Noise-Induced Instability in the ENSO Recharge Oscillator

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ABSTRACT

The conceptual El Niño–Southern Oscillation (ENSO) recharge oscillator model is used to study the linear stability of ENSO under state-dependent noise forcing. The analytical framework developed by Jin et al. is extended to more fully study noise-induced instability of ENSO. It is shown that in addition to the noise-induced positive contribution to the growth rate of the ensemble mean (first moment) evolution of the ENSO cycle, there is also a noise-induced instability for the ensemble spread (second moment). These growth rates continue to increase as the strength of the multiplicative noise increases. In both the analytical solution and the numerical model, the criticality threshold for instability of the second moment occurs at a lower value of the parameter that measures multiplicative forcing than the threshold for the first moment. The noise-induced instability not only enhances ENSO activity but also results in a large ensemble spread and thus may reduce the effectiveness of ENSO prediction. As in the additive noise forcing case, the low-frequency variability in the forcing is the important part for forcing El Niño events and the high-frequency forcing alone cannot effectively excite ENSO.

1. Introduction

Over the course of the study of El Niño–Southern Oscillation (ENSO), models of varying amounts of complexity have been used. These results suggest that the deterministic part of ENSO is described by the first few coupled modes of variability in the equatorial Pacific (e.g., Anderson and McCreary 1985; Cane and Zebiak 1985; Suarez and Schopf 1988; Battisti and Hirst 1989; Philander 1990; Jin 1997; Neelin et al. 1998; Bejarano and Jin 2008). However, ENSO itself is irregular, not entirely deterministic. Partly these irregularities may be explained by stochastic forcing (Penland 1996; Penland and Sardeshmukh 1995; Kleeman and Moore 1997; Moore and Kleeman 1999; Flügel and Chang 1996; Flügel et al. 2004; Zavala-Garay et al. 2003, 2004, 2005, 2008). The stochastic forcing represents the assumed random variability of the atmosphere in events like the MJO or westerly wind bursts (WWBs) (e.g., Zhang and Gottshalk 2002; Hendon et al. 2007). For example, the 1997/98 El Niño was shown to be excited by three sequential events (McPhaden 1999; McPhaden and Yu 1999; Vialard et al. 2001). If these high-frequency phenomena are not affected by the underlying state of ENSO, then they may be treated as simple additive noise to excite ENSO. This has led to ENSO being viewed as a weakly damped linear oscillator with stochastic forcing that is coherent in space. (Penland and Sardeshmukh 1995; Penland 1996; Federov et al. 2003; Flügel et al. 2004; Kleeman and Moore 1997; Moore and Kleeman 1999, 2001).

Recently, it has been suggested that the high-frequency events that the stochastic forcing simulates are not independent of the background state of ENSO (Keen 1982; Kessler et al. 1995; Kessler and Kleeman 2000; Vecchi and Harrison 2000; Yu et al. 2003; Eisenman et al. 2005; Perez et al. 2005; Gebbie et al. 2007; Tziperman and Yu 2007). Some of these findings (Keen 1982; Lengaigne et al. 2004) suggest that a nascent El Niño event may increase the likelihood of an additional MJO–WWB event that would act to enhance the El Niño event by further shifting the western Pacific warm pool eastward and initiating more of the downwelling Kelvin waves that create the leveled equatorial thermocline associated with an El Niño event. Thus, if the fast atmospheric and slow ENSO time scales are coupled, as indicated by these studies, then changes in ENSO stability and predictability also occur (Jin et al. 2007).

Specifically, in continuation of the Jin et al. (2007) study, the aims of this study will be to further examine the theoretical framework put forth by Jin et al. (2007)
by expanding the ensemble-mean dynamical equations from second- to fourth-order closure. Moreover, different types of noise (peaked and skewed) will be introduced and the ways in which these different types of state-dependent (multiplicative) noise alter ENSO stability and predictability will be examined.

The rest of this paper is organized as follows. In section two, the ensemble-mean dynamical framework will be further developed and the growth rate of the system solved analytically. In section three, with a numerical ensemble approach, the impacts of multiplicative noise on the stability of different statistical moments will be explored. In section four, the effectiveness of different kinds of noise will be explored. In section five, the effect of a simple nonlinearity will be examined. Finally, section six contains a summary and discussion of the work.

2. The ensemble-mean dynamical framework

a. The recharge oscillator with multiplicative noise

This study will use the multiplicative noise approach, which was added symbolically to the recharge oscillator equations in Jin et al. (2007) and is equivalent to the correlated additive and multiplicative (CAM) noise described by Sardeshmukh and Sura (2009):

\[
\begin{align*}
\frac{dT}{dt} &= -\lambda T + \omega h + \sigma \xi(t) G \\
\frac{dh}{dt} &= -\omega T \\
\frac{d\xi}{dt} &= -r\xi + w(t)
\end{align*}
\]

(1)

In these equations, \(w(t)\) denotes white noise with a Gaussian distribution. Then \(\xi(t)\) is a red noise with decorrelation time scale of \(r^{-1}\). That the red noise forcing is modulated by ENSO SST anomalies is simply expressed as \(G = 1 + BT\). It is found that when \(B \neq 0\), or when the noise is modulated by ENSO, the dynamic of ENSO itself is altered [Jin et al. 2007; more information on stochastic systems can be found in references such as Gardiner (1982) and van Kampen (2007)]. The values of the constants in this study are set to \(\lambda = 1/6\) month\(^{-1}\), \(r = 1/1.5\) month\(^{-1}\), \(\alpha = 1/\sqrt{24}\) month\(^{-1}\), and \(\omega = 2\pi/48\) month\(^{-1}\).

Recent observational evidence suggests that the simple coupling factor \(B\) (representing the multiplicativeness of the noise) used in this model is significant and has increased over the past few decades (Kug et al. 2008). With the support of a relevant simple model represented by Eq. (1), the goal of this work is to further examine the behavior of this simple conceptual ENSO model under modulated stochastic forcing.

b. The first-moment equations and noise-induced instability of ENSO

As shown in Jin et al. (2007), the modulated noise has a profound effect on the ENSO stability. There, they analytically depicted the ensemble-mean dynamics of the system under second-order closure. Using the approach outlined by Jin et al. (2006), starting from Eq. (1), which describes the recharge oscillator, the variables are split into ensemble mean and departure terms,

\[
\begin{align*}
T &= \langle T \rangle + T', \\
h &= \langle h \rangle + h',
\end{align*}
\]

(2)

where \(\langle T \rangle\) and \(\langle h \rangle\) depict the ensemble means and \(T'\) and \(h'\) depict the departure from the means for \(T\) and \(h\) respectively, as has been done for many other studies (e.g., Lau 1988).

As shown in Jin et al. (2007), the resultant ensemble mean can be written as follows:

\[
\begin{align*}
\frac{d\langle T \rangle}{dt} &= -\lambda \langle T \rangle + \omega \langle h \rangle + \sigma B \langle \xi T' \rangle \\
\frac{d\langle h \rangle}{dt} &= -\omega \langle T \rangle
\end{align*}
\]

(3)

and

\[
\begin{align*}
\frac{d\langle \xi T' \rangle}{dt} &= -(\lambda + r)\langle \xi T' \rangle + \omega \langle \xi h' \rangle \\
&\quad + \sigma (1 + B(T)) + \sigma B \langle \xi^2 T' \rangle \\
\frac{d\langle \xi h' \rangle}{dt} &= -r \langle \xi h' \rangle - \omega \langle \xi T' \rangle.
\end{align*}
\]

Jin et al. (2007) proposed a second-order closure assumption of \(\langle \xi^2 T' \rangle = 0\). In continuation of the approach used in Jin et al. (2007) and outlined previously in the literature (Jin et al. 2006; Sardeshmukh et al. 2003), the system of equations can be expanded to fourth-order closure. Other approaches to similar closure problems can be found in Baerentsen and Berkowitz (1984), Weil (1990), Du et al. (1994), and Luhar et al. (1996). In this higher-order system, the equations for the third-order terms, starting with \(\langle \xi^2 T' \rangle\) and \(\langle \xi^2 h' \rangle\), were derived and examined:

\[
\begin{align*}
\frac{d\langle \xi^2 T' \rangle}{dt} &= -(\lambda + 2r)\langle \xi^2 T' \rangle + \omega \langle \xi^2 h' \rangle + \sigma B \langle \xi^3 T' \rangle, \\
\frac{d\langle \xi^2 h' \rangle}{dt} &= -2r \langle \xi^2 h' \rangle - \omega \langle \xi^2 T' \rangle.
\end{align*}
\]

(5)

Here we have followed Jin and Lin (2007) and Sardeshmukh et al. (2003) to adopt the fourth-order
sets of strongly negative values, relating to other order that is near zero or positive, increasing with increasing steady state, the equations can be solved analytically. Solving for the eigenvalues of the system shows one set that is near zero or positive, increasing with increasing values of $B$, largely relating to the first moments, and two sets of strongly negative values, relating to other order moments. The growth rate for ENSO can be expressed approximately as follows:

$$\text{Gr}_4 \approx \frac{\lambda^2 B^2}{2r}.$$  \hfill (6)

The growth rate shows only small changes from the solution of the system under the second order closure conditions derived in Jin et al. (2007). The growth rate is a function of $B^2$. As $B$ increases, the growth rate increases. Since $B$ represents the strength of the multiplicative noise in the system, the stronger the multiplicative noise is, the more unstable the system is. The differences between the numerical and analytical solutions are negligibly small (Fig. 1).

c. The second-moment equations and noise-induced instability for ENSO spread

The covariance matrix was also derived in Jin et al. (2007):

$$\frac{d\langle T^2 \rangle}{dt} = -2\lambda \langle T^2 \rangle + 2\omega \langle h'T' \rangle + 2\sigma(1 + B(T))\langle \xi T' \rangle + 2\sigma B\langle \xi T^2 \rangle,$$

$$\frac{d\langle h^2 \rangle}{dt} = -2\omega \langle h'T' \rangle,$$

$$\frac{d\langle h^2 T' \rangle}{dt} = -\lambda \langle h'T' \rangle + \omega(\langle h^2 \rangle - \langle T^2 \rangle) + \sigma(1 + B(T))\langle \xi h' \rangle - \langle \xi h'T' \rangle.$$  \hfill (7)

The covariance matrix produces two third-order terms, $\langle \xi T^2 \rangle$ and $\langle \xi T'h' \rangle$, whose governing equations can be derived as follows:

$$\frac{d\langle \xi T^2 \rangle}{dt} = -(2\lambda + r)\langle \xi T^2 \rangle + 2\omega \langle \xi h'T' \rangle + 2\sigma(1 + B(T))\langle \xi^2 T' \rangle - 2\sigma B\langle \xi^2 T^2 - \langle \xi T^2 \rangle \rangle,$$

$$\frac{d\langle \xi h^2 \rangle}{dt} = -2\omega \langle \xi h'T' \rangle - r\langle \xi h^2 \rangle,$$

$$\frac{d\langle \xi h^2 T' \rangle}{dt} = -\lambda \langle \xi h'T' \rangle + \omega \langle \xi h^2 \rangle + \sigma(1 + B(T))\langle \xi^2 h' \rangle + \sigma B(\langle \xi^2 h'T' \rangle - \langle \xi h' \rangle \langle \xi T' \rangle) - r\langle \xi h'T' \rangle - \omega \langle \xi T^2 \rangle.$$  \hfill (8)

There are now a total of an extra three fourth-order variables that need to be solved for: $\langle \xi^2 T^2 \rangle$, $\langle \xi^2 h^2 \rangle$, $\langle \xi^2 h'T' \rangle$, and $\langle \xi^2 T' \rangle$. Again following Jin and Lin (2007), we adopt the following fourth-order closure conditions for the three new variables:

$$\langle \xi^2 T^2 \rangle = 2\langle \xi^2 \rangle \langle T^2 \rangle = 2\sigma \langle T^2 \rangle,$$

$$\langle \xi^2 h^2 \rangle = 2\langle \xi^2 \rangle \langle h^2 \rangle = 2\sigma \langle h^2 \rangle,$$

$$\langle \xi^2 h'T' \rangle = 2\langle \xi^2 \rangle \langle h'T' \rangle = 2\sigma \langle h'T' \rangle.$$  \hfill (9)

Then Eqs. (3), (4), (5), (7), and (8) form a closed system. Using the numerical ensemble described in section 3, we found that $a = 0.7$. The linear set of Eqs. (3), (4), (5), (7), and (8) under fourth-order closure shall be referred to as the EMD equations.

The EMD equations can be separated into two sets of six equations. As stated previously, the first set of six equations [(3), (4), and (5)] essentially describe the first-moment (ensemble mean) evolution of the system. These equations are independent of the rest of the equations. It should be pointed out that Eqs. (4) and (5) are dominated by the fast time scale related to the noise auto-decay time scale $1/r$. Therefore, using the so-called quasi-equilibrium approach proposed in Jin et al. (2006) and Jin and Lin (2007), we set $d/dt = 0$ in Eqs. (4) and (5). These equations can be reduced to a set of two equations, whose eigenvalues can be solved analytically. These solutions have previously been discussed (see Fig. 1).

The second set of six equations relates to the second-moment (ensemble variance) evolution of the ensemble system and is contained in Eqs. (7) and (8). These six equations are not independent of the first-moment evolution equations. Since the six first-moment evolution equations are independent of the second-moment equations, the ensemble-mean evolution of the system only
affects the steady-state solutions of Eqs. (7) and (8). However, the eigenvalue solution about the steady state of the covariance matrix is independent of the ensemble-mean evolution of the system. Again, using the quasi-equilibrium approximation by setting $d/dt = 0$ in Eq. (8), the six equations can be reduced to three and the eigenvalues of the system may be solved analytically. However, the solutions are complicated. Assuming that the growth/damping rate of the ENSO oscillator is close to zero, we can obtain the following approximate solution for the growth rate:

$$Gr_c \approx -\lambda + \frac{2\alpha r^2 B^2}{r}.$$  \hspace{1cm} (10)

In the simple analytically solved cases, one of the three solutions has a zero frequency similar to the numerical solution. However, the three solutions all have the same growth rate. These solutions lead to a growth rate that changes as a function of $B^2$, similar to the first moment equations. In addition, when comparing Eqs. (10) and (6), we found that as $B$ increased, the second moment becomes supercritical first. In other words, the noise-induced instability for the second moment increases faster than for the first moment as $B^2$ increases, which is a new finding, the consequences of which will be discussed later.

Without the quasi-equilibrium approximation, we can solve the eigenvalues of the EMD equations numerically. The result shows that the eigenvalues eliminated by the quasi-equilibrium approximation are strongly damped with a time scale on the order of the noise time scale, $1/r$, for all values of $B$ (not shown), which confirms the validity of the quasi-equilibrium approach. There are three sets of eigenvalues that become supercritical as $B$ increases (Fig. 2). These eigenvalues relate to the slow time scales that come from the slow dynamics of the system and are related to first and second moments of the system. These sets of eigenvalues eventually become unstable with increasing values of $B$. Upon closer inspection (not shown here), the zero-frequency eigenvalue has a positive growth rate at $B = 1.18$. This mode is mostly related to the mixed heating-temperature term in the covariance matrix.

The second set of eigenvalues to become positive is related to the ensemble variances of the temperature and heat content. This growth rate becomes positive at $B = 1.33$. Although these two growth rates are not exactly the same, they are similar as expected from the approximate analytical solution in Eq. (10) (Fig. 3). This instability of the second moment explains the large increase in the ensemble variance, which can also be seen by the time-integrated EMD equations for $B = 1.4$, at which point the second moment shows exponential growth (not shown).

The third set of eigenvalues has a steadily increasing growth rate with increasing values of $B$; however, its growth is much slower than the set of eigenvalues associated with the second moment, starting with less damping at $B = 0$ and not becoming positive until $B = 1.70$ (not shown here). This set of eigenvalues relates to the first moment of the oscillation. These results are consistent with the results first derived by Jin et al. (2007).

These different criticalities for the first and second moments derived from the linear eigenanalysis are unexpected. It begs the question “What are the impacts of these instabilities on the ensemble mean and variance evolutions?”, which will be addressed in the later sections.

3. The numerical ensemble

a. Ensemble simulations

Following the same approach as Jin et al. (2007), we further examine and compare the fourth-order EMD equations with the numerically integrated ensemble simulations. The Gaussian red noise in this study is generated numerically by the same method used in Jin et al. (2007) and Jin and Lin (2007). An ensemble of a large number of members (5000) of red noise time series $[\xi_j(t),$ where $j$ denotes the realization] is generated. The red noise time series is 10 years long.

Starting a year into the numerically generated red noise realization to account for spinup time, we integrate Eq. (1) for 9 yr to create a 5000-member ensemble. The initial conditions are chosen to be $T = 0$ and $h = 2$ as specified by Jin et al. (2007). The initial conditions are chosen for the onset of a warm ENSO phase.
The model was integrated using both a simple forward stepping method and the Heun method. The differences between these two integration methods were negligible and all the subsequent results are created using the Heun method.

Based on the eigenanalysis of the EMD equations, the eigenvalues display critical values at different values of $B$ (1.18, 1.33, and 1.70), which guides us to set the values of $B$ that will be used throughout the experiments. The values of $B$ are chosen in order to examine the transitions from either side of these critical values. The values of $B$ chosen to do this are 0, 0.7, 1, 1.2, 1.4, 1.7, and 1.8. We chose $B = 0$ in order to examine the system without the multiplicative noise; $B = 0.7$ is approximately the observed value (Kug et al. 2008). The parameter range of $B = 1$ to $B = 1.2$ effectively brackets the first critical point. The parameter range of $B = 1.2$ to $B = 1.4$ effectively brackets the second critical point. Finally, the parameter range of $B = 1.7$ to $B = 1.8$ accounts for the effects of the unbounded growth in the ensemble-mean evolution, bracketing the third critical value. At these chosen values of $B$, the EMD equations are also integrated with a simple forward time step; thus, under each set of conditions, the series of different values of $B$ are used to examine the effect of the coupling on the results and time evolution of the system.

The time evolution of the ensemble mean for $B = 0, 1, 1.4,$ and 1.8 are shown in Fig. 4a. As the value of $B$ increases, the growth rate of the system increases. This is evident from the increase in the magnitude of the initial El Niño event that peaked at about 400 days, the second El Niño that peaks around 1800 days, and the magnitude of the La Niña event that occurs between the two warm events when values of $B$ are large. Indeed, as $B$ becomes supercritical ($B > 1.7$), ensemble-mean evolution becomes unstable, consistent with the eigenanalysis.

Jin et al. (2007) also suggested that there exists increasing spread between the different runs and that the spread increases most markedly in the first year. This can be confirmed by examining the ensemble standard deviation time evolution in Fig. 4b. Again, as predicted in section 2, the second moment explodes by $B = 1.4$. This noise-induced instability of the second moment is intriguing and requires further examination.

Noise-induced instability in the ensemble variance can be seen by examining the partial distribution function (PDF) for the system, which is approximated by taking the histogram of the temperature anomaly and of its associated second-moment PDF, which is defined as $f(x) \times x^2$, where $f(x)$ is the PDF; thus, $\int f(x) \times x^2 \, dx$ is the second moment (Fig. 5). For these PDFs, a 50,000-member ensemble was used. The third-moment PDF, which is similarly defined as $f(x) \times x^3$, where $f(x)$ is the PDF—thus, $\int f(x) \times x^3 \, dx$ is related to the skewness and asymmetry of the ensemble—is also shown. The histogram of the temperature anomaly shows that the distribution is Gaussian when $B = 0$ and increasingly non-Gaussian when $B$ increases, as suggested by Sardeshmukh and Sura (2009). The positive tails in the temperature anomaly histogram are hard to separate. In contrast, the histogram of the second-moment tails becomes larger with larger values of $B$, clearly indicative of the noise-induced instability of the second moment and other higher moments. The magnitude of the third-moment PDF increases as $B$ increases and the positive values of third moment are larger in absolute value than
the negative values. This suggests an increase of the skewness toward positive El Niño events of the ENSO system. In both the second- and third-moment PDFs, for the higher values of $B$, the PDF does not return to the $x$ axis. As such, the integral is no longer finite, which is equivalent to a positive growth rate for these moments and is indicative of noise-induced instability.

b. Super El Niño events

The noise-induced instability of the second- and higher-order moments suggest that the system might be prone to producing some ENSO events that are significantly larger than the average one as an example of noise-induced ENSO asymmetry. Because the global repercussions of ENSO do not scale linearly with the size of the event, these stronger events require further examination. To further explore the possibility of these events, dubbed “super” events, the model was integrated in a different setup than the ensemble described earlier in this section. Instead of having a 5000-member ensemble of 9-yr realizations, individual 5000-yr realizations were examined. An example of 100 years from this realization can be seen in Fig. 6. The threshold for an event was defined to be $T > 0.5$. Further, to be counted as an event, the temperature had to remain above the El Niño event threshold for 180 days around the time of the maximum value. The threshold temperature for a super El Niño event was $T > 4.0$ and it only had to remain above the super threshold at the peak of the event.

In agreement with the noise-induced instability of the second moment from the analytical solutions and the ensemble, the standard deviation of this long time integration increases slowly for $B < 1$ but increases three orders of magnitude from $B = 1.4$ to $B = 1.8$. The total number of El Niño events increases with increasing value of $B$, and the number of super El Niño events increases even faster. The mean peak anomaly for both an event and a super event increases with increasing $B$. The largest increases in these measures occur after the mean value has become unstable from $B = 1.7$ to $B = 1.8$, adding additional evidence for the noise-induced instability of the first moment around the values of $B$ predicted by the analytical solution (Table 1). This occurs because of the noise-induced instability of the second- and higher-order moments. At higher values of $B$, the noise and the background state become more interdependent. This allows an El Niño event to begin under less favorable conditions than would occur at
lower values of $B$ and to persist until the conditions become favorable and the event becomes a super event. These El Niño and super El Niño events are composited around their peak anomaly; the mean event and super event are described by Fig. 7. Since red noise contains patterns at all different frequencies, the low-frequency part of the noise allows for long buildups of positive forcing episodes. This is what triggers an event or super event in this conceptual model. The lower the value of $B$, the longer the buildup required to force an event. However, the multiplicative part of the noise forcing means that the largest values of forcing are recorded with the largest values of $B$. Further, the positive skewness of the model is confirmed with the number of El Niño events exceeding the number of La Niña events at all values of $B$ (not shown).

4. The effect of other types of noise

The simplest analogue to the model is the forcing from atmospheric processes as red noise (e.g., Penland 1996). However, there are phenomena that may not be simply regarded as a red noise process. For instance, the MJO, as an organized, propagating convective system has a distinct period of 30–60 days. This can more generally be classified as an intraseasonal oscillation. The MJO has been shown to influence El Niño events (e.g., McPhaden 1999; van Oldenborgh 2000). Further, westerly wind bursts are more common and stronger than easterly ones, so one must ask whether the skewness of

<table>
<thead>
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<th>$B$</th>
<th>Standard deviation</th>
<th>Number of events</th>
<th>Number of super events</th>
</tr>
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<tr>
<td>0.0</td>
<td>0.54</td>
<td>728</td>
<td>0</td>
</tr>
<tr>
<td>0.7</td>
<td>0.67</td>
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<tr>
<td>1.2</td>
<td>1.56</td>
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<td>839</td>
<td>266</td>
</tr>
<tr>
<td>1.7</td>
<td>88.1</td>
<td>889</td>
<td>462</td>
</tr>
<tr>
<td>1.8</td>
<td>2230</td>
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the noise plays an important role in the modulation of ENSO. In general, it is worth understanding what would happen to the recharge oscillator if it were excited by other types of noise forcing.

Two different types of noise were examined to see their different effects on the numerically solved recharge oscillator model. A positive skewed noise, \( \xi^2 - 1 \), was created algebraically from the red noise, \( \xi \). This skewed noise is meant to mimic the nonsymmetric forcing from westerly wind bursts. Also, a peaked noise was considered by assessing \( \frac{d\xi}{dt} = -(r + i\omega_0)\xi + w(t) \), where \( \omega_0 = 2\pi/60 \text{ days}^{-1} \) and \( w(t) \) is white noise; thus, the peaked noise peaks smoothly at the MJO frequencies. It contains only a small amount of energy at very low frequencies. All of the different noises were renormalized so that they are of unit variance. The power spectrums of red noise and skewed noise are very similar, with skewed noise containing slightly less power at low frequencies and more at high frequencies. They both have white noise–like spectrums at frequencies below the decorrelation time scale \( 1/r \) and have negative slopes after. The peaked noise has little power in the low frequencies, rising quickly to a peak at \( \omega_0 \) and then descending to similar power as red and skewed noise. The partial distribution functions for red and peaked noise are the same broad Gaussian distribution with a mean of 0. The partial distribution function of skewed noise has a sharp peak between 0 and \( -1 \), which decays quickly toward 0 for increasingly negative numbers. However, the positive tail of the skewed noise partial distribution function decays much more gradually and is broadened at the large positive values of \( \xi \). The power spectrums and partial distribution functions of the different types of noise can be found in Figs. 8a and 8b, respectively.

For low values of \( B \), the effects of the different types of noise on ensemble mean evolution are all weak. With increasing values of \( B \), the ensemble-mean evolution of the skewed noise forced realizations show increasingly large temperature anomalies in the initial El Niño and the following La Niña. In general, the skewed noise runs follow the same overall evolutionary pattern as the red noise (Fig. 9).

Overall, the introduction of the different types of noise reduces the variability of the ensemble. The difference is more marked both at the larger values of \( B \), where the changes in the noise affect the evolution to a greater extent, and later in the evolution, where the effect of noise-induced growth becomes clear (Fig. 10). The differences between the ensemble-mean and variance growth rates are more easily examined in the

![Fig. 7. (top) The mean event with its peak temperature displacement set at 0 on the x axis. (bottom) The corresponding mean forcing \([\xi(1 + BT)]\) for the events. Super El Niño events are represented by the bold lines.](image1)

![Fig. 8. (a) Spectrum of the three different types of noise used. (b) The partial distribution function for the three different types of noise used. Red and peaked noises have the same PDF.](image2)
signal-to-noise ratio (Fig. 11). As the value of $B$ increases, the signal-to-noise ratio decreases. This is a manifestation of the decreasing predictability with increasing multiplicativeness. This is true for all types of noise. Peaked noise forcings have a drastically reduced effect (Figs. 9 and 10). This is because only the low-frequency parts of the noise spectrum are effective in exciting ENSO. In fact, we also considered blue noise (not shown), which has no low-frequency component in its power spectrum. We found that it is entirely ineffective in exciting ENSO. It is well known that this is the case for the additive noise case, (e.g., Roulston and Neelin 2000) and it is also true for the multiplicative noise cases.

Using the same definitions and procedures as in section 3b, 5000-yr realizations with both skewed and peaked noises were also examined. The skewed noise remains fairly similar to the red noise case, while the lack of variance in the peaked noise case is further evident. In the additive noise case, the skewed noise (Table 2) has fewer total El Niño events but a larger number of super events. As the strength of the multiplicative noise increases, both the total number of El Niño events and number of super events increases, similar to the red noise case. However, whereas at lower values of $B$, there are more super events for skewed noise, at the larger values of $B$, there are fewer. This transition happens at $B = 1.4$, when the noise-induced instability of the second moment occurs for the red noise realizations. Since the skewed noise realizations do not undergo this same transition, this change can be explained by the noise-induced instability of the second moment. Similarly, as predicted by the examination of the 5000-member ensemble standard deviation, the standard deviation of the skewed noise long realizations does not grow as large as the red noise case for the largest values of multiplicative noise forcing, nor at any point are there as many El Niño events. The skewed noise also requires less buildup time of positive noise forcing to force either an El Niño or super El Niño event (not shown). The peaked noise does not produce any El Niño events.

5. The effect of nonlinearity

a. A symbolic nonlinear damping term

Following Jin (1997), symbolic nonlinearities are further taken into account through the addition of a nonlinear
damping term to the temperature tendency equation in Eq. (1), which takes the form of

\[
\frac{dT}{dt} = -\lambda T + \omega h + \sigma \xi (1 + BT) + \left( \frac{1}{C} T \right)^3. \tag{11}
\]

The value of the constant, \( C \), is determined to be \( 3 \text{ yr}^{-1} \) using a heat budget analysis, such that the amplitude of ENSO is on the order of \( T = \sqrt{\lambda} \). Experiments were performed with different values of \( C \) and the results are not qualitatively different.

**b. Nonlinearity and noise-induced instability**

The addition of a nonlinear damping term acts to effectively control the linear growth rate of the system. Still, at larger values of \( B \), the ensemble mean takes more than one oscillation to damp out the initial perturbation (Fig. 12a). The standard deviation increases rapidly in response to the initial perturbation, but then slowly declines as time increases for all values of \( B \) (Fig. 12b). Examining the effects of the nonlinear damping term through the moment PDFs confirms the conclusions from the mean and standard deviation plots. The first, second, and third moments are finite even at the large values of \( B \), where they were not in the case without nonlinear damping (Fig. 13).

**c. Nonlinearity and super El Niños**

As demonstrated in the previous subsection, the addition of a nonlinear damping term acts to inhibit exponential growth in the system. With the simple...
nonlinearity, there is a sharp decrease in the standard deviation in both the red and skewed noise forcing scenarios. Similarly, a large effect of the nonlinear damping term is seen in the peak magnitudes of the average event and super event, which are drastically reduced from the realization without the damping at the largest values of $B$ (Table 3). The nonlinear damping term is cubic in temperature, so its effects are magnified as the temperature increases. Effectively, this term is in competition with the multiplicative noise term, which is linearly dependent on temperature. Through the nonlinear rectification, the nonlinear damping stabilizes the system.

6. Summary and discussion

a. Summary

The eigenvalue analysis of the EMD equations shows that there are two different types of noise-induced instabilities: the instabilities for the first and second moments. As the parameter $B$, the measure for the strength of the multiplicative noise, increases, the eigenvalues that control the second moment become supercritical and the covariance field of the system becomes unstable. This implies that noise-induced instability leads to larger ensemble spreads and thus degrades the predictability of the system. As $B$ further increases, the first moment
evolution becomes increasingly unstable with the strengthening of multiplicative noise in addition to ENSO, its spread, and its asymmetry. The ensemble variance evolution becomes increasingly unstable with the strengthening of multiplicative noise in addition to the ensemble mean cycle, as was discussed in Jin et al. (2007). Practically speaking, the state-dependent coupling of ENSO and shorter-lived events, such as the MJO and WWBs, represented by the multiplicative noise in the system, acts to increase the instability of the system, making larger El Niño events. This coupling can lead to the repeated occurrence of individual forcing events, as seen in the initial stages of the 1997/98 El Niño event (McPhaden 1999; McPhaden and Yu 1999; Vialard et al. 2001), to create the super events created in this study. However, this same coupling can negatively affect the predictability of an event by increasing the variability of the system because an additional WWB or MJO event affects ENSO to a much greater degree and the likelihood of another strong WWB is increased with an increased value of multiplicative noise constant. This is consistent with what Lengaigne et al. (2004) found in examining the differences between the strong and moderate events after inserting a westerly wind event into a coupled global model.

Examination of the third moment of the system suggests an increase in the skewness of the system or ENSO asymmetry, in agreement with Sardeshmukh and Sura (2009). An increase in skewness of the already positively skewed system combined with the increased variance from second-moment instability may lead to a larger increase in the number and magnitude of large El Niño events, as Lengaigne et al. (2004) found. We suggest that this finding should be further investigated in a more complex model for a better determination of how the strength of the state-dependent noise affects the size and rate of occurrence of these strong El Niño events.

Further, the different types of noise used to force this conceptual model show that the low-frequency part of the noise forcing is important to the generation of El Niño events and the multiplicative blue and peaked noise forcings are as ineffective as the additive noise in exciting El Niño events. While many studies suggest that short time scale forcings can modify El Niño events in models (Roulston and Neelin 2000) and observations (Zhang and Gottshalk 2002; McPhaden 1999, etc.), the actual noise must not solely be confined to the short periods associated with these forcings. This study agrees with the conclusion of many other studies (Roulston and Neelin 2000; Zavala-Garay et al. 2003, 2005, etc.) that the low-frequency tail of the noise forcing is responsible for triggering of El Niño events. As such, we suggest that the strength of the nonlinear rectification of WWBs and the MJO be further studied in an atmospheric general circulation model to ascertain the strength of the low-frequency tail created by the nonlinear dynamic processes of the atmosphere.

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REFERENCES


