Resonant interactions among equatorial waves in the presence of a diurnally varying heat source are studied in the context of the diabatic version of the equatorial β-plane primitive equations for a motionless, hydrostatic, horizontally homogeneous and stably stratified background atmosphere. The heat source is assumed to be periodic in time and of small amplitude [i.e., O(ε)] and is prescribed to roughly represent the typical heating associated with deep convection in the tropical atmosphere. In this context, using the asymptotic method of multiple time scales, the free linear Rossby, Kelvin, mixed Rossby–gravity, and inertio-gravity waves, as well as their vertical structures, are obtained as leading-order solutions. These waves are shown to interact resonantly in a triad configuration at the O(ε) approximation, and the dynamics of these interactions have been studied in the presence of the forcing.

It is shown that for the planetary-scale wave resonant triads composed of two first baroclinic equatorially trapped waves and one barotropic Rossby mode, the spectrum of the thermal forcing is such that only one of the triad components is resonant with the heat source. As a result, to illustrate the role of the diurnal forcing in these interactions in a simplified fashion, two kinds of triads have been analyzed. The first one refers to triads composed of a k = 0 first baroclinic geostrophic mode, which is resonant with the stationary component of the diurnal heat source, and two dispersive modes, namely, a mixed Rossby–gravity wave and a barotropic Rossby mode. The other class corresponds to triads composed of two first baroclinic inertio-gravity waves in which the highest-frequency wave resonates with a transient harmonic of the forcing. The integration of the asymptotic reduced equations for these selected resonant triads shows that the stationary component of the diurnal heat source acts as an “accelerator” for the energy exchanges between the two dispersive waves through the excitation of the catalyst geostrophic mode. On the other hand, since in the second class of triads the mode that resonates with the forcing is the most energetically active member because of the energy constraints imposed by the triad dynamics, the results show that the convective forcing in this case is responsible for a longer time scale modulation in the resonant interactions, generating a period doubling in the energy exchanges. The results suggest that the diurnal variation of tropical convection might play an important role in generating low-frequency fluctuations in the atmospheric circulation through resonant nonlinear interactions.

1. Introduction

The atmospheric flow is characterized by the existence of low-frequency fluctuations with time scales ranging from 20 to around 100 days (Madden and Julian 1972, 1994; Hayashi and Golder 1993; Ghil and Mo 1991). The dominant component of this intraseasonal variability of the atmospheric circulation in the tropics is the 40–50-day Madden–Julian oscillation (MJO), which is characterized in the troposphere by a planetary-scale (wavenumber 1–2) wave envelope having an eastward propagation with a velocity on the order of 5 m s⁻¹ over strongly convective regions, such as the eastern Indian and the western Pacific Oceans, and of 15 m s⁻¹ outside of these convective regions (Wheeler and Hendon 2004; Kayano and Kousky 1999; Hendon and Salby 1994). However, embedded in this planetary-scale envelope are supercloud clusters associated with synoptic-scale convectively coupled equatorial waves, some of them propagating westward, as well as smaller-scale squall line clusters (Nakazawa 1988; Hendon and Liebmann 1994). The MJO modulates the monsoon circulation (Wang et al. 2005) and the tropical cyclone activity (Maloney and Hartmann 2000) and acts...
as a stochastic forcing for the El Niño–Southern Oscillation (ENSO; Zavala-Garay et al. 2005). The intraseasonal variability of the atmospheric flow is also characterized by the existence of teleconnection patterns from tropics to extratropics due to the propagation of barotropic Rossby wave trains that are responsible for the MJO impact on the circulation and predictability in the middle latitudes (Mo and Higgins 1998; Jones et al. 2004). Present-day atmospheric general circulation models (AGCMs) typically poorly represent the MJO and the associated intraseasonal variability of the atmospheric circulation (Sperber et al. 1997), and the reason for this poor performance is one of the greatest scientific challenges in atmospheric science. The goal of this work is to theoretically explore a particular physical mechanism that might be important for generating intraseasonal oscillations in the atmospheric flow: the diurnal cycle of convective heating.

Several theoretical models have been proposed throughout the last decades to explain the origin and dynamics of the MJO. They can be grouped into some categories depending on which physical mechanisms they are analyzing regarding their potential role in the MJO. The simplest theoretical model for the MJO is given by a planetary-scale response to a moving heat source with a velocity prescribed at the MJO phase speed through the linear shallow-water equations with the equivalent depth of the first baroclinic mode (Gill 1980; Chao 1987). This category of theories, in which the MJO is generated by independently existing forcing mechanisms, also includes the lateral forcing theory in which the MJO is forced by extratropical disturbances (Kiladis and Weickmann 1992; Matthews and Kiladis 1998) and the stochastic forcing mechanism proposed by Salby and Garcia (1987). Another category refers to models that explore the role of the interaction between large-scale equatorial wave dynamics and moist convection in the generation of the MJO. These models consider the MJO as an intrinsic mode that results from atmospheric instability and include the theories based on evaporation–wind feedback (Emanuel 1987; Neelin et al. 1987), linear and nonlinear wave convective instability (Chang and Lim 1988; Lau and Peng 1987), boundary layer frictional convective convergence (Wang and Rui 1990), and radiation instability (Raymond 2001), as well as the recent theoretical models for convective parameterizations carrying two baroclinic modes that take into account the multicloud nature of the convection organization (Mapes 2000; Khouider and Majda 2006; 2007; 2008; Majda et al. 2007). These recent theoretical models presented by Khouider and Majda (2006, 2007, 2008) and Majda et al. (2007) are shown to be able to explain some key observed features of the organization of convection in convectively coupled equatorial waves and in the MJO.

With regard to the nonlinear dynamics of the MJO, Majda and Biello (2004) and Biello and Majda (2005) developed asymptotic multiscale models in which the MJO is generated by the upscale transference of energy from thermally driven synoptic-scale equatorial waves. The Majda–Biello model is also based on two baroclinic modes and the upscale transport is directly related to the wave tilting due to the progressive deepening of convection. An observational evidence for this upscale transport of momentum from synoptic-scale to intraseasonal time scale motions can be found in Krishnamurti and Chakraborty (2005), who, using a dataset from the European Center for Medium-Range Weather Forecasts (ECMWF) reanalysis, performed a Fourier analysis and showed that during periods of intense activity of the MJO there is a significant flux of kinetic energy from synoptic-scale modes to modes with periods on the order of 30–60 days. Recent observational studies also demonstrate the modulation of synoptic-scale convectively coupled equatorial waves by the MJO (Mapes et al. 2006; Roundy 2008).

There are also some theoretical models that relate the intraseasonal oscillations to energy (amplitude) modulations associated with the weakly nonlinear dynamics of the large-scale atmospheric circulation. Kartashova and L’vov (2007) have studied resonant triad interactions among Rossby waves in a barotropic nondivergent model in spherical coordinates and pictured the intraseasonal oscillations in the atmosphere as a result of the energy exchanges within these triad interactions. In addition, Raupp and Silva Dias (2006) studied the resonant triad interactions among waves in the equatorial $\beta$-plane shallow-water equations and pointed out the importance of the inertio-gravity modes for the generation of low-frequency (intraseasonal and/or even longer-term) oscillations in the atmospheric model adopted. Because the inertio-gravity waves are directly linked to diurnal variations of tropical deep convection (Yang and Slingo 2001), this result of Raupp and Silva Dias suggests that the diurnal cycle of tropical convection might play an important role in the dynamics of the intraseasonal oscillations in the atmospheric flow. In fact, Misra (2005) showed that the intraseasonal variance of the outgoing longwave radiation (OLR) over convective regions of South America is better simulated when the coarse-resolution National Centers for Environment Prediction (NCEP) reanalysis or the Center for Ocean–Land–Atmosphere Studies (COLA) AGCM integrations are downscaled to take into account the effects of the diurnal cycle of convection over some regions of the continent. This indicates that there might be a certain mechanism of interaction between the diurnal cycle of tropical convection over continental areas and the intraseasonal variability of the atmospheric circulation.
2. Theoretical development

The model equations adopted here are the same as those used in Raupp et al. (2008) governing equatorially trapped large-scale perturbations of tropospheric motions embedded in a motionless, horizontally homogeneous, hydrostatic and stably stratified background atmosphere. The only difference is the inclusion of the diabatic term in the right-hand side of the thermodynamics equation. This model is governed by the following diabatic version of the primitive equations with the equatorial $\beta$-plane approximation:

$$\frac{\partial u}{\partial t} + \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + F \frac{\partial u}{\partial p} \right) - v \frac{\partial \phi}{\partial x} = 0, \quad (2.1a)$$

$$\frac{\partial v}{\partial t} + \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + F \frac{\partial v}{\partial p} \right) + v \frac{\partial \phi}{\partial y} = 0, \quad (2.1b)$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial y} + F \frac{\partial \phi}{\partial p} = 0, \quad \text{and}$$

$$\frac{\partial \phi}{\partial t} + \left( \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + F \frac{\partial \phi}{\partial p} \right) + F \frac{\partial \phi}{\partial p} = \frac{F \bar{\omega}}{\bar{\rho}} - \frac{\kappa}{\bar{\rho}^2} \bar{\phi} Q. \quad (2.1d)$$

In the equations above, $\phi$ is the geopotential perturbation; $(u, v)$ are the velocity perturbations in the $(x, y)$ coordinate directions; $\omega = \text{Dp}/\text{Dt}$ is the vertical velocity in pressure coordinates; $Q$ represents the prescribed heat source; and $\bar{\rho} = \rho_0/(\bar{\rho}_0 \beta^2 L^4)$, where $\bar{\rho}_0 = \rho(\bar{\rho})$ refers to the background density at the earth’s surface. The quantity $\epsilon = \text{Ro} = U(\beta L^2)$ is the equivalent for the equatorial region of the Rossby number in midlatitudes and corresponds to a measure of the typical amplitude of the perturbations associated with the horizontal flow; $\beta$ is the gradient of the Coriolis parameter at the equator and is assumed here to be a constant; $F = \Theta/\text{Ro}$, where $\Theta = \Omega/\beta L \rho_0$ is a measure of the typical amplitude associated with the vertical motion; $\kappa = \text{R}/C_p$, with $R$ and $C_p$ being the gas constant for dry air and the thermal capacity of dry air at constant pressure, respectively; and $\sigma = \bar{\sigma}(\beta^2 L^4)$, where $\sigma$ is the static stability parameter of the background atmosphere, given by

$$\sigma = \frac{R}{p^2} \left( \frac{RT}{p^2 C_p} - \frac{dT}{dp} \right). \quad (2.2)$$

In (2.2) above, $\overline{T} = \overline{T}(p')$ represents the background temperature. The static stability parameter is positive for a stably stratified atmosphere and will be assumed here to be a constant with its typical tropospheric value of $\sigma = 2 \times 10^{-6} \text{ m}^2 \text{ s}^{-2} \text{ kg}^{-1}$. Equations (2.1) are non-dimensionalized as follows:

$$(u', v') = U(u, v), \quad (x', y') = L(x, y), \quad t' = (1/\beta L)t,$$

$$p' = \rho_0 p, \quad \omega' = \Omega \omega, \quad \phi' = \beta L^2 U \phi, \quad Q' = \Omega \rho_0^{-1} Q \quad (2.3)$$
The boundary conditions for Eqs. (2.1) are identical to those used in Raupp et al. (2008); that is, we have assumed periodic solutions in the x direction, bounded solutions as |y| tends to infinity, and rigid lid boundary conditions in the vertical direction. As in Raupp et al. (2008), we have performed a Taylor expansion in the vertical boundary conditions around the isobaric surfaces $p' = p_0$ and $p'' = p_T$ close to the earth’s surface and to the top of the troposphere, respectively. The final expressions for the vertical boundary conditions can be written as

$$\frac{\partial \phi}{\partial t} + \varepsilon \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + F \omega \frac{\partial \phi}{\partial p} \right) - \frac{F \omega}{\bar{p}_0} + \varepsilon \bar{p}_0 \phi(x, y, 1, t) \frac{\partial \phi}{\partial p} \left( \frac{\partial \phi}{\partial t} + \varepsilon \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + F \omega \frac{\partial \phi}{\partial p} \right) - \frac{F \omega}{\bar{p}} \right) = 0 \quad \text{at}$$

$$p = 1, \quad \text{and}$$

$$\frac{\partial \phi}{\partial t} + \varepsilon \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + F \omega \frac{\partial \phi}{\partial p} \right) - \frac{F \omega}{\bar{p}_T} - \varepsilon \bar{p}_T \phi(x, y, \bar{p}_T, t) \frac{\partial \phi}{\partial p} + \varepsilon \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + F \omega \frac{\partial \phi}{\partial p} \right) = 0 \quad \text{at}$$

$$p = \bar{p}_T,$$

where $\bar{p}_T = p_T / p_0$, $\bar{p} = \beta^2 L^4 / p_0$, and $\bar{p}_T = \bar{p}(\bar{p}_T)$. Equations (2.1c) and (2.1d) can be combined to give

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial p} \left( \frac{1}{\sigma} \frac{\partial \phi}{\partial p} \right) + \varepsilon \left( \frac{\partial}{\partial p} \left( \frac{\partial \phi}{\partial p} \right) + \frac{\partial}{\partial y} \frac{\partial \phi}{\partial y} + F \omega \frac{\partial^2 \phi}{\sigma^2 \partial p^2} + \frac{F \omega}{\sigma} \left( 1 - \kappa \right) \frac{\partial \phi}{\partial p} \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -F \kappa \sigma \frac{\partial}{\partial p} \left( \frac{Q}{\sigma^2} \right)$$

(2.5)

Because in the present work we are interested in studying resonant triads that resonate with the diabatic forcing $Q$, we look for solutions of system (2.1)–(2.5) in the weakly nonlinear and weak forcing regime. Thus, we have assumed that the heat source $Q$ is of $O(\varepsilon)$ magnitude and both $Ro$ and $\Theta$ are small and proportional to each other; that is, (i) $0 < \varepsilon \ll 1$ and (ii) $F$ is a constant, although it can be small for large-scale atmospheric disturbances. In fact, the assumption that $F$ is fixed is necessary to allow all the possible equatorial wave types to formally represent the leading-order solution in the limit $\varepsilon \to 0^+$, yet for large-scale atmospheric perturbations with $U \sim 5$ m s$^{-1}$, $L \sim 1500$ km, $\Omega \sim 0.1$ Pa s$^{-1}$, and $\beta = 2.3 \times 10^{-11}$ m$^{-1}$ s$^{-1}$, it follows that $\varepsilon \approx 0.1$ and $F \approx 0.3$. Therefore, we have kept $F$ fixed in our formal asymptotic development and then used the small value $F \approx 0.3$ emerging from the conservative choice of $\varepsilon \approx 0.1$ to physically interpret the solutions in section 4. Similarly, the assumption that the heat source $Q$ is of $O(\varepsilon)$ is needed in order for the free linear equatorial waves to formally represent the leading-order solution in the limit $\varepsilon \to 0^+$. In addition, with the conservative choice of $\varepsilon \approx 0.1$, the prescribed heating adopted in this paper yields solutions with physically reasonable magnitudes, as will be shown in section 4.

Therefore, with the assumptions described above, to seek solutions of system (2.1)–(2.5) in the limit $\varepsilon \to 0^+$ we have adopted the multiple-time-scale asymptotics in which the dependent variables are assumed to have uniformly valid asymptotic expansions of the forms shown below:

$$u = u^{(0)}(x, y, p, t, \tau) + \varepsilon u^{(1)}(x, y, p, t, \tau) + O(\varepsilon^2),$$

$$v = v^{(0)}(x, y, p, t, \tau) + \varepsilon v^{(1)}(x, y, p, t, \tau) + O(\varepsilon^2),$$

$$\phi = \phi^{(0)}(x, y, p, t, \tau) + \varepsilon \phi^{(1)}(x, y, p, t, \tau) + O(\varepsilon^2), \quad \text{and}$$

$$\omega = \omega^{(0)}(x, y, p, t, \tau) + \varepsilon \omega^{(1)}(x, y, p, t, \tau) + O(\varepsilon^2).$$

(2.6)

Substituting the ansatz (2.6) into the governing equations (2.1a,b), (2.4), and (2.5), it follows that the leading-order solution is written according to

$$\left[ u^{(0)}(x, y, p, t, \tau) \right] = \sum_a A_a(r) \xi_a(y) e^{ikx + imy + iG_a(p)} + \text{c.c.},$$

where c.c. denotes the complex conjugate of the previous term and $\tau = \varepsilon t$ is the long time scale. In (2.7), the subscript $a = (m, k, n, r)$ refers to a particular expansion mode characterized by a vertical mode $m$, a zonal wavenumber $k$, a meridional index $n$ distinguishing the meridional structure of the equatorial waves, and the wave type represented by $r$: $r = 1$ for Rossby waves (RWs), $r = 2$ for westward propagating inertia-gravity waves (WGWs), and $r = 3$ for eastward propagating inertia-gravity waves (EGWs). The mixed Rossby–gravity waves (MRGWs) are associated with the $n = 0$ meridional index and are included in the $r = 1$ (for $k > 2^{-1/2}$) and $r = 2$ (for $k < 2^{-1/2}$) categories. The Kelvin waves are represented by $n = -1$ and $r = 3$. 
In (2.7), \( G_a(p) \) represents the vertical structure functions that distinguish the vertical structure of the linear eigenmodes. These vertical structure functions are the eigenfunctions of the following Sturm–Liouville problem:

\[
\frac{d}{dp} \left( \frac{1}{\sigma} \frac{dG}{dp} \right) + \frac{1}{c^2} G = 0, \quad \text{and} \quad \frac{dG}{dp} + \sigma \tilde{\sigma} G = 0 \quad \text{at} \quad p = 1 \quad \text{and} \quad p = \tilde{p}_T, \tag{2.8a}
\]

\[
\frac{dG}{dp} + \sigma \tilde{\sigma} G = 0 \quad \text{at} \quad p = 1 \quad \text{and} \quad p = \tilde{p}_T, \tag{2.8b}
\]

where \( c \) is the separation constant. For the constant static stability parameter \( \sigma \) assumed in this work, the eigenfunctions \( G_m(p) \) are given by a combination of sines and cosines and the eigenvalues \( \lambda_m = \sqrt{\sigma} c_m \) are determined by the following transcendental equation:

\[
\lambda^2 \sin[(1 - \tilde{p}_T)\lambda] - \lambda \sigma (\tilde{p}_0 - \tilde{p}_T) \cos[(1 - \tilde{p}_T)\lambda] + \sigma^2 \tilde{p}_T \tilde{p}_0 \sin[(1 - \tilde{p}_T)\lambda] = 0. \tag{2.9}
\]

The background density profile adopted in this work is determined from the background temperature profile by using the ideal gas law for dry air, with the temperature background profile being determined by Eq. (2.2) for constant \( \sigma \). The illustration of the background temperature and density profiles for \( \sigma = 2 \times 10^{-6} \text{ m}^4 \text{ s}^2 \text{ kg}^{-2} \) adopted in this work can be found in Fig. 1 of Raupp et al. (2008). The \( m = 0 \) mode is associated with the first \( \lambda \neq 0 \) root of (2.9) and corresponds to the barotropic mode because its eigenfunction has no phase inversion and is almost constant throughout the troposphere. The \( m > 0 \) modes are usually referred to as internal or baroclinic modes and correspond to the oscillations associated with the rigid lid boundary conditions. Figure 1 illustrates the vertical structure eigenfunctions \( G_m(p) \) for the first three vertical eigenmodes.

The meridional structure functions \( \xi_a(y) = [u_a(y), v_a(y), \phi_a(y)]^T \) were first obtained by Matsuno (1966) and are given by

\[
\xi_a(y) = \begin{bmatrix} u_a(y) \\ v_a(y) \\ \phi_a(y) \end{bmatrix} = \begin{bmatrix} c_a & 0 & 0 \\ 0 & c_a & 0 \\ 0 & 0 & c_a^2 \end{bmatrix} \begin{bmatrix} 1/2 (\tilde{\sigma}_a - \tilde{k}_a) H_{n+1} \left( \frac{y}{\sqrt{c_a}} \right) + n(\tilde{\sigma}_a + \tilde{k}_a) H_{n-1} \left( \frac{y}{\sqrt{c_a}} \right) \\ i(\tilde{\sigma}_a^2 - \tilde{k}_a^2) H_n \left( \frac{y}{\sqrt{c_a}} \right) \\ 1/2 (\tilde{\sigma}_a - \tilde{k}_a) H_{n+1} \left( \frac{y}{\sqrt{c_a}} \right) - n(\tilde{\sigma}_a + \tilde{k}_a) H_{n-1} \left( \frac{y}{\sqrt{c_a}} \right) \end{bmatrix} \frac{e^{-y^2/2c_a}}{\|\xi_a\|} \tag{2.10}
\]

\[
\xi_a(y) = \begin{bmatrix} c_a & 0 & 0 \\ 0 & c_a & 0 \\ 0 & 0 & c_a^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{e^{-y^2/2c_a}}{(2\sqrt{\pi})^{3/2}}. \tag{2.12}
\]

In the leading-order solution represented by (2.7) are also degenerate eigenmodes associated with the eigenfrequency \( \sigma = 0 \). These modes have a \( k = 0 \) zonal structure and are characterized by a perfect geostrophic balance and the absence of a meridional circulation \( (v_a = 0) \) (Silva Dias and Schubert 1979). The \( k = 0 \) Kelvin modes are also included in this category. The meridional structure functions \( \xi_a(y) \) associated with \( \sigma_a = 0 \) are given by

\[
\xi_{m,0,n,1}(y) = \lim_{k \to 0} \begin{bmatrix} u_a(y) \\ v_a(y) \\ \phi_a(y) \end{bmatrix} = \begin{bmatrix} c_a & 0 & 0 \\ 0 & c_a & 0 \\ 0 & 0 & c_a^2 \end{bmatrix} \begin{bmatrix} n \left( 2(n+1) H_{n-1} \left( \frac{y}{\sqrt{c_a}} \right) - H_{n+1} \left( \frac{y}{\sqrt{c_a}} \right) \right) \\ 0 \\ n \left( -2(n+1) H_{n-1} \left( \frac{y}{\sqrt{c_a}} \right) - H_{n+1} \left( \frac{y}{\sqrt{c_a}} \right) \right) \end{bmatrix} \frac{e^{-y^2/2c_a}}{[2^n n! \sqrt{\pi} 4n(n+1)(2n+1)]^{1/2}} \tag{2.13}
\]
The finite amplitude and forcing corrections for the leading-order solution (2.7) are obtained from the \( O(\varepsilon) \) problem, which is governed by the linear and inhomogeneous system

\[
\mathcal{A} \xi^{(1)} = N^{(0)} + S, \tag{2.14}
\]

with the vertical boundary conditions expressed according to

\[
\frac{\partial \phi^{(1)}}{\partial t} - \frac{F \omega^{(1)}}{\bar{p}_0} = - \left[ \frac{\partial \phi^{(0)}}{\partial \tau} + u^{(0)} \frac{\partial \phi^{(0)}}{\partial x} + v^{(0)} \frac{\partial \phi^{(0)}}{\partial y} + F \omega^{(0)} \frac{\partial \phi^{(0)}}{\partial \rho} - \bar{p}_0 \phi^{(0)} \left( \frac{\partial \phi^{(0)}}{\partial t} - \frac{F \omega^{(0)}}{\bar{p}} \right) \right] \quad \text{at } p = 1 \tag{2.15a}
\]

and

\[
\frac{\partial \omega^{(1)}}{\partial t} - \frac{F \omega^{(1)}}{\bar{p}} = - \left[ \frac{\partial \phi^{(0)}}{\partial \tau} + u^{(0)} \frac{\partial \phi^{(0)}}{\partial x} + v^{(0)} \frac{\partial \phi^{(0)}}{\partial y} + F \omega^{(0)} \frac{\partial \phi^{(0)}}{\partial \rho} - \bar{p} \phi^{(0)} \left( \frac{\partial \phi^{(0)}}{\partial t} - \frac{F \omega^{(0)}}{\bar{p}} \right) \right] \quad \text{at } p = \bar{p}. \tag{2.15b}
\]

In (2.14), \( \xi^{(1)} = [u^{(1)}, v^{(1)}, \phi^{(1)}]^T \), \( \mathcal{A} \) is the linear operator given by

\[
\mathcal{A} = \begin{bmatrix}
\frac{\partial}{\partial t} & -y & \frac{\partial}{\partial x} \\
y & \frac{\partial}{\partial t} & \frac{\partial}{\partial y} \\
-\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial t} \frac{1}{\partial \rho} \frac{\partial}{\partial \rho}
\end{bmatrix} \tag{2.16}
\]

and the vector \( N^{(0)} \) contains the leading-order contribution of the nonlinear terms in the governing Eqs. (2.1a), (2.1b), and (2.5) and the long time evolution of \( u^{(0)}, v^{(0)}, \) and \( \phi^{(0)} \). The forcing term \( S \) in (2.14) is defined as

\[
S = \begin{bmatrix}
0 \\
0 \\
-F \kappa \frac{\partial}{\partial \rho} \left( \frac{Q}{\sigma p} \right)
\end{bmatrix}. \tag{2.17}
\]
The heat source $Q$ adopted in this work is specified to roughly simulate the typical convective heating over the Amazon region during the austral summer period, which is characterized by a strong diurnal variation (Silva Dias et al. 1987). Thus, the heat forcing $Q$ is defined as

$$ Q(x, y, p, t) = e Q_0 H(x, y) q(p) f(t), $$

where

$$ H(x, y) = \exp\left[ -\left( \frac{x - x_0}{a} \right)^2 - \left( \frac{y - y_0}{a} \right)^2 \right] $$

and

$$ q(p) = \sin\left[ \pi \left( \frac{p - p_T}{p_0 - p_T} \right) \right], $$

with $y_0$ and $x_0$ corresponding approximately to 15°S and 65°W, respectively, and $a = 0.53$ (≈800 km for $L = 1500$ km). These values are the same as those assumed by Silva Dias et al. (1983, 1987) to simulate the impact of the Amazonian heat source on the large-scale atmospheric circulation. The time-dependent part of the thermal forcing $Q$, given by the function $f(t)$, is a half-sine during half a day (in dimensionless units) and zero otherwise, repeating the cycle forever to simulate the diurnal variation of convection. The parameter $Q_0$ in (2.18a) is chosen to result in a total heating ratio of 20 K day$^{-1}$ integrated throughout the entire vertical column over $(x_0, y_0)$ or a precipitation ratio of 80 mm day$^{-1}$. These values are characteristic of typical convective episodes throughout the Amazon region (see, e.g., Halverson et al. 2002; Anagnostou and Morales 2002; Laurent et al. 2002). The function $f(t)$ in this case can be written in terms of the following Fourier cosine series:

$$ f(t) = \sum_{j=0}^{\infty} \frac{2}{\pi(1 - 4j^2)} \cos\left( \frac{2\pi j t}{\Delta} \right), $$

where $\Delta$ is the duration of a day in dimensionless units.

The vertical dependence of heating $q(p)$ simulates the typical vertical profile of the tropical deep convection heating (DeMaria 1985; Silva Dias and Bonatti 1985). The function $q(p)$ in (2.18c) is normalized so that $\int_{-\infty}^{\infty} q(p) = 1$.

Therefore, with the specification of the heat source in terms of (2.18) and (2.19), the terms $N^{(0)}$ and $S$ appear as periodic forcings in (2.14). Consequently, the case of interest in this paper is when both of these forcings resonate with one of the linear eigenmodes. In this case, a necessary (but, as will be shown later, not sufficient) condition to eliminate the secular solutions is given by

$$ \xi^{(a)}_y(y) = \left[ \begin{array}{c} u^{(a)}_y(y) \\ v^{(a)}_y(y) \\ -\phi^{(a)}_y(y) \end{array} \right], $$

where the meridional structure function $\xi^{(a)}_y(y)$ can be written in terms of the meridional structure functions of the original $O(1)$ problem according to

$$ \xi^{(a)}_y(y) = \left[ \begin{array}{c} u^{(a)}_y(y) \\ v^{(a)}_y(y) \\ -\phi^{(a)}_y(y) \end{array} \right]. $$

Integrating by parts the left-hand side of (2.20), using (2.15) and the thermodynamics Eq. (2.1d) for $O(\varepsilon)$, as well as the continuity Eq. (2.1c) and the boundary conditions of the leading-order solution, it is possible to rewrite (2.20) as

$$ \frac{\mathbf{F} \kappa}{\sigma p} \nabla Q \right) \frac{1}{p_T} dt \ dy \ dx = \lim_{\chi \to \infty} \frac{1}{2\chi L_x} \int_{-L_y}^{L_y} \left[ \int_{-\infty}^{\infty} \phi^{(a)}_y(y) e^{-ikx - im\gamma} G_a(p) \left( -\frac{\partial \phi^{(0)}}{\partial t} + \frac{1}{\sigma p^2} \partial^2 \phi^{(0)} - \Lambda \partial \phi^{(0)} \left( \frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right) \right) \right] \ dx \ dy \ \ (2.23) \]
where $\tilde{\Gamma} = [(1 - \kappa) / \rho] \tilde{p}^2 + \tilde{\sigma} \tilde{p}^2 (1 + \Lambda) - \tilde{\Lambda} (dp/\rho, dp)$ and $\Lambda$ is defined according to

$$\Lambda = \begin{cases} 
1, & \text{if } p = 1, \\
-1, & \text{if } p = \bar{p}_T.
\end{cases}$$

Inserting (2.7), (2.18), and (2.19) into (2.23) and holding the slowly varying wave amplitudes $A_{\delta(t)}$ constant during the integration in the fast time scale $t$, it follows that only the triads satisfying the resonance conditions $\sigma_a = \pm \sigma_b \pm \sigma_c$, $k_a = \pm k_b \pm k_c$, and $n_a + n_b + n_c = \text{odd}$ contribute to the integrals in (2.23). Thus, if one expresses $u^{(0)}$, $v^{(0)}$, and $\phi^{(0)}$ in terms of (2.7) truncated in such a way to consider only a triad of modes $a$, $b$, and $c$ satisfying the resonance conditions $\sigma_a = \sigma_b + \sigma_c$, $k_a = k_b + k_c$, and $n_a + n_b + n_c = \text{odd}$, Eq. (2.23) can be written as

$$c^2_a \frac{dA_a}{d\tau} = A_y A_c \eta^{bc} + q_a g \left( \sigma_a \pm \frac{2\pi f_1}{\Delta} \right),$$  \hspace{1cm} (2.24a)

$$c^2_b \frac{dA_b}{d\tau} = A_y A_c \eta^{bc} + q_b g \left( \sigma_b \pm \frac{2\pi f_2}{\Delta} \right), \hspace{1cm} (2.24b)
$$

$$c^2_c \frac{dA_c}{d\tau} = A_y A_c \eta^{ab} + q_c g \left( \sigma_c \pm \frac{2\pi f_3}{\Delta} \right). \hspace{1cm} (2.24c)$$

In (2.24), $f_1$, $f_2$, and $f_3$ refer to three arbitrary harmonics of the time-dependent part $f(t)$ of the heat source and $\eta_a^b$, $\eta_b^c$, and $\eta_c^a$ are the nonlinear coupling coefficients given by

$$\eta_a^b = - \langle B \cdot \xi_a(y) \rangle - \int_{-\infty}^{+\infty} \phi_a(y) \left[ \phi_b i\sigma_c \phi_c \left( \frac{\tilde{p}}{c_b} + \tilde{\Gamma} \right) - \tilde{\Lambda} \phi_b \left( ik_c u_c + \frac{dv_c}{dy} \right) + CP \right] \frac{G_a(p)G_b(p)G_c(p)}{||G_a||^2} \left( \tilde{p} \right) dy$$

with the inner product $\langle \cdot \rangle$ defined according to (2.21). The vector $B$ in (2.25a) is defined by

$$B = \left[ \begin{array}{c}
\left( u_a i\sigma_c u_c + v_b \frac{du_c}{dy} \right) \alpha_{ab}^{bc} + i\sigma_b \phi_b u_b \mu_{ab}^{bc} - \left( ik_a u_a + \frac{dv_a}{dy} \right) u_a \theta_{ab}^{bc} + CP \\
\left( u_a i\sigma_c u_c + v_b \frac{du_c}{dy} \right) \alpha_{ab}^{bc} + i\sigma_b \phi_b u_b \mu_{ab}^{bc} - \left( ik_a u_a + \frac{dv_a}{dy} \right) v_a \theta_{ab}^{bc} + CP \\
\left( u_a i\sigma_c u_c + v_b \frac{dv_c}{dy} \right) \phi_a \phi_c \Psi_{ab}^{bc} - \left( ik_a u_a + \frac{dv_a}{dy} \right) \phi_a s_{ab}^{bc} + CP
\end{array} \right], \hspace{1cm} (2.25b)$$

where $\alpha_{ab}^{bc}$, $\mu_{ab}^{bc}$, $\theta_{ab}^{bc}$, $\Psi_{ab}^{bc}$, and $s_{ab}^{bc}$ represent the interaction coefficients among the vertical eigenfunctions of modes $a$, $b$, and $c$ and are expressed by

$$\alpha_{ab}^{bc} = \frac{1}{||G_b||} \int_{\tilde{p}_T}^{\tilde{p}_T} G_b(p) G_c(p) \frac{dG_c}{dp} \frac{G_a(p)}{dp}, \hspace{1cm} \mu_{ab}^{bc} = \frac{1}{||G_a||} \int_{\tilde{p}_T}^{\tilde{p}_T} \frac{dG_a}{dp} G_b(p) \frac{dG_c}{dp} \frac{G_c(p)}{dp}.$$

$$\theta_{ab}^{bc} = \frac{1}{||G_a||} \int_{\tilde{p}_T}^{\tilde{p}_T} \frac{dG_a}{dp} \left[ \int_{\tilde{p}_T}^{\tilde{p}_T} G_b(p) \frac{dG_c}{dp} \frac{G_c(p)}{dp} \right] G_b(p) \frac{dG_c}{dp} \frac{G_c(p)}{dp}, \hspace{1cm} \Psi_{ab}^{bc} = \frac{1}{||G_a||} \int_{\tilde{p}_T}^{\tilde{p}_T} \left( \frac{dG_b}{dp} - \frac{1}{\tilde{p}_T} \frac{dG_b}{dp} \frac{dG_c}{dp} \right) G_b(p) \frac{dG_c}{dp} \frac{G_c(p)}{dp}, \hspace{1cm} (2.25c)$$

$$s_{ab}^{bc} = \frac{1}{||G_a||} \int_{\tilde{p}_T}^{\tilde{p}_T} \left[ \int_{\tilde{p}_T}^{\tilde{p}_T} \frac{dG_b}{dp} + \frac{1}{\tilde{p}_T} \frac{dG_b}{dp} \frac{dG_c}{dp} \right] G_b(p) \frac{dG_c}{dp} \frac{G_c(p)}{dp}.$$
The term CP in (2.25) indicates cyclical permutations between the superscripts $b$ and $c$ and $\|G_x\|$ corresponds to the norm of the vertical eigenfunctions. The coefficients $q_a, q_b, \text{ and } q_c$ in (2.24) refer to the projection of the diabatic term in the right-hand side of (2.5) onto modes $a, b, \text{ and } c$, respectively, given by

$$ q_a = \frac{-1}{2L} \int_{-L}^{L} \left\{ \int_{-\infty}^{\infty} \frac{Q_0}{\pi(1-4^2)} \left[ \frac{1}{\|G_x\|^2} \int_{p_1}^{p_2} \frac{\kappa \gamma}{\sigma p} \left( \frac{q}{\sigma p} \right) G_a(p) \, dp \right] H(x,y) \phi_a(y) \, dy \right\} e^{i k_x x} \, dx $$

In (2.24), the function $g$ contains the limit of the fast time integral involving the forcing term and is defined according to

$$ g(\delta) = \lim_{\delta \to 0} \frac{1}{\sqrt{\Delta}} \int_0^\infty e^{\delta t} \, dt = \begin{cases} 1, & \text{for } \delta = 0, \\ 0, & \text{for } \delta \neq 0. \end{cases} (2.27) $$

### 3. Determination of possible resonant triads

As shown in Eqs. (2.24) and (2.27), in the small amplitude and weak forcing limit ($\gamma \ll 1$), the diurnally varying heat source adopted in this work affects the energy exchanges among the waves in a resonant triad if and only if at least one of the triad components resonates with one of the harmonics of the forcing. In this sense, the goal of this section is to find possible triads of waves satisfying the relations $\sigma_a = \sigma_b = \sigma_c, \, k_a = \pm k_b = \pm k_c, \text{ and } n_a + n_b + n_c = \text{odd}$ that can be potentially affected by the diurnal heat source. It is important to mention that because of the discrete spectrum of zonal wavenumbers that results from the periodic boundary condition in the $x$ direction, the resonance condition for the time frequencies is not easily satisfied for dispersive waves, making its occurrence the exception instead of the rule. This is also the case for the resonance relation $\sigma_a = \pm 2\pi j/\Delta$ involving a particular wave and a harmonic of the forcing. Nevertheless, exact resonances are difficult to be satisfied in practice, near-resonances satisfy the relation $\sigma_a = \sigma_b = \sigma_c = \delta = O(\epsilon)$, which is the actual condition for a significant interaction involving three quantized wavenumbers to take place at $O(\epsilon)$ [for a further discussion of this point, see Bretherton (1964)]. Similarly, in the weak forcing limit, the resonance requirement between a harmonic of the diurnal heat source and one of the wave modes may be relaxed so that the relation $\sigma_a = \pm 2\pi j/\Delta = \delta = O(\epsilon)$ is the actual condition for a particular wave to be significantly excited by the prescribed forcing.

As demonstrated in Raupp et al. (2008), the resonant triads whose wave components satisfy the relation $\lambda_a = \lambda_b \pm \lambda_c$ for their vertical eigenvalues have the most significant coupling among their vertical structure eigenfunctions and consequently undergo the most significant interactions, although this resonance condition is no longer excluding; that is, for the resonant triads whose wave components do not satisfy this condition, their coupling coefficients are small but not zero. On the other hand, the vertical profile of the heat source given by (2.18c) fully projects onto the first baroclinic mode (DeMaria 1985). In addition, a significant portion of the energy in the atmosphere is in equivalent barotropic Rossby modes that are associated with global teleconnection patterns that characterize the low-frequency variability of the atmospheric circulation (Mo and Higgins 1998 and references therein). Therefore, we will focus our analysis here on resonant triads containing two first baroclinic equatorial wave modes and one barotropic Rossby mode.

Table 1 shows the first harmonic frequencies associated with the time-dependent part of the heat source. The first harmonic ($j = 0$) of the forcing has a zero time frequency and consequently is resonant with the $k = 0$ geostrophic modes. Raupp et al. (2008) show that the first baroclinic $k = 0$ geostrophic modes with an odd meridional index act as catalyst modes in the resonant triads containing a first baroclinic mixed Rossby–gravity wave mode and a barotropic Rossby mode with meridional index $n = 2$, enabling energy to be efficiently exchanged between these modes but without being affected by the propagating waves. On the other hand, as the transient harmonics $j \geq 1$ have periods equal or shorter than 1 day, they are only resonant with high-frequency equatorial waves (Kelvin and inertio-gravity waves). The Kelvin waves are nondispersive and the inertio-gravity waves tend to become nondispersive as $k \to \pm \infty$. Because in a nondispersive wave pack all the components are resonant with each other, their interaction is rather complex and produces effective energy cascades toward small-scale wave modes. As a consequence, with regard to resonant triads involving two high-frequency first baroclinic wave modes and one
barotropic Rossby mode, we have focused our attention in this work only on triads composed of planetary-scale inertio-gravity waves (zonal wavenumber $l$ in the range $-4 \leq l \leq 4$ or $-1 \leq k \leq 1$, with $k = 2\pi l/L_x$) because this refers to the most dispersive range in the inertio-gravity wave dispersion relation and thus constitutes the most likely set of inertio-gravity waves to undergo sparse resonant triad interactions. On the other hand, in our search for such resonant triads composed of two planetary-scale first baroclinic inertio-gravity wave modes and one barotropic Rossby mode, we have found that only one of the inertio-gravity modes is nearly resonant with the forcing, since for both the inertio-gravity modes to be nearly resonant with the forcing, the frequency of the barotropic Rossby mode would have to be smaller than the upper limit tolerance $\delta = O(\varepsilon)$ within which the resonance conditions must be satisfied.

Therefore, for the resonant triads containing two first baroclinic equatorially trapped wave modes and one barotropic Rossby mode analyzed here, the spectrum of the diurnal heat source is such that only one of the triad components is resonant (or nearly so) with the forcing. Consequently, to illustrate the role of the diurnal forcing in these interactions in a simplified fashion, two classes of triads have been analyzed. The first one refers to the triads composed of a $k = 0$ first baroclinic geostrophic mode, which is resonant with the stationary component of the diurnal heat source, a first baroclinic mixed Rossby–gravity wave mode and a barotropic Rossby mode with meridional index $n = 2$. The other class corresponds to triads composed of two planetary-scale first baroclinic inertio-gravity waves in which the highest-frequency wave resonates with a transient harmonic ($j \approx 1$) of the forcing. Examples of nearly resonant triads of these two categories are displayed in Table 2. Table 2 shows the triad components and their respective eigenfrequencies and coupling coefficients. The resonant triads shown in Table 2 were found setting $\mathcal{P} = 2 \times 10^{-6}$ m$^4$ s$^{-2}$ kg$^{-1}$, $L = 1.5 \times 10^6$ m, $\beta = 2.3 \times 10^{-11}$ m$^{-1}$ s$^{-1}$, $p_0 = 1000$ hPa, and $p_T = 100$ hPa. These values are fixed in all the calculations displayed hereafter in this paper.

Because in the first class of triads the mode resonating with the forcing is only a catalyst component and in the second class the mode that resonates with the forcing is the most energetically active member (the one with the highest absolute value coupling coefficient), it is expected that the stationary component and the transient harmonics of the diurnal heat source impact the resonant energy exchanges among the waves in different ways because of the different triad components with which they resonate. This will be addressed in the next section.

### 4. Dynamics of the forced resonant interactions

To illustrate the role of the diabatic heat forcing in the resonant wave interactions in a simplified fashion and, consequently, to shed some light on the possible effect of the diurnal cycle of tropical convection on the low-frequency variability of the atmospheric circulation, in this section we show results of the integration of the asymptotic reduced Eqs. (2.24) for the selected triads presented in Table 2. As discussed in the previous section, for these resonant triads the spectrum of the diurnally varying thermal forcing given by (2.19) is such that only one of the triad components is resonant or nearly resonant with the heat source.

The $j = 0$ harmonic of the time-dependent part $f(t)$ of the heat source is resonant with the first baroclinic zonally symmetric geostrophic modes because these modes have a zero time frequency ($\sigma = 0$). As illustrated in Table 2,
in a resonant triad involving a zonally symmetric geostrophic mode and two dispersive equatorial waves, the coupling coefficient of the geostrophic mode is zero. As a consequence, for the resonant triads 1 and 2 of Table 2, the asymptotic reduced Eqs. (2.24) can be written as

\[ \frac{dA_j}{dt} = A_j A_k \hat{q}_b c_j, \quad (4.1a) \]

\[ \frac{dA_b}{dt} = A_a A_c \hat{q}_b c_j, \quad (4.1b) \]

\[ \frac{dA_c}{dt} = q_c. \quad (4.1c) \]

In this case, Eq. (4.1c) decouples and integrates exactly, yielding the amplitude of the first baroclinic \( k = 0 \) geostrophic mode (mode \( c \)) as a linear function of the long-time scale; that is, \( c_j^2 \dot{A}(\tau) = c_j^2 A_j(t=0) + q_c \tau \). The remaining two-by-two system formed by Eqs. (4.1a) and (4.1b) in turn becomes a harmonic oscillator with the frequency increasing linearly with \( \tau \). Thus, one notices from Eqs. (4.1) in this case that for the resonant triads 1 and 2 of Table 2, the heat source resonantly excites the zonally symmetric geostrophic mode, whose energy (amplitude) grows in time without any bound. As a consequence, as the convective forcing excites these stationary modes, the energy exchanges between the resonant first baroclinic mixed Rossby–gravity and the stationary modes, the energy exchanges become so fast that the asymptotic reduced Eqs. (2.24) can be written as

\[ \frac{dA_j}{dt} = A_j A_k \hat{q}_b c_j, \quad (4.2a) \]

\[ \frac{dA_b}{dt} = A_a A_c \hat{q}_b c_j, \quad (4.2b) \]

\[ \frac{dA_c}{dt} = A_a A_b \hat{q}_b c_j. \quad (4.2c) \]

In (4.2), the mode resonating with the heating is the highest absolute frequency mode of the triad, which is specified as mode \( a \). To illustrate the dynamics of the resonant interactions in triads 3 and 4 of Table 2 in the presence of the diurnal heat source, we have numerically integrated Eqs. (4.2) by using the Matsuno (predictor–corrector) scheme at the first step with a time step of \( \Delta t/6 \) and the leapfrog scheme at the remaining integration steps with a time step of \( \Delta t \). In most of our numerical integrations we have set \( \Delta t = 10 \) min.

Results of numerical integrations of (4.2) are illustrated in Figs. 2 and 3, which show the time evolution of the modes’ energy \( c_j^2 |A_j|^2 \), \( j = a, b, \) and \( c \), for the resonant triad 4 of Table 2. This triad is composed of an inertia-gravity wave with zonal wavenumber-2 and meridional mode \( n = 2 \) (mode \( a \)), a \( k = 0 \) inertia-gravity mode with meridional mode \( n = 1 \) (mode \( b \)), both having the first baroclinic mode vertical structure, and a barotropic zonal wavenumber-2 Rossby wave with meridional mode \( n = 2 \) (mode \( c \)). The initial amplitudes in the numerical simulation shown in Fig. 2 are set as \( |A_a(t=0)| = 0, |A_b(t=0)| = 0.02, \) and \( |A_c(t=0)| = 0.03 \). As illustrated in Fig. 2, for the resonant triads 3 and 4 of Table 2 containing two first baroclinic inertia-gravity waves and one barotropic Rossby mode, the diurnally varying heat forcing resonates through its second harmonic \( (j = 1) \) with the highest-frequency inertia-gravity mode, which is always the most energetically active member of the triad (i.e. the triad component whose energy always grows or decays at the expense of the other waves). As a consequence, although the highest-frequency inertia-gravity mode is resonantly excited by the thermal forcing, its energy no longer grows indefinitely in time. Instead, the energy constraints imposed by the triad dynamics yield saturation for the energy of the excited mode, and the thermal forcing in this case produces some interesting features for the dynamics of the energy exchanges. As observed in Fig. 2, mode \( a \), whose energy is zero initially, is resonantly excited by the heat source and attains an energy level such that it becomes unstable with regard to the other two triad components. At this stage, there is an energy exchange among the modes, with mode \( a \) supplying energy to and then receiving energy from modes \( b \) and \( c \). After this periodic energy exchange, the energy of mode \( a \), as well as total energy, decreases with time until the initial value and begins growing again due to the resonant forcing, with the cycle repeating itself. Thus, it is possible to note from Fig. 2 that the thermal forcing in this case is responsible for a longer time scale modulation in the energy exchanges.

Figure 3 illustrates another example of the thermal forcing modulating the energy exchanges among the modes of this resonant triad. This figure shows the time evolution of the modes’ energy for the same resonant triad of Fig. 2, but with the initial mode amplitudes being...
In this simulation, because the initial amplitudes of modes $b$ and $c$ are higher than in the numerical simulation shown in Fig. 2, it is possible to note from Fig. 3 that the nonlinear saturation of the energy of mode $a$ occurs earlier than in the previous case. As a result of this earlier saturation, the energy level achieved by the modes in this simulation is lower in comparison with the simulation shown in Fig. 2, yielding a more realistic magnitude of the wind field associated with this solution.

Another interesting feature to be observed in Fig. 3 is that the energy surplus due to the forcing that remains after mode $a$ saturation goes to modes $b$ and $c$, leading to an oscillation in the total energy. As a result of this total energy modulation, there are alternations of large and small energy peaks in modes $b$ and $c$ on a time scale of 160 days.

To analyze the implications of the energy exchanges observed in Fig. 3 for the solution in physical space, Figs. 4–6 illustrate some aspects of the physical space solution referred to the interaction shown in Fig. 3. The quantities displayed in Figs. 4–6 are obtained by (2.7) truncated in such a way as to consider only the triad components of Fig. 3. The dimensional variables displayed in Figs. 4–6 are obtained from (2.3) for $U = 5\text{ m s}^{-1}$, $L = 1500\text{ km}$, and $\beta = 2.3 \times 10^{-11}\text{ m}^{-1}\text{ s}^{-1}$. Figure 4 displays the horizontal wind (vector) and horizontal divergence (contour) fields at $p = 1000\text{ hPa}$ referred to the same numerical simulation of Fig. 3 at $t = 0$ (Fig. 4a), $t = 42$ days (Fig. 4b), $t = 62$ days (Fig. 4c), and $t = 85$ days (Fig. 4d). Because the barotropic Rossby waves are almost nondivergent, the divergence field illustrated in Fig. 4 essentially results from the activity of the inertio-gravity modes of the triad. At $t = 0$ (Fig. 4a), the energy of mode $a$ is zero and the flow pattern illustrated in Fig. 4a results from the superposition of modes $b$ and $c$. The divergence field is essentially due to the activity of...
mode $b$, showing a zonally symmetric structure and a meridional structure that is symmetric about the equator and trapped in the tropical region. The contribution of the barotropic Rossby mode for the wind field at $t = 0$ can be clearly noticed in Fig. 4a through the vortices centered on $45^\circ$ (south and north) with a wavenumber-2 zonal structure. The global structure of this mode is more clearly observable in Fig. 5, which shows the relative vorticity at $p = 500$ hPa associated with the same numerical simulation of Figs. 3–4 at $t = 0$. Because the relative vorticity of the inertio-gravity modes is much smaller than that of the quasigeostrophic waves, the pattern displayed in Fig. 5 is due to the activity of the barotropic Rossby mode (mode $c$). Figure 5 shows that this mode is no longer trapped in the equatorial region and instead has a significant midlatitude projection. Consequently, in the real atmosphere this mode is believed to play an important role in the teleconnection patterns from tropics to midlatitudes, as well as in the midlatitude influence on the tropical wave dynamics.

At $t = 42$ days, the energy of mode $b$ is minimal and the energy of mode $a$ is maximal (Fig. 3). Consequently, the divergence field displayed in Fig. 4b basically results from the activity of mode $a$, showing a wavenumber-2 zonal structure and an antisymmetric about the equator meridional structure that is trapped in the equatorial region. At $t = 62$ days, Fig. 4c clearly shows the divergence pattern due to the superposition of modes $a$ and $b$ because these two inertio-gravity modes have almost the same energy level at this stage (Fig. 3). At $t = 85$ days, the energy of mode $a$ is minimal and the energy of mode $b$ is maximal (Fig. 3). As a consequence, the divergence pattern displayed in Fig. 4d is entirely due to the activity of mode $b$. Nevertheless, because of the large energy peak in mode $b$ due to the total energy modulation, the magnitude of the divergence field shown in Fig. 4d is larger than the magnitude of the divergence field shown in Fig. 4a.

Figure 6 shows the time evolution of the 1000-hPa horizontal divergence at $60^\circ$W, $0^\circ$ (Fig. 6a) and at $60^\circ$W, $12^\circ$S (Fig. 6b). Because the horizontal divergence magnitude associated with mode $a$ is maximal at around $12^\circ$ (south and north) and the horizontal divergence signal associated with mode $b$ is maximal on the equator, the time evolution of the horizontal divergence shown in Figs. 6a and 6b illustrates the activity of modes $b$ and $a$, respectively, of the resonant triad of Fig. 3. The time evolution of the horizontal divergence illustrated in Fig. 6 exhibits local oscillations with a period of the order of 1 day. These local oscillations are due to the phase propagation of the waves: the first baroclinic zonal wavenumber-2 inertio-gravity wave with meridional mode $n = 2$ has a period of approximately 1 day, whereas the first baroclinic zonal wavenumber-0 inertio-gravity mode with meridional mode $n = 1$ has a period of approximately
FIG. 4. The horizontal wind (vector) and divergence (contour) fields at $p = 1000$ hPa referred to the same numerical simulation of Fig. 3 at $t = (a) 0$, (b) 42, (c) 62, and (d) 85 days. The wind and divergence fields are displayed in m s$^{-1}$ and in $10^{-6}$ s$^{-1}$, respectively.
1.4 days. Apart from these high-frequency local oscillations, a longer time scale modulation in the amplitude of these local oscillations is also observed. Comparing Figs. 6 and 3 reveals that this longer time scale modulation with a period on the order of 80 days results from the energy exchanges among the modes of this resonant triad. Also observable in Fig. 6a is the alternation of small- and large-amplitude peaks in the horizontal divergence in a time scale of 160 days, which results from the period doubling in the time evolution of mode b energy. The alternation of small- and large-amplitude peaks on a time scale of 160 days is also evident in the time evolution of the geopotential and the relative vorticity in midlatitudes (figures not shown) and is a result of the period-doubling modulation in the energy of the barotropic Rossby mode.

Therefore, Figs. 4–6 illustrate the periodic changes of regime of the physical space solution due to the energy exchanges among waves constituting a resonant triad whose highest-frequency component is resonantly excited by the thermal forcing. Such changes of regime occur on a longer time scale than the period of the local oscillations resulting from the phase propagation of the waves. As observed in Figs. 4–6, the initial amplitudes set in Fig. 3 reproduce realistic magnitudes of weather and climate anomalies. Thus, with the initial amplitudes characterizing realistic magnitudes of atmospheric flow perturbations, Fig. 3 shows that the waves of the resonant
triad 4 of Table 2 exchange energy on an intraseasonal time scale, with the alternation of small and large energy peaks on a semiannual time scale. In a more realistic atmospheric model, which takes into account the effects related to the interaction between the large-scale waves and moist convection, the horizontal divergence is closely related to the convective activity in the tropics. As a consequence, the results presented here suggest that the diurnal variation of tropical heat sources might be important for the semiannual modulation of the intraseasonal oscillations both in the convective activity in the tropics associated with the equatorially trapped inertio-gravity modes and in the extratropical circulation associated with barotropic Rossby modes.

![Graph](https://example.com/graph.png)

**Fig. 6.** Time evolution of the 1000-hPa horizontal divergence at (a) 60°W, 0° and (b) 60°W, 12°S, associated with the solution of Figs. 3–5. The divergence is displayed in this figure in 10^{-6} s^{-1}. 
5. Concluding remarks

Weakly nonlinear interactions among equatorial waves in the presence of a diurnally varying heat source have been explored in this paper in the context of the diabatic version of the equatorial $\beta$-plane primitive equations for a basic state characterized by a motionless, hydrostatic, horizontally homogeneous and stably stratified atmosphere. The heat source was assumed to be periodic in time and of small amplitude [i.e., $O(\epsilon)$] and was prescribed to roughly represent the typical heating associated with deep convection in the Amazon region during the austral summer period. In this context, using the asymptotic method of multiple time scales, the free linear Rossby, Kelvin, mixed Rossby–gravity, and inertio–gravity waves, as well as their vertical structures, have been obtained as leading-order solutions. These waves are shown to interact resonantly in a triad configuration at the $O(\epsilon)$ approximation, and we have studied the dynamics of these interactions in the presence of the forcing.

For the diurnal variation of the heat source considered here, it was found that for the planetary-scale wave resonant triads composed of two first baroclinic equatorially trapped waves and one barotropic Rossby mode, the spectrum of the thermal forcing is such that only one of the triad components is resonant with the heat source. As a result, to illustrate the role of the diurnal forcing in these interactions, two kinds of triads have been analyzed. The first one refers to triads composed of a $k = 0$ first baroclinic geostrophic mode, which is resonant with the stationary component of the diurnal heat source, and two dispersive modes, namely, a first baroclinic mixed Rossby–gravity wave and a barotropic Rossby mode with meridional index $n = 2$. The other class corresponds to triads composed of two first baroclinic inertio–gravity waves in which the highest-frequency wave resonates with a transient harmonic of the forcing. The integration of the asymptotic reduced equations for these selected resonant triads shows that the stationary component and the transient harmonics of the forcing modulate the energy exchanges among the waves in different ways because of the different triad components with which these harmonics resonate. In the first class of triads analyzed, the stationary component of the diurnal thermal forcing resonantly excites the first baroclinic zonally symmetric geostrophic modes. These modes, in turn, allow the two other waves with nearly equal time frequencies to exchange energy, but they are not affected by the propagating waves. As a result, the energy of the zonally symmetric geostrophic modes grows in time without any bound. Consequently, the energy exchanges between the propagating waves become faster as time evolves and the asymptotic solution based on the assumption of multiple scales in time breaks down. On the other hand, in the second class of triads, the transient harmonics resonate with the highest-frequency modes of the resonant triads, which are always the modes having the coupling coefficient with the highest absolute value. As a result, because of the energy constraints imposed by the triad dynamics, the results show that the convective forcing in this case is responsible for a longer time scale modulation in the resonant interactions, generating a period doubling in the energy exchanges.

For the example illustrated in this paper, in which the highest-frequency inertio–gravity mode is excited by the second harmonic (i.e., the harmonic referred to the period of 1 day) of the forcing, if the initial amplitudes of the triad components are set in such a way as to yield realistic magnitudes of atmospheric flow perturbations for the physical space solution, the waves exchange energy on a time scale of 80 days, with modes $b$ and $c$ alternating small and large energy peaks on a time scale of 160 days. This long time scale energy exchange modulated by the thermal forcing leads both the horizontal divergence and the extratropical circulation to undergo this double-period amplitude modulation.

Therefore, this paper has demonstrated in a simplified fashion how the diurnally varying tropical heating associated with deep convection can modulate the resonant wave interactions involving equatorially trapped first baroclinic wave modes and barotropic Rossby modes. Based on the results presented here, the importance of the diurnal variation of tropical deep convection for the intraseasonal variability of the atmospheric circulation verified in some observational and modeling studies is quite plausible. Moreover, since the horizontal divergence associated with the activity of the first baroclinic inertio–gravity waves is believed to be closely related to the convective activity in the tropics, the results suggest that the diurnal variation of tropical deep convection might play an important role in the semiannual modulation of the intraseasonal oscillations of tropical convection. In this context, the resonances explored here involving planetary-scale first baroclinic inertio–gravity waves and barotropic Rossby modes can be a possible dynamical mechanism responsible for the influence of the diurnal cycle of tropical convection on the intraseasonal variability of the atmospheric flow. Furthermore, because the barotropic Rossby modes are believed to play an important role in the teleconnection patterns from tropics to midlatitudes, as well as in the midlatitude influence on the tropical wave dynamics, the results suggest that the planetary-scale first baroclinic inertio–gravity modes having a time period close to 1 day are crucial for the low-frequency modulation of these teleconnection patterns.
Furthermore, the results presented here have some possible implications for the predictability of the intraseasonal oscillations both in the tropics and in the extratropics. The results suggest that the correct representation of the intraseasonal oscillations in atmospheric general circulation models (AGCMs) depends on the correct representation of the rapidly propagating first baroclinic inertia-gravity waves and, consequently, on the correct representation of the diurnal cycle of tropical convection. Thus, according to our results, a possible reason for the poor performance of AGCMs in simulating the intraseasonal variability of the atmospheric circulation is the difficulty these global models have with representing the small-scale processes that yield the diurnal variation of tropical convection.

Acknowledgments. The work of Carlos F. M. Raupp was supported by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) through post-doctoral fellowship 06/53606-0. Pedro L. Silva Dias is supported by grants from PROSUR/IRI and the Moore Foundation. We would also like to thank the anonymous reviewers for their useful suggestions to improve this manuscript.

REFERENCES


