atmosphere, where there is horizontal random straining (e.g., Shepherd et al. 2000). In the geophysical literature, the hyperbolic instability, which underlies the pressureless growth mechanism, has received less attention than other instability mechanisms; however, an analysis of solvability conditions for the “balance equations” (McWilliams et al. 1998) supports the claim that it can lead to a loss of balance [cf. their Eq. (7) and our (A13)].

From a practical perspective, there are several implications. First, the effects of imbalance might not be difficult to parameterize. A \( \nu_{\text{eddy}} \) that is strongly peaked at small \( k_h \) is good news for parameterization efforts because it means that high resolution (in the horizontal) might not be required in a parameterized model. If the buoyancy and dissipation scales are widely separated, the feedback of the ageostrophic modes on the geostrophic flow simplifies.

These results are consistent with recent results on the parameterization of gravity wave drag in middle atmosphere models (McLandress and Scinocca 2005). It has been shown that the precise details of the nonlinear dissipation mechanism are secondary as long as the momentum balance is respected. Although the gravity wave–drag problem is much more complicated than our idealized one—we do not need to contend with the propagation of gravity waves, and the resulting deposition of (pseudo) momentum, from a source region to a dissipation region—the simplified interaction seems analogous.

Second, our results provide guidance about appropriate resolution for direct numerical simulation. While there are no special requirements for the horizontal resolution, the vertical resolution should be sufficient to resolve the buoyancy scale. This suggests an alternative view of synoptic-scale flow. Small vertical scales, with \( k_z > k_h \), can have an important influence on geostrophic–ageostrophic interactions. If the vertical grid spacing is coarser than the buoyancy scale, vertical overturning might not be properly represented, and an inverse energy cascade could develop (Bartello 2000; Waite and Bartello 2006). In state-of-the-art numerical weather prediction models, the buoyancy scale is marginally resolved (\( \Delta z \approx H_b \sim 1 \text{ km} \)). In climate models, however, resolution of the buoyancy scale could be an important issue.

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APPENDIX A

GGG Triads and Quasigeostrophy

The basic states are generated by setting the amplitude of the wave modes to zero. At each time step, the dynamical fields \((\omega, b_k)\) are reconstructed from \([A_k^{(0)}, A_k^{(2)} = 0]\). In particular, \((\xi_k, D_k, T_k) = (\xi_k, D_k, T_k)_{G}\).

A number of authors have shown that zeroing the wave modes yields quasigeostrophic dynamics. This has been done for the shallow-water equations by Salmon (1998) and for the hydrostatic Boussinesq equations by Leith (1980). Leith also shows that a single iteration of nonlinear normal-mode iteration yields quasigeostrophic [i.e., GGG flow plus the \(\omega\) equation, (16)]. Recall that the vertical velocity has no influence on OG dynamics.

We now demonstrate this equivalence for nonhydrostatic, Boussinesq flow. We only consider modes satisfying \((k_z \neq 0, k_h \neq 0)\). The GGG flow is obtained by truncating the Boussinesq equations to

\[
\frac{d}{dt} A_k^{(0)} = \sum_{p+q=k} \Gamma_{kpq}^{(0)} A_p^{(0)} A_q^{(0)}. \tag{A1}
\]

It was shown in Bartello (1995) that the linear quasigeostrophic potential vorticity, \(q\), can be expressed as

\[
\hat{q} = F \left( \xi + \frac{f}{N^2} \frac{\partial b}{\partial z} \right) = \frac{\sigma_k k A_k^{(0)}}{k_h}, \tag{A2}
\]

where \(F\) is Fourier transform and the inviscid, real space quasigeostrophic equation

\[
\frac{\partial \hat{q}}{\partial t} = -\mathbf{u}_h \cdot \nabla_h \hat{q}. \tag{A3}
\]

Recall some properties of the flow truncated to the zero modes. From Eq. (5a) in Bartello (1995)

\[
\xi_k = \frac{N^2 k_h A_k^{(0)}/\sigma_k}{k}, \quad \delta_k = 0, \quad b_k = -if N^2 k_z A_k^{(0)}/\sigma_k \sigma_k \sigma_k \tag{A4}
\]

(to within a factor of \(N\)). The velocity is perfectly geostrophic, the flow is hydrostatic and therefore there is thermal wind balance. From thermal wind,

\[
\frac{\partial}{\partial z} (f \mathbf{z} \times \mathbf{u}) = \frac{\partial}{\partial z} \left(-\frac{1}{\rho_o} \mathbf{V} \mathbf{p}'\right), \quad \text{or} \quad f \mathbf{z} \times \frac{\partial \mathbf{u}_h}{\partial z} = -\mathbf{V} \mathbf{p}, \tag{A5}
\]

since \(\partial p'/\partial z = -g p'/\rho_o = b\). Since we have zero horizontal divergence, we can introduce a vertically varying streamfunction such that \(u_k = -ik_z \psi_k = i k_z \xi_k \sigma_k \) and \(\mathbf{u}_k = -i k_z \psi_k = i k_z \xi_k \sigma_k \). Both components of the thermal wind relation imply that \(ik_z \xi_k = -k_z^2 b_k\), which is satisfied by the above expressions for \(\xi_k\) and \(b_k\).

Now, from the Boussinesq equations

\[
\frac{\partial q}{\partial t} = -(\mathbf{u} \cdot \nabla)q + (\omega \cdot \nabla)w - \frac{f}{N^2} \frac{\partial}{\partial z} (\mathbf{u} \cdot \nabla b). \tag{A6}
\]

Recall that the linear term is zero for this mode. The stretching term is also zero, since \(w\) is zero for geostrophic modes. Also, the latter implies that \(\mathbf{u} \cdot \nabla = \mathbf{u}_h \cdot \nabla_h\). That means we have

\[
\frac{\partial q}{\partial t} = -(\mathbf{u}_h \cdot \nabla_h)q - \frac{f}{N^2} \frac{\partial}{\partial z} (\mathbf{u}_h \cdot \nabla_h b). \tag{A7}
\]

The last term can be obtained by taking the scalar product of the thermal wind relation (A5) and \(\partial \mathbf{u}_h / \partial z\); that is,

\[
-\frac{\partial \mathbf{u}_h}{\partial z} \cdot \nabla_h b = \frac{\partial \mathbf{u}_h}{\partial z} \left(f \mathbf{z} \times \frac{\partial \mathbf{u}_h}{\partial z}\right) = 0, \tag{A8}
\]

whence the quasigeostrophic potential vorticity Eq. (A3) is recovered.

APPENDIX B

Time-Dependent Hyperbolic Instability

a. Unstratified flow

In P1 we showed that decaying two-dimensional turbulence three-dimensionalizes through a time-dependent version of the hyperbolic instability (e.g., Klaassen and Peltier 1985; Caulfield and Kerswell 2000). Physically, the instability arises from straining of the 3D perturbation by the 2D base flow—horizontal vorticity, in the form of so-called rib vortices, is generated. Mathematically, the hyperbolic instability may be analyzed with a “pressureless” approximation (e.g., Leblanc and Cambon 1997). From the linearized perturbation equations for unstratified, nonrotating flow,

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u} = -\nabla p - \mathbf{u} \cdot \nabla \mathbf{U}, \quad \nabla \cdot \mathbf{u} = 0, \tag{B1}
\]
where we have taken the constant density $\rho_0 = 1$. The horizontal pressure gradient may be neglected if the horizontal scale of the perturbation is large compared to the vertical scale, that is, if $\varepsilon_a < 1$, $\varepsilon_a := H/L = k_h/k_z$ being the ratio of vertical and horizontal scales, or alternatively, a spectral aspect ratio. More precisely,

$$\frac{|\nabla_h P|}{|\mathbf{u} \cdot \nabla_h \mathbf{U}|} = O(\varepsilon_a^2), \quad (B2)$$

and $\varepsilon_a \to 0$ corresponds to the hydrostatic limit, which suggests applicability to large-scale geophysical flows. On occasion we refer to “pressureless modes.”

In the hydrostatic limit the perturbation equation reduces to the compact form

$$\frac{D \mathbf{u}_h}{Dt} = -\mathbf{u}_h \cdot \nabla_h \mathbf{U}, \quad (B3)$$

where $D/Dt = \partial/\partial t + \mathbf{U} \cdot \nabla$. We emphasize that there is no inconsistency between “pressureless dynamics” and the existence of a vorticity equation, in which pressure terms do not appear explicitly. A similar analysis could be performed on the perturbation vorticity equation by taking $\varepsilon_a \to 0$; however, on account of the vortex tilting term, this equation is not as convenient to analyze.

The perturbation Eq. (B3) is formally analogous to the equation for a material line element

$$\frac{D \mathbf{L}}{Dt} = \mathbf{L} \cdot \nabla_h \mathbf{U}. \quad (B4)$$

For slowly varying (i.e., frozen) strain, it is clear that there is exponential stretching. Following the pioneering work of Cocke (1969) on the stretching of material lines in statistically isotropic flows, Kraichnan proved that material lines stretch exponentially if the correlation time of the turbulent fluctuations is infinitesimally small compared to the eddy turnover time of the turbulence (Kraichnan 1974). As is intuitively clear (the minus sign should not make a difference), and as shown explicitly by Kraichnan, this analysis also applies to equations of the form (B3). Therefore $\mathbf{u}_h$—and hence the 3D perturbation, $\mathbf{u}$—grows exponentially if there is a separation in time scales between the 2D and 3D motion. This is “pressureless growth.”

The analogy with material line stretching can be pursued further. Following work on the characterization of tracer gradients (Lapeyre et al. 2001; Klein et al. 2000), we can obtain estimates of the (linear) growth rate along particle trajectories (Straub 2003):

$$\Gamma = \pm \frac{1}{2} \sqrt{\sigma^2 - (\omega - 2\phi)^2}, \quad \Gamma_0 = \pm \frac{1}{2} \sqrt{\sigma^2 - \omega^2}, \quad (B5)$$

where $\sigma = [(U_x - V_x)^2 + (V_x + U_y)^2]^{1/2}$ and $\omega = V_x - U_y$ denote the strain and vorticity of the base flow, and $-\phi$ is the rotation rate of the strain axes in the Lagrangian frame. These estimates are spatially local: variations of the strain and vorticity along particle trajectories are neglected. Here $\Gamma_0$ is the well-known Weiss criterion for the existence of coherent structures in 2D turbulence, while $\Gamma$ is an extension that takes the rotation of the strain axes into account. With background rotation of angular velocity $2\Omega$ or Coriolis parameter $f$,

$$\Gamma = \pm \frac{1}{2} \sqrt{\sigma^2 - (\omega + 2\phi)^2}, \quad \Gamma_0 = \pm \frac{1}{2} \sqrt{\sigma^2 - (\omega + 2f)^2}. \quad (B6)$$

With strong rotation, $Ro \ll 1$, growth via random straining is suppressed.

If $\langle \Gamma_0 \rangle > 0$ the flow is strain dominated and characteristic solutions are hyperbolic, as in the steady hyperbolic instability. This condition also defines one of the limits to balance described by McWilliams et al. (1998).

Applicability to decaying 2D turbulence was verified numerically in P1 and P2. In particular, the growth of the perturbation is strongly determined by the (effectively) pressureless modes and the initial growth rate is bracketed by the linear estimates, $\Gamma$ and $\Gamma_0$. The analysis applies to flow in isotropic and anisotropic domains.

Nonlinear dynamics were studied in P2. We showed that the “return to isotropy” is extremely slow in thin domains. We also showed that the eddy viscosity is strongly peaked at low wavenumbers, which indicates that the perturbation–base flow interaction is confined to the largest (horizontal) scales, reflecting the relevance of the pressureless growth mechanism even after the saturation of the perturbation. Growth of the perturbation via random straining by the base flow represents a damping on large horizontal scales, that is, an energy sink or a positive eddy viscosity.

b. Extension to rotating stratified turbulence

Although the discussion in appendix Ba is specific to unstratified flow, the time-dependent hyperbolic instability could represent a route by which unbalanced motion is generated by the balanced flow. For the analysis to generalize it is necessary that there be (i) dominance of the production term, $-\mathbf{v}_a \cdot \nabla_h \mathbf{v}_a$; and (ii) advection via random straining by a two-dimensional base flow. Here we relate the pressureless growth mechanism to the qualitative arguments of section 2d.

The validity of the pressureless approximation (B2)
depends on the weakness of the stratification: the condition \( \varepsilon \) does not guarantee (effectively) pressureless dynamics. In other words, how weak must the stratification be in order for there to be pressureless dynamics at a given vertical scale, \( H \)? Stratified turbulence phenomenology suggests that buoyancy effects are relatively weak below the buoyancy scale (cf. section 2d). In spectral space, the buoyancy wavenumber \( k_b := N/U \) (it is conventional to exclude the factor of \( 2\pi \)) and there are effectively pressureless dynamics for vertical wavenumbers satisfying

\[
|k_z| > k_b,
\]

(For brevity we generally omit the absolute value.) This defines the “weakly stratified range.”

With respect to the second condition, the basic state must be quasi-two-dimensional if the problems are to be isomorphic. Departures from the previous, unstratified results may arise from baroclinicity in the base flow. For rapid rotation, there is quasi-2D flow on scales where rotation dominates, that is,

\[
\frac{L_d}{L} < 1, \quad L_d = \frac{NH}{f},
\]

for given \( H \) and \( L \). There is some freedom in the definition of \( L_d \); it is possible to choose a vertical scale corresponding to the perturbation or the base flow, that is, to different baroclinic Rossby radii (Gill 1982, section 7.5). Choosing \( L \sim L_0, H \sim H_0 \) suggests a connection with the first baroclinic deformation radius of a continuously stratified fluid. It yields the condition (9) and ensures that base flow is quasi 2D when \( L_d/L_0 < 1 \).

It is important to keep in mind that (B8) is a sufficient condition: the ageostrophic perturbation could still grow if it were violated. However, without random straining by a quasi-2D flow, the geostrophic-ageostrophic interactions might differ qualitatively from those described in P1 and P2.

Although synoptic flow is one of the sufficient conditions for the previous results to be recovered, the instability occurs irrespective of \( L_d \) and the global Froude number. Thus it may complement asymptotic analyses of spontaneous imbalance that require small \( Fr \) (e.g., Ford et al. 2000).

REFERENCES


