Piecewise Potential Vorticity Inversion and Vortex Interaction

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ABSTRACT

Piecewise potential vorticity inversion (PPVI) aims at an interpretation of the interaction of potential vorticity (PV) anomalies. The total flow domain $D$ is divided into subdomains $D_1$ and $D_2$ with $q_1$ and $q_2$ as PV anomalies. PPVI estimates the height tendency in $D$ that results if $q$ is transported by the winds obtained by an inversion of a PV field with $q = q_1$ in $D_1$ and $q = 0$ in $D_2$. Tests compare the actual height tendency to that obtained via PPVI. This method is applied to the interaction of vortices in quasigeostrophic shallow water, a problem generally accepted as a paradigm for PPVI. Indeed, results are excellent for vortex pairs. Combinations of more vortices, including vortex clusters akin to Rossby waves, are also exposed to tests. The utility of PPVI tendencies depends strongly on the arrangement of the vortices in that case. Very good agreement is found as well as cases with quite low skill. The reasons for this scatter are discussed and a dynamic interpretation of PPVI is proposed.

1. Introduction

Piecewise potential vorticity inversion (PPVI) is an accepted tool in atmospheric diagnostics (Hoskins et al. 1985; Hakim et al. 1996) to explore the interaction of potential vorticity (PV) anomalies. The basic idea behind this technique can be explained in terms of the quasigeostrophic $\beta$-plane potential vorticity

$$q = \nabla^2 \psi - \lambda^2 \psi + f_0 + \beta y$$

(1.1)

of shallow water flow ($\psi$ streamfunction; $f_0$ Coriolis parameter; $\lambda^{-1} = R_o$ Rossby radius). PPVI deals with anomalies $q'$ of $q$ where the Coriolis term or other background PV distributions are removed in (1.1) so that

$$q' = \nabla^2 \psi' - \lambda^2 \psi'$$

(1.2)

is the PV anomaly. In what follows the primes will be omitted. Given an observed distribution $q$ of potential vorticity anomalies in a flow domain $D$, (1.2) can be inverted to obtain $\psi$ and, thus, the geostrophic wind $v_g$ in $D$ for specific boundary conditions. The streamfunction is then predicted via

$$\frac{\partial}{\partial t} \psi = F(q, v_g),$$

(1.3)

where $F$ is an operator that also acts on background fields. PPVI assumes a modified PV distribution $q_p$ where $q_p$ equals the observed PV anomaly in a selected region $D_1(q_p = q)$ and where $q_p = 0$ in the region $D_2$, where $D = D_1 + D_2$. Inversion of $q_p$ yields $v_{gp}$, and with (1.3) we obtain the tendency

$$\frac{\partial \psi_p}{\partial t} = F(q, v_{gp}),$$

(1.4)

where we assume that boundaries do not cause problems. Note that the PV in (1.4) is the observed one and not the modified PV. Thus, the tendency (1.4) results from the transport of the observed PV with winds obtained from PPVI. In the same way, we can apply PPVI to the PV in $D_2$ to obtain a tendency $\partial \psi_{p2}/\partial t$. The linearity of (1.2) and (1.4) implies

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi_p}{\partial t} + \frac{\partial \psi_{p2}}{\partial t}.$$  

(1.5)

The key question, of course, is what can be learned from subdividing $D$ into two or more parts $D_i$ and performing
PPVI? There appears to be general agreement that by calculating the balanced winds associated with PV anomalies “we can deduce which anomalies figure prominently in the amplification of others” (Davis and Emanuel 1991, p. 1936). This statement implies that PV anomalies “induce” flows at some distance that can transport other PV anomalies. This idea is attractive and has been applied widely, although the explicit tendency calculations have not been carried out in most cases [however, see Hakim et al. (1996) and Nielsen-Gammon and Lefevre (1996)]. For example, stratosphere–troposphere interaction has been investigated using PPVI (Hartley et al. 1998). Problems in hurricane dynamics (Mölle and Jones 1998) and cyclone research (Davis and Emanuel 1991) have been addressed. Feldmann and Davies (1997) applied a specific version of PPVI tailored to forecast problems. Hakim et al. (1996) diagnosed the interaction of two observed cyclonic disturbances using PPVI. Cycloysis has been dealt with by Martin and Marsili (2002).

Nevertheless, it is not clear which processes are responsible for this induction. A well-known answer to this question has been proposed by Bishop and Thorpe (1994, hereafter BT94) by invoking the analogy of electrostatics and quasigeostrophic dynamics (see also Hoskins et al. 1985). This analogy is thought to “provide a theoretical basis for the concept of ‘action-at-a-distance’ which is a cornerstone of PV-thinking” (BT94, p. 713). The atmosphere acts according to this view as if it were composed of particles, or charges of PV. PPVI “seeks to attribute to a feature on the weather map, such as a vorticity anomaly, a unique influence on the rest of the atmosphere” (BT94, p. 713). There is no doubt that electric charges induce electric fields and that these fields exert forces on other charged particles. However, PV anomalies are not elementary particles. Flows do not exert forces on PV; rather, they transport other PV anomalies. The validity of the electrostatics analogy is as yet unclear.

A simple example may shed some light on this problem. Consider a Rankine vortex ($R_o = \infty$; $\beta = 0$; infinite $f$ plane) of radius $r_1$ and constant vorticity $\zeta_1$ such that

$$\psi = \frac{1}{2} \zeta_1 r_1^2 \ln(r/r_1)$$ (1.6)

for radii $r > r_1$ and

$$\psi = \frac{1}{4} \zeta_1 (r^2 - r_1^2)$$ (1.7)

for $r \leq r_1$. Simple calculations reveal that the total energy and angular momentum of (1.6) are infinite (see also section 4). Thus, the flow around the center cannot be induced by the vorticity in $r \leq r_1$. Instead, (1.6) and (1.7) determine the flow needed to establish the vorticity of the Rankine vortex.

It becomes rather unlikely in view of this example that PV in more general fluid systems induces flows at a distance. As a consequence, a more pragmatic view of PPVI is adopted in this paper. PPVI is seen as a method to estimate the actual tendency of $\psi$ on the basis of winds, which can be understood more easily than the complete flow field. Symmetric vortices are perfect candidates for such investigations because the related circulations are well understood. It is the main purpose of this paper to apply PPVI to vortex interaction. The barotropic tests of PPVI in Egger (2008) were conducted for one-dimensional models where vortex interaction cannot be fully dealt with. Nielsen-Gammon and Lefevre (1996) and Wu et al. (2003) applied PPVI to vortex interactions in case studies, so tests of PPVI for vortex pairs are also of practical relevance.

### 2. The model

The model to be used contains only standard features. Nevertheless, a brief description will be given for the sake of clarity. We consider quasigeostrophic $\beta$-plane shallow water flow of depth $h = \bar{H} + \eta$ (where $H$ is constant). The only prognostic equation to be solved is the equation of continuity

$$\frac{\partial}{\partial t} \eta - w = 0,$$ (2.1)

where $w$ is the vertical velocity at the fluid’s upper surface. Note that height advection by the geostrophic wind drops out in (2.1). The vertical velocity is obtained from the “omega equation”

$$\nabla^2 w - \lambda^2 w = -\frac{f}{g} \beta v + \frac{f}{g} \frac{\partial}{\partial y} \left( u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} \right)$$

$$-\frac{f}{g} \frac{\partial}{\partial x} \left( u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} \right),$$ (2.2)

which is equivalent to the vorticity equation but formulated diagnostically [$g$ is reduced gravity; $\mathbf{v} = \mathbf{k} \times \nabla \psi$, with $\psi = g f^{-1} \eta$; $R_o = (gH)^{1/2} f_o$]. The model (2.1) and (2.2) can be seen as a mass-predicting balance model (e.g., Allen 1993; Warn et al. 1995), which is not based on the inversion of PV. One may evaluate PV at any time using (1.1) but that is not necessary to run the model. Of course, this model is equivalent to the more elegant PV model.
\[
\frac{\partial q}{\partial t} + \mathbf{v}_g \cdot \nabla (\nabla^2 \psi + \beta y) = 0. \tag{2.3}
\]

Interpretations of flow evolutions in terms of the mass field are equivalent to those in terms of PV.

Because PPVI is a diagnostic technique, we do not have to run the model in time and may concentrate on selected “typical” flow configurations. Nevertheless, (2.1) and (2.2) or (2.3) are needed to calculate the tendencies. More specifically, we prescribe in a vortex experiment the height field associated with various patches of potential vorticity. The solution of (2.2) yields the vertical motion and, thus, the related height tendency \( \partial \eta/\partial t \). PPVI assumes \( q = 0 \) in \( D_2 \) and provides an estimate \( \partial \psi_v/\partial t \) of the tendency, which must then be compared to the true one.

There is some freedom with respect to the choice of the boundary conditions. Periodic conditions are chosen here but Dirichlet or Neumann conditions could be imposed as well at walls encircling \( D \). Moreover, an infinite plane may be selected. Many choices have the problem that a single vortex cannot be dealt with in a strict sense (as is also true for the sphere). Integration of (1.2) over \( D \) gives

\[
C_g - \lambda^2 \overline{\psi_p} = \overline{\psi}_p, \tag{2.4}
\]

where the overbar indicates the area integral over \( D \) and \( C_g \) is geostrophic circulation around \( D \). The geostrophic circulation has to vanish for double periodic flow but not necessarily in all other cases. There is, however, no guarantee that \( \overline{\psi}_p = 0 \). This is unacceptable if PV is thought to induce winds because PV would have also to induce (or reduce) mass. However, (2.4) does not pose a problem for our more pragmatic interpretation of PPVI. The winds calculated via PPVI in the doubly periodic domain cannot exist in an atmosphere of mean depth \( H \) but in an atmosphere of mean depth:

\[
H_i = H + \overline{\psi}_p H f^{-1}. \tag{2.5}
\]

We are free to use these winds for tendency estimates.

As stated above, the calculation of tendencies in PPVI requires that we apply the winds obtained via PPVI from the inversion of the modified PV \( q_p \) (see also Hakim et al. 1996). Thus, (2.2) is to be replaced by

\[
\nabla^2 w_p - \lambda^2 w_p = f v_g^{-1} \left[ \frac{\partial}{\partial y} (v_{g_p} \cdot \nabla u_p) - \frac{\partial}{\partial x} (v_{g_p} \cdot \nabla v_p) \right] - f v_g^{-1} \beta v_{g_p}, \tag{2.6}
\]

where \( w_p \) is the related vertical velocity. The tendencies resulting from (2.6) have to be compared to those obtained from (2.2). Note that (2.6) is not the omega equation of a real flow because the transporting winds differ from those transported. However, (2.6) is mathematically correct and quantifies that part of the total vertical motion that balances the “forcing” on the righthand side of (2.6).

Altogether we see that PPVI tendency calculations are fairly complicated. They require us to find the winds that support the modified PV field. Then we have to calculate the tendency of the streamfunction on the basis of these winds transporting the total PV. PPVI tries to “explain” the observed tendencies. The selection of the region \( D_1 \) is helpful with respect to this explanation if much of the total tendency is captured by the result of PPVI. The helpfulness of PPVI can be gauged by comparing visually the “observed” vertical motion \( w \) for the full problem with \( w_p \) as obtained via PPVI. In addition, a more quantitative score is provided by calculating correlation coefficients. For example,

\[
C_i(w, w_p) = \frac{\overline{ww_p}^2 - (\overline{w^2} - \overline{w^2})^{1/2}}{\overline{w^2} - \overline{w^2}} \tag{2.7}
\]

provide a gross measure of the success of PPVI with respect to tendency amplitudes in the domains \( D_i (i = 1) \), \( D_2 (i = 2) \), and \( D_{12} (i = 12) \).

All inversions of the Laplacian are carried out spectrally whereas the terms on the right-hand side of (2.2) and also (2.7) are evaluated in a grid and then transformed to the spectral domain.

3. Results

We proceed from a vortex pair (section 3a) to situations with more vortices (section 3b). The choice of the domains \( D_1 \) and \( D_2 \) automatically affects the result of PPVI. If the size of \( D_1 \) is much larger than that of \( D_2 \), the result of PPVI with respect to \( D_2 \) is almost trivial. If, on the other hand, \( D_1 \) is quite small, PPVI is pointless. It appears best to divide the quadratic domain \( D \) of side length \( L \) in two subdomains of equal size and shape.

a. Vortex pair

Here and in all following calculations we prescribe circular PV anomalies surrounded by fluid with \( q = 0 \) and assume \( \beta = 0 \) if not mentioned otherwise. PV decays linearly with distance to the vortex center to vanish at the surrounding circle. The basic test assumes two circular vortices C (cyclogon) and A (anticyclonic) with radius
with a distance $\Delta x = R_o$ of their centers (see Fig. 1). Hence, the vortices are fairly close and strong interaction can be expected. The related height field is obtained by inversion of (1.2). Of course, we could also have prescribed a height field with deep circular lows and highs, but the related potential vorticity would not be strictly localized and vortices could not be identified well. Thus, a specification of the flow state in terms of PV anomalies is better suited to our purpose than the equivalent formulation of height anomalies. On the other hand, if the height field were specified initially as needed in the model version (2.1) and (2.2), nobody would claim that the height field in $D_1$ induces winds in $D_2$.

In a first test we choose a vortex pair with maximum $|\text{PV}| = 2 \times 10^{-4} \text{s}^{-1}$ in the respective centers; $r_1 = 400 \text{ km}$, $R_o = 1000 \text{ km}$. The domain size is quite large with $L = 12 \times 10^6 \text{ m}$ to exclude boundary effects. The standard numerical resolution is $25 \times 25$ wavenumbers. The flow field related to $C$ and $A$ is displayed in Fig. 1a. Strong southerlies are found, of course, in between the vortex centers. Note that the condition

$$\bar{\varrho} = 0$$

(3.1)

is satisfied for this configuration because the intensity and shape of both vortices are the same. The vertical velocity $w$ in Fig. 1b shows a quadrupole pattern with rising motion south (north) of $C$ ($A$) and descent north (south) of $C$ ($A$). Clearly, the vortices are going to move toward the north but will be deformed as well because the extrema of vertical motion are located east (west) of the center of $C$ ($A$).

The western half of the domain $D$ in Fig. 1 is chosen as subdomain $D_1$ with the cyclonic vortex $C$. The separating
Table 1. Scores $C_i$ and $R_i$ [see (2.7) and (2.8)] for the reference experiment in Fig. 1, with $\beta = 1.6 \times 10^{-11}$ m$^{-1}$ s$^{-1}$ and for variations of the reference experiment with respect to $R_0$ (in 10$^6$ m) and vortex distance $D_1$ in 10$^6$ m; $\bar{g} = 1$ m s$^{-1}$ and $\beta = 0$ if not specified otherwise.

<table>
<thead>
<tr>
<th></th>
<th>$C_2$</th>
<th>$C_1$</th>
<th>$C_{12}$</th>
<th>$R_2$</th>
<th>$R_1$</th>
<th>$R_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference</td>
<td>0.95</td>
<td>0.31</td>
<td>0.69</td>
<td>0.95</td>
<td>0.46</td>
<td>0.72</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.73</td>
<td>0.48</td>
<td>0.60</td>
<td>0.94</td>
<td>0.72</td>
<td>0.83</td>
</tr>
<tr>
<td>$\bar{g} = 10$; $R_0 = 3.2$</td>
<td>0.95</td>
<td>0.22</td>
<td>0.67</td>
<td>0.98</td>
<td>0.45</td>
<td>0.74</td>
</tr>
<tr>
<td>$\bar{g} = 0.1$; $R_0 = 3.2$</td>
<td>0.92</td>
<td>0.55</td>
<td>0.75</td>
<td>0.81</td>
<td>0.55</td>
<td>0.66</td>
</tr>
<tr>
<td>$\Delta x = 2$</td>
<td>0.80</td>
<td>0.69</td>
<td>0.74</td>
<td>0.72</td>
<td>0.62</td>
<td>0.67</td>
</tr>
<tr>
<td>Two cyclones</td>
<td>0.93</td>
<td>0.39</td>
<td>0.71</td>
<td>0.91</td>
<td>0.46</td>
<td>0.70</td>
</tr>
</tbody>
</table>

The application of PPVI to two vortex pairs is discussed here because it sheds light on an aspect of PPVI that has received little attention so far. What happens if there is more than one PV anomaly in D2? Some results are summarized in Table 2. We consider the interaction of two cyclone–anticyclone pairs in D1 and D2. The meridional distance of the members of a pair is $\Delta y$; the zonal distance is $\Delta x$. Note that the mean PV in D1 vanishes in this case. The schematic in the upper left corner of Table 2 depicts the vortex arrangement. The pair in D1(D2) would move eastward (westward) in the absence of the other pair. Table 2 lists the scores ($C_2$, $C_1$, $R_2$) for various choices of the distances $\Delta x$ and $\Delta y$. The result, $C_2 = 0.25$, is surprisingly small if $\Delta x = \Delta y = R_0$. The winds associated with the vortex pair in D1 do not contribute to the westward motion of the pair in D2. Thus, PPVI misses the most important part of the motion in D2. On the other hand, $C_1 = 0.53$ is relatively large because PPVI captures the westward motion in D1. An increase of $\Delta x$ leads to a substantial increase of $C_2$, but $R_2$ is now so small that the result of PPVI is not helpful either. Results become much better if $\Delta y$ is increased. After all, we have just two reference pairs, as in Table 1, if $\Delta y$ is large. Indeed, the scores $C_2 = 0.94$ and $C_1 = 0.32$ are fairly close to those in Table 1 for the reference run. Note that these tests are invariant with respect to a change of $q_p$. If $q_p = 0$ is assumed in D1 and $q_p = q$ in D2, the scores will be exactly the same. This does not contradict (1.5) but tells us that skill scores are not additive.

c. \textit{“Wave” structures}

By assembling many vortices we can approximate any PV structure. This can be done exactly by introducing vortices with a grid square PV distribution. It is, however,
better in line with the calculations presented so far to use our standard vortices to build simple wave structures. In a first experiment we place eight cyclonic (anticyclonic) vortices in $D_1(D_2)$ to prescribe a rather simple version of a Rossby wave of zonal wavenumber 1 (see Fig. 2a). The amplitudes of these vortices vary approximately as $\sin(2\pi x/L)$. The southerlies near $x = 0$ transport air of low PV northward. This leads to a height rise (Fig. 2). The various vortices induce local variations. We expect PPVI to work rather well in this case because the PV structure in $D_2$ is so simple. Indeed, PPVI produces with $C_2 = 0.96$, $C_1 = 0.99$, and $C_{12} = 0.94$ an excellent result where the score for $D_1$ is even better than that for $D_2$. The value of $R_2 = 0.37$ is fairly low because the winds associated with the anticyclonic vortices in $D_2$ are missing. The pattern of vertical motion $w_p$ (Fig. 2b) is fairly similar to the original one in $D_1$ but provides only a weak copy of $w$ in $D_2$ corresponding to the low value of $R_2$. Nevertheless, the basic pattern is correct.

Next we reduce the wavelength to $L/2$ so that cyclonic and anticyclonic vortices are both in $D_1$ and $D_2$ (Fig. 3). As one would expect, there are two regions of upward motion east of the two troughs and corresponding domains with downward motion (Fig. 3a). PPVI yields $C_2 = 0.39$, $C_1 = 0.89$, and $R_1 = 0.46$. Clearly, this type of forecast is not useful. As can be seen from Fig. 3b, $w_p$ misses most of the “observed” vertical motion $w$ in $D_2$. This failure will be explained below.

4. Discussion and conclusions

The consideration of the Rankine vortices provided a starting point for the interpretation of PPVI. (Potential) vorticity does not induce flow in its environment. Instead, these flows are needed to establish the PV anomaly in the center as well as the vanishing of PV elsewhere. There is no causal relationship between PV and $\psi$. Instead, knowledge of $\psi$ is sufficient for a calculation of $q$. In turn, $\psi$ can be derived from the PV field. These basic facts suggest that we can view PPVI as a method for tendency estimation. If high scores are found, we learn that the total advecting wind in $D_2$ can be replaced by that resulting from $q_p = q$ in $D_1$ and $q_p = 0$ in $D_2$. Conversely, this wind is not important if scores are low. Of course, results depend on the chosen separation but an interesting separation is often suggested by the observations.

Interaction of two vortices has been chosen as a first paradigmatic example. It is found that PPVI produces tendencies in $D_2$ that are quite close to those obtained for the complete PV distribution. PPVI correctly predicts northward motion of the anticyclonic vortex in $D_2$ for $\Delta x \sim R_\psi$. The scores go down if $\beta$ is included because a single vortex has a tendency field of its own in that case. The success of PPVI is fairly robust with respect to variations of the Rossby radius. The scores decrease with increasing vortex distance.

The calculations for four vortices reveal a fairly wide range for the test scores. The performance of PPVI is worst for the small meridional distance $\Delta y = 1$ and is quite good for $\Delta y = 2$ and 3 and $\Delta x = 1$. This means that the eventual success of PPVI is difficult to foresee in a situation with observations. The tests for waves provide
a good result if just one wave is prescribed whereas a fairly complete failure is found for a sequence of two waves. PPVI has difficulties in capturing Rossby wave dynamics (see also Egger 2008).

There remains the nagging question why PPVI performs so well in some cases while the results are unsatisfactory in others. A fairly simple one-dimensional example will help us to understand this result. Height tendencies vanish in the one-dimensional case except if $b \neq 0$. Nevertheless, important properties of the inversion procedures can be demonstrated by looking at this relatively simple situation. It is straightforward to show that a constant PV anomaly $q = q_1$, located in the interval $|x - x_1| < a$ and with $q = 0$ outside the interval is associated in an infinite domain with the streamfunction

$$
\psi_1 = -q_1 R_o^2 \{1 - Q \cosh[\lambda(x - x_1)]\} \tag{4.1}
$$

in $|x - x_1| < a$ with

$$
Q = [\cosh(\lambda a) + \sinh(\lambda a)]^{-1} \tag{4.2}
$$

and

$$
\psi_1 = -q_1 R_o^2 Q \sinh(\lambda a) \exp(\lambda a) \exp(-\lambda |x - x_1|) \tag{4.3}
$$

in $|x - x_1| > a$. For $q_1 > 0$ we have a trough centered at $x = x_1$ (Fig. 4; dashed) and negative values of $\psi_1$ for all $x$. In particular, $\partial \psi_1 / \partial x > 0$ for $x > x_1$.

We assume now that $D_1(D_2)$ contains all points with $x \leq 0 (x \geq 0)$. Thus, (4.1) and (4.3) is the streamfunction resulting from PPVI if $x_1 < -a$ and if there is no other PV anomaly in $D_1$. We assume two further anomalies in $D_2$, one with $q = q_2$ in $|x - x_2| < a$ and another with $q = q_3$ in $|x - x_3| < a$, where we make sure that there is no overlap. The total streamfunction related to all anomalies is then

$$
\psi = \psi_1 + \psi_2 + \psi_3, \quad \tag{4.4}
$$

where $\psi_2$ and $\psi_3$ follow immediately from (4.1) and (4.3) by replacing $(x_1, q_1)$ by $(x_2, q_2)$ and $(x_3, q_3)$, respectively. Instead of evaluating the vertical motion for $\beta \neq 0$, we adopt a simpler approach more in line with what is generally done in PPVI. The meridional wind at the centers of the anomalies indicates their expected meridional motion. Thus,

$$
\nu_2 = \left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_3}{\partial x}\right)|_{x=x_2} \tag{4.5}
$$

is the observed wind at the center of the second anomaly. One finds

$$
\nu_2 \sim \{q_1 \exp[-\lambda(x_2 - x_1)] - q_3 \exp[-\lambda(x_3 - x_2)]\} Q R_o \tag{4.6}
$$

and, analogously,

$$
\nu_3 \sim \{q_1 \exp[-\lambda(x_3 - x_1)] + q_2 \exp[-\lambda(x_3 - x_2)]\} Q R_o \tag{4.7}
$$
PPVI tries to predict these motions on the basis of the first anomaly alone. Thus, PPVI stipulates

\begin{align}
\frac{\partial \psi}{\partial x} & > 0, \\
\frac{\partial \psi}{\partial y} & > 0.
\end{align}

The forecast (4.8) is exact if \( q_3 = 0 \). Thus, PPVI can handle vortex pairs rather well. The forecasts (4.8) and (4.9) are not helpful if \( q_3 = q_1, q_2 = -q_1 \) and if \( (x_3 - x_2) < -x_3 \) because both (4.6) and (4.7) give northerly winds whereas PPVI predicts southerlies. This situation is depicted in Fig. 4. There is a “vortex” \( Z_1 \) in \( D_1 \), and anticyclonic (\( A_2 \)) and cyclonic (\( Z_2 \)) vortices in \( D_2 \). All have the same intensity \( q_1 \) and width \( 2a = 10^6 \text{ m} \). PPVI gives minima at \( Z_1 \) and \( Z_2 \) and a maximum of the streamfunction at \( A_2 \). Note, however, that the peak of the streamfunction is located west of the center of \( A_2 \) (cross) and the minimum is east of the center of \( Z_2 \) (cross) so that northerlies are found in the centers. PPVI gives the dashed curve with westerlies. Thus, Fig. 4 shows a situation in which PPVI is not helpful, but we could just as well have chosen a situation where PPVI would be helpful (\( q_2 = q_1, q_3 = -q_1 \)). This result is fully corroborated by two-dimensional calculations in which three vortices are aligned zonally, \( Z_1 \) in \( D_1 \), \( A_2(2Z_2) \) at a distance \( 1.5R_o \) in \( D_2 \), and \( Z_2(A_2) \) further east at a distance \( R_o \) to \( A_2(Z_2) \). The result is \( C_2 = -0.53, C_1 = 0.51, \) and \( R_2 = 0.49 \) in the first case where the vortex pair in \( D_2 \) moves southward, whereas \( C_2 = 0.85, C_1 = 0.65, \) and \( R_2 = 0.49 \) in the second case, where the pair’s northward motion concurs with the southerlies east of \( Z_1 \). It is the location and type of anomalies in \( D_2 \) that decides in favor or against the tendency estimates via PPVI. Thus PPVI is applied best if there is only one dominant anomaly in \( D_2 \), preferably an axisymmetric one.

Although the induction of flows by PV is ruled out by the Rankine vortices, there may be the possibility of a “virtual induction” in terms of geostrophic adjustment (e.g., Blumen 1972). Consider vortex \( C \) in the reference experiment (see Fig. 1 and Table 1) and restrict \( D_1 \) to the circular domain of \( C \). For the sake of simplicity, a homogeneous PV anomaly \( q_1 \) is ascribed to this circle. This anomaly is assumed to be due to an unbalanced height anomaly

\[ \eta_1 = -q_1 H f^{-1} \]

at rest. The domain \( D_2 \) with \( q = 0 \) extends to infinity. We speculate that geostrophic adjustment will produce a balanced final state with streamfunction \( \psi_a \) solving

\[ r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial \psi_a}{\partial r} \right) - \lambda^2 \psi_a = q, \]

where \( r \) is the distance to the vortex center, \( q = q_1 \) for \( r \leq r_1 \) (radius of C) and \( q = 0 \) for \( r > r_1 \). Thus, \( \psi_a \) is exactly the same streamfunction that would be obtained by PPVI with respect to the PV anomaly \( q = q_1 \) in \( D_1 \). The idea of interpreting PPVI in terms of geostrophic adjustment is based on solutions of the one-dimensional geostrophic adjustment problem where indeed the one-dimensional version of (4.11) gives the final state (e.g., Gill 1982). If these examples can be extended to the axisymmetric case, the winds in \( D_2 \) as obtained via PPVI are those which would result from geostrophic adjustment if the PV in \( D_1 \) were represented initially by an unbalanced height perturbation (4.10). This way, the issues of “attribution” and “induction” could be clarified. This proposition can be tested by numerical calculations of the adjustment process or, as will be done here, by invoking the conservation of total energy and angular momentum during adjustment. In other words, the total angular momentum \( M_o \) and energy \( E_o \) of the “initial” state (4.10) must equal

\[ M = \int_0^{r_o} \left[ \frac{\partial \psi_a}{\partial r} r^2 (h + \eta) + \frac{1}{2} \eta^3 \right] dr \]

and

\[ E = \frac{1}{2} \int_0^{r_o} \left[ \left( \frac{\partial \psi_a}{\partial r} \right)^2 (H + \eta) + \bar{g}(H + \eta)^2 \right] r \, dr, \]

where \( M \) and \( E \) are related to the adjusted state. Insertion of (1.7) in (4.12) with \( \eta = 0 \) shows immediately that
$M$ is finite for the Rankine vortex. The equivalence of $M$ and $M_o$ can be demonstrated by multiplying (4.11) by $-(\frac{1}{2})r^3H$ to obtain

$$
-\frac{1}{2}Hr^2 \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{2} \frac{r^3}{f} \eta_a = \frac{1}{2} \frac{r^3}{f} \eta_1. \quad (4.14)
$$

Integration over $r$ using

$$
\frac{1}{2} r^3 \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = \frac{1}{2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) - r \frac{\partial \psi}{\partial r} \quad (4.15)
$$

gives

$$
\int_0^\infty \left( \frac{\partial \psi}{\partial r} r^2 H + \frac{1}{2} \eta_a r^3 \right) dr = M_o \quad (4.16)
$$

for a sufficiently fast drop of velocities with increasing $r$. Thus $M \sim M_o$ except for the small term

$$
M - M_o = \int_0^\infty \left( \frac{\partial \psi}{\partial r} \eta_a r^2 \right) dr. \quad (4.17)
$$

Actual evaluations of (4.17) for a wide range of the parameter $r_1/R_o$ show that the ratio $M/M_o$ deviates from unity by a few percent if $\alpha = 0.1$ for $q_1 = \alpha f_o$. Of course, the deviations increase with growing $\alpha$. Nevertheless, the agreement is quite good. The corresponding calculation for the energy is more tedious but shows also that the ratio $E/E_o$ is close to unity. We can, therefore, conclude that geostrophic adjustment would indeed transform (4.10) into the result of PPVI provided $|q_1| \ll f_o$. Of course, this adjustment does not take place in reality but we have here nevertheless a kind of virtual causality and induction. This interpretation breaks down if strong vortices are considered or if (4.11) is unstable with respect to two-dimensional perturbations. The calculations of Ford (1994) indicate that growth rates are quite small for axisymmetric vortices as considered here. Ford (1994), however, did not investigate the profile (4.10). Note that there is no geostrophic adjustment in nondivergent flow. It is clear that these calculations do not provide more than a suggestion.

REFERENCES


