The Continuous Spectrum in Baroclinic Models with Uniform Potential Vorticity Gradient and Ekman Damping

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(Manuscript received 4 March 2009, in final form 10 April 2009)

ABSTRACT

Analytic solutions of the continuous spectrum are obtained for quasigeostrophic models in which the basic-state meridional potential vorticity (PV) gradient is uniform but nonzero. The modes that form the continuous spectrum—continuum modes—are baroclinically neutral, singular solutions of the linear dynamical equations, which have an interior critical level. Although most inviscid shear flows support continuous spectra, previous studies have largely been restricted to models in which the interior PV gradient is zero. In that situation, continuum modes are formed by a superposition of boundary potential temperature anomalies and an interior PV δ function at the critical level. Here it is shown how a nonzero interior PV gradient and Ekman damping nontrivially affect the vertical structure of continuum modes. A consideration of the perturbation PV further rationalizes their vertical structure.

1. Introduction and model equations

It has long been recognized that the continuous spectrum (CS) plays a role in the barotropic and baroclinic initial-value problem (Orr 1907; Case 1960a,b; Pedlosky 1964; Burger 1966; Farrell 1982). The CS has been investigated previously for geometries in which the basic-state potential vorticity (PV) gradient vanishes in the interior, such as the Eady (1949) model (e.g., Chang 1992; Jenkner and Ehrendorfer 2006; De Vries and Opsteegh 2007). Some studies have addressed the CS in models where the interior PV gradient does not vanish, but these did not look at the structure of the continuum modes that underly the CS (Pedlosky 1964; Burger 1966; Boyd 1983; Bishop and Heifetz 2000). Hirota (1968) numerically investigated the normal-mode spectra of the Green (1960) model (Eady model with nonzero planetary vorticity gradient β). This note presents some analytical solutions of the CS for baroclinic flows with constant but nonzero interior PV gradient. The method used to obtain the results is similar to that used in Case (1960a).

The evolution of small disturbances on a zonal shear flow \( \mathbf{f}(y, z) \) can be described by the linearized quasigeostrophic potential vorticity equation

\[
\frac{\partial q}{\partial t} + \pi \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} = 0,
\]

where \( q \) is the perturbation PV, \( \pi \) the basic-state zonal wind, \( v = \partial \psi / \partial x \) the perturbation meridional wind, and \( \psi \) the perturbation streamfunction. Equations have been made nondimensional using scalings for shear \( L_0 \), planetary vorticity gradient \( \beta_0 \), Coriolis parameter \( f_0 \), and nondimensional buoyancy frequency \( N_0^2 \). This gives a height scale \( H = (f_0/N_0)^2 \beta_0 \) and a length scale \( L = (N_0/f_0)H \). The perturbation PV expressed in terms of streamfunction reads

\[
q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{\tilde{\rho}} \frac{\partial}{\partial z} \left( \frac{\tilde{\rho} \partial \psi}{N_0^2} \right),
\]

where \( \tilde{\rho} = \exp(-sz) \) is a density profile of the reference state with \( s = H/H_\rho \) (\( H_\rho \sim 7 \) km), and \( N^2 \) is a nondimensional buoyancy frequency profile. The basic-state PV gradient is given by

\[
\frac{\partial q}{\partial y} = \beta - \frac{\partial^2 \pi}{\partial y^2} - \frac{1}{\tilde{\rho}} \frac{\partial}{\partial z} \left( \frac{\tilde{\rho} \partial \pi}{N_0^2} \right),
\]

Rigid-lid boundary conditions at \( z = H_\pm \) are implemented by setting the vertical velocity to zero in the thermodynamic equation, such that
\[ \frac{\partial \theta}{\partial t} + \nabla \cdot \theta + v \cdot \frac{\partial \theta}{\partial y} = \pm \alpha \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right), \quad (z = H_c), \quad (4) \]

where \( \theta = \frac{\partial \psi}{\partial z} \) is proportional to potential temperature. Ekman damping is included by the term involving \( \alpha \), the Ekman parameter (Pedlosky 1987). In vertically unbounded domains, the perturbation streamfunction vanishes at infinity.

2. Normal modes

Normal modes are solutions of (1)–(4) of the form

\[ \psi(x, y, z, t) = \phi(y, z)e^{ik(x-c)} , \quad (5) \]

where \( c \) is the complex phase speed. Substituting (5) into (1) gives

\[ (\overline{\nu} - c) \left( D^{(2)} + \frac{\overline{q}_y}{\overline{\nu} - c} \right) \phi(y, z) = 0, \quad (6) \]

where (in the case of uniform \( \nu^2 = 1 \))

\[ D^{(2)} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \beta \frac{\partial}{\partial z} - k^2 \]

is the differential operator relating streamfunction to PV; \( q(y, z) = D^{(2)} \phi(y, z) \). The normal modes are grouped into the discrete spectrum (DS) and the continuous spectrum (CS). The DS contains the nonsingular solutions of (6). Allowable phase speeds \( c \) follow from applying the boundary conditions. To be nonsingular, the neutral discrete modes must have a steering level outside the range of \( \overline{\nu} \). The CS is formed by singular solutions of (6) called continuum modes (CMs). CMs, for which \( c = \overline{\nu}(yc, z_c) \), satisfy

\[ \left( D^{(2)} + \frac{\overline{q}_y}{\overline{\nu} - c} \right) \phi(y, z) = \delta(z-z_c)\delta(y-y_c) \quad (7) \]

and the boundary conditions (Case 1960a,b; Pedlosky 1964); \( \delta(z) \) denotes the Dirac delta function. An important difference between the DS and CS is that the latter consists of nondispersive modes, which have a phase speed that is known a priori. Equation (6) shows that the PV of a CM increases indefinitely as the critical surface \( \overline{\nu}(yc, z_c) \) is approached. The physical reason for this unbounded increase of the PV amplitudes is that the circulation is generally nonzero near and at the critical surface. Since the PV anomalies in the vicinity of the critical surface need to propagate only weakly relative to the local background flow—at the critical surface itself, they should be propagating exactly with the background flow—the only way to prevent them from propagating too fast relative to the flow is to increase their amplitude.

a. Continuum modes for zonal flows of the form \( \overline{\nu}(z) \)

From here on, we consider \( y \)-independent zonal flows \( \overline{\nu}(z) \), such that \( \phi(y, z) = \phi(z) \sin(ly) \). The level \( z = z_c \) where \( \overline{\nu}(z_c) = c \) is called the critical level (for simplicity it is assumed that there is only one critical level for a given \( c \)). Equation (7) becomes an inhomogeneous second-order ordinary differential equation. The homogeneous equation [cf. (6)] has two independent solutions, which are denoted by \( \phi \pm \). Using \( \phi \pm \), the solution of (7) can be written as

\[ \phi(z) = H(z_c - z)\phi^+ + H(z - z_c)\phi^-, \quad (8) \]

where \( H(x) \) is the Heaviside step function and

\[ \phi^+ = c_+^\ast \phi_+ + c_-^\ast \phi_-, \quad \phi^- = c_+ \phi_+ + c_- \phi_- . \quad (9) \]

The four constants \( c_{\pm} \) are fixed by imposing the two boundary conditions as well as a matching condition (streamfunction continuity at the critical level). This matching condition is required for the solution to make sense physically. In this way the CM is determined up to an arbitrary multiplicative factor.

b. Analytic solution for the case \( \overline{\nu} = z \)

The case of a constant shear flow \( \overline{\nu} = z \) and \( \overline{q}_y = \beta \) is analytically tractable. CMs propagating with speed \( c = h \) have a single critical level at \( z_c = h \). As outlined in the previous section, so as to determine the CM, first the homogeneous form of (7) is solved. The transformation

\[ \phi_\pm(z) = f_\pm(z) \exp \left[ \frac{s}{2} + \frac{1}{2} \frac{1}{r} (z - c) \right] , \]

where \( 1/2r = [k^2 + l^2 + (s/2)^2]^{1/2} \), and the coordinate change \( \xi = (z - c)/r \) reduce (6) to Kummer’s equation:

\[ \frac{\xi^2}{\partial \xi^2} f_\pm + \xi \frac{\partial f_\pm}{\partial \xi} + (r \beta_c) f_\pm = 0 . \quad (10) \]

For each sign in (10) there are two independent solutions. We choose those two solutions that become constant in the limit \( \beta_c \to 0 \). This gives

\[ f_\pm(z) = U \left[ \pm r \beta_c, 0, \frac{(z - c)}{r} \right] , \]

with \( U [a, b, c] \) the confluent hypergeometric function of the second kind (Abramovitz and Stegun 1970). The rigid-lid conditions and matching condition determine
the constants $c_\infty$, giving the solution of the CM. The following sections discuss the solution for various geometries.

3. Vertically unbounded domain

As a first example, consider a flow $u = z$ in a vertically unbounded domain. Shear and buoyancy frequency are set to their scaling values. We also set $s = 0$ and $\alpha = 0$ (constant density, no Ekman damping). The structure of a CM with critical level at $z = h$ is given by

$$\phi(z) = H(h - z)\phi_+ + H(z - h)\phi_-. $$

where $a_\pm = \phi_\pm(h)/\phi_\pm(h)$ such that $\phi$ is continuous. On the $f$ plane the CM has the structure shown by the dashed line in Fig. 1. Since $\phi_\pm$ reduce to exponential functions, $\log(\phi) = -|z - h|/(2r)$ up to an arbitrary normalization [we set $\phi(z = h) = 1$ and $r = 0.5$ in Fig. 1]. The perturbation PV takes the form $q(z) = \delta(z - h)$.

$\beta$ plane

On the $\beta$ plane, the CM structure is different in at least two aspects (solid line in Fig. 1). First, the streamfunction reaches its extremum above (rather than at) the steering level and decays more rapidly downward than on the $f$ plane, but less rapidly upward. Second, the PV is nonzero everywhere and becomes unbounded and changes sign at the steering level.

Further examination shows that above the steering level negative PV is collocated with positive streamfunction (implying cyclonic motion around positive PV), whereas below the steering level positive PV is collocated with positive streamfunction (cyclonic motion around negative PV). This configuration can be understood from a PV perspective. Above the steering level, the interior PV needs to propagate westward relative to the flow (i.e., to counterpropagate), whereas below the steering level, the interior PV has to propagate eastward relative to the flow (i.e., to propagate). To ensure that this relation between streamfunction and PV is satisfied, the PV should generally have larger amplitudes above the steering level than below. Incidentally, this also explains why the streamfunction maximum is found above the steering level.

Figure 2 shows how the CM structure varies as the horizontal wavenumber $\sim 1/r$ is changed. As $r \to 0$ the CM streamfunction becomes increasingly similar to that obtained using the $f$-plane approximation, confirming that the effects of $\beta$ become less relevant at small horizontal scales. However, as $r$ increases (i.e., longer wavelengths) the streamfunction maximum moves upward, away from the steering level $z = h$. The streamfunction also develops new extrema each time $r\beta_c$ passes an integer value (see also section 4b).

4. Semi-unbounded domain

A lower boundary at $z = 0$ is added to the basic state of the previous section. The CM structure then becomes

$$\phi(z) = H(h - z)(a_+\phi_+ + a_-\phi_-) + H(z - h)\phi_-. \quad (11)$$

The constants $a_\pm$ follow from the boundary conditions. In addition to interior PV, the CM also has a boundary potential temperature contribution (edge wave) that propagates on the meridional gradient of basic-state potential temperature. This edge wave is interpreted as a boundary PV $\delta$ function as in Bretherton (1966). On the $f$ plane, the model is generally known as the semi-infinite Eady model (Eady 1949; Chang 1992). Each CM consists of a single interior PV $\delta$ function and an edge wave. A few examples (normalized to have $q_b = -1$) are shown by dashed lines in Fig. 3. Note how the lower boundary for this wavenumber ($r = 0.5$) has a profound effect on the CM structure for steering levels up to $z = 2H$. This emphasizes that, despite its nondispersive propagation behavior, the CM is not simply a slightly modified interior PV $\delta$ function in models with rigid boundaries: the boundary edge wave contribution is often essential.

To investigate the role of the boundary PV more quantitatively, boundary and interior PV are considered...
separately. Their amplitude ratio (interior PV over boundary PV) and phase difference as a function of the steering level of the CM are shown by the dashed lines in Fig. 4. Again, the PV perspective provides a simple framework to interpret these results. For a CM with a critical level above/below $z_{ew}$, the steering level of the free edge wave ($z_{ew} = 1$ in Fig. 4), the phase difference must be zero/π since only in that configuration does the interior PV function act to increase/decrease the speed of the edge wave (required to maintain uniform speed at all levels). Furthermore, CMs with a critical level $h/C_29/2 < z_{ew}$, the edge wave will have a negligible amplitude because the interior PV function induces almost no surface circulation. However, as the critical level $h/C_29/2 > z_{ew}$, the amplitude ratio must tend to zero (no interior PV function) to form a consistent neutral1 mode. As already stated, for all $h < z_{ew}$, the boundary and interior PV have to be π out of phase so as to reduce the speed of the edge wave. In the limit $h \rightarrow 0$, the constraint of zero phase speed can only be met if interior and boundary PV have equal but opposite amplitude, such that $\phi(0) = 0$.

a. β plane

With β included, the model is known as the constant density Charney (1947) model (Kuo 1973). A pair of growing and decaying normal modes exists for all wavelengths, except at a number of neutral points. Examples of CMs are shown in Fig. 3 (full lines). They differ in a similar way from their $f$-plane counterparts, as in the unbounded geometry. The interior PV is nonzero everywhere and has a positive and negative spike near the steering level (not shown).

Because the interior PV is vertically distributed, it makes no sense to compare the amplitude of the boundary PV to the PV at an interior level, such as the critical level. It is more useful to compare it to the vertically integrated interior PV. If boundary PV and vertically integrated PV have the same sign, the two components are said to be “in phase,” whereas if they have opposite sign, they are “π out of phase,” in reference to the situation on the $f$ plane where the two components are just PV functions. The amplitude ratio and phase difference thus defined are shown by the solid lines in Fig. 4. Notice how the β-plane results are very similar to the conceptually much simpler $f$-plane situation. The most important difference is that the location at which the vertically integrated interior PV is zero is shifted downward because the interior PV is not localized into a PV function.

b. Neutral points (for integer values of $r\beta_c$)

This section discusses a special case. It can be shown that $\phi_+(h) = 0$ if $r\beta_c = n$ ($n \in 1, 2, \ldots$). Since $\phi_+(h) \neq 0$ for these values of $r\beta_c$, it follows from (11) that the CM has zero amplitude for $z \leq h$ (this also holds for the unbounded problem). The circulations associated with the PV function at $z = h$ and all other interior PV aloft completely cancel for $z \leq h$. The critical level may even be at $h = 0$. In that case the PV function becomes a

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1 If an interior PV anomaly is located right at the steering level, a linear resonance occurs (Thorncroft and Hoskins 1990; Chang 1992; De Vries and Opsteegh 2005). This case is not further discussed here.
boundary edge wave and the solution thereby a non-singular discrete mode. Indeed, one has arrived at a neutral point in the Charney dispersion relation (Miles 1964). By rewriting $r \beta_e = n$, it is easily shown that the nth neutral point is reached at a total horizontal wavenumber:

$$k^2 + \ell^2 = \frac{1}{n^2} \left( \frac{\beta + s}{2} \right)^2 - \left( \frac{s}{\pi} \right)^2 \geq 0.$$

Since $k^2 + \ell^2 \geq 0$, there is a maximum to the number of neutral points. Were density constant ($s = 0$), there would be infinitely many neutral points but, for a more realistic density profile ($s \approx 2$ for $H_r \approx 7$ km), only the first neutral point remains (Lindzen and Farrell 1980). An example is shown in Fig. 5.

5. Ekman damping at the surface

This section briefly discusses how Ekman damping modifies the CM structure. Boundary Ekman damping is included so as to crudely represent frictional effects. It is well known that sufficiently strong Ekman damping can completely neutralize the basic states of the Eady and Charney model. Nevertheless, significant transient development does occur in these models for favorably configured initial conditions, as shown for example by Farrell (1985), emphasizing that the CS can be important even if Ekman damping is present.

Figure 6 shows the amplitude ratio and phase difference of the interior and boundary PV as a function of the steering level height for $r = 0.5$ and various values of the Ekman parameter $\alpha$ for the $f$ plane. The most important effect of Ekman damping is that the CM develops a westward tilt with height between the boundary PV anomaly and the interior PV $\delta$ function to compensate
for the energy lost due to surface friction. The phase difference decreases rapidly when the critical level of the CM moves through $z_{ew}$, from $\pi$ (interior PV hindering the propagation of the boundary PV anomaly) at low altitudes to 0 (interior PV helping the boundary PV to propagate) at high altitudes. The Ekman damping also affects the amplitude ratio with the interior PV amplitude for nonzero $\alpha$ being larger for all steering levels compared to the undamped situation. CMs with a steering level near $z_{ew}$ (in the figure at $h = 1$) are changed most.

In the undamped case, PV at $z_{ew}$ produces a resonance, resulting in sustained linear amplification of the surface edge wave $\pi/2$ downshear of the interior PV anomaly (Chang 1992). With Ekman damping included, there is no resonant solution and a neutral mode can be formed. Note that a neutral CS can still exist in the presence of Ekman damping because there is no interior dissipation. If interior damping is present, the CMs will be nonneutral and decay with time (not shown).

The addition of $\beta$ leads to similar modifications of the CS as in the previous sections and is not shown.

Acknowledgments. Discussions with John Methven and Tom Frame are appreciated. The author is jointly funded through the National Environmental Research Council (Grant NE/D011507/1) and the Netherlands Organisation for Scientific Research (Rubicon grant).

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