Gravity Waves, Dynamical Resistance, and Forcing Efficiency

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(Manuscript received 30 June 2009, in final form 21 October 2009)

ABSTRACT

The effect of the dynamical response associated with high-frequency gravity waves on the total energy generated by imposed heating is examined in a 2D linear compressible model. The work performed by waves against a sustained forcing is defined as the dynamical resistance. The dynamical resistance is minimized and forcing efficiency maximized for basic-state and forcing configurations that yield a wave response whose phase varies minimally relative to the forcing. When generated against a forcing-relative background flow, waves that have a deep vertical scale relative to the forcing depth impose less resistance than waves of a shallow vertical scale. The efficiency of an ensemble of forcing elements is shown to differ significantly from that corresponding to an isolated forcing. If the forcing elements are all of the same sign (e.g., are all warmings), then the efficiency increases with decreasing separation between elements.

1. Introduction

Gravity waves in the atmosphere and ocean have been studied extensively, and much is known about how they propagate, evolve, and affect the ambient flow [e.g., see texts by Lighthill (1978), Gill (1982), Lindzen (1990), and Nappo (2002) for a review of basic gravity wave dynamics]. Particular attention has been devoted to understanding how gravity waves interact with the mean flow in the stratosphere where vertically propagating waves may impose a drag due to convergence of momentum flux [see review by Fritts and Alexander (2003) and references therein]. A problem that is complimentary to the wave–mean flow interaction problem is the wave–forcing interaction problem. The latter has received relatively less scrutiny in the literature (exceptions noted below) but may be of significance to mesoscale dynamics in the troposphere. The wave–forcing interaction problem is of particular relevance to convectively generated waves that, unlike topographically forced waves, originate from sources which are themselves dynamic. Given that space and time scales characterizing waves and convection are similar, source and spectrum may be mutually dependent.

For convenience, the relationship between gravity waves and their sources is usually analyzed as though it proceeds in one direction. Either one analyzes the waves generated by convection, which are assumed to have no subsequent influence on the convection itself (e.g., Bretherton and Smolarkiewicz 1989; Nicholls et al. 1991; Chagnon and Bannon 2005a,b), or one examines the role of waves in the initiation of convection but not in the subsequent evolution of convection (e.g., Marsham and Parker 2006). There have been a few exceptions in which the mutual interaction has been considered. The wave-CISK hypothesis of Lindzen (1974) could be considered the most extreme type of interaction in which no dynamical distinction between waves and convection is made. Cohen and Craig (2004) hypothesized that gravity waves play an important role in establishing equilibrium spatial and temporal scales in fields of convection simulated using a cloud-resolving model but stopped short of identifying the actual mechanisms involved. Lin (1987) demonstrated that the linear wave response to a diabatic heating applied in a sheared flow could produce a maximum vertical velocity that is either in or out of phase with the heating, depending on the Richardson number of the background flow. Most recently, Robinson et al. (2008) found that the amplitude and depth of the response to patches of surface heating can be optimized for specific horizontal scales of the heating that excite a resonant response. A question that follows from these studies is to what extent an evolving internal forcing such as convection might be influenced by interfering tendencies between dynamical response and forcing.

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DOI: 10.1175/2009JAS3244.1

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This paper considers the relationship between high frequency waves and the forcing from which they originate. The effect of waves on forcing (and vice versa) may be thought of as a problem complimentary to the effect of the environment (e.g., wind shear) on waves. As discussed above, the latter has received much attention for its significance to stratospheric dynamics, while the former is relevant to the dynamics of the troposphere. The problem examined in this paper considers a wave forcing mechanism such as that due to the release of a local instability (e.g., convection). The nature of this instability and the origin of the forcing mechanism are not addressed in this paper. If the environment is statically stable, then the forcing will generate a dynamical response involving waves. If the response induces dynamical perturbations in the vicinity of the forcing, then the response may subsequently do work on the forcing itself, thereby acting as either a resistance or conductance to the forcing. Under statically stable conditions, the effect is usually a resistance. This dynamical resistance depends on the properties of the forcing. A forcing that generates a wave response that minimizes the dynamical resistance imposed by the resulting wave field may be characterized as efficient. The analysis contained in this paper explores the effect of the dynamical response on the energetic efficiency of the forcing across a wide parameter space describing the forcing and environment. Although the scope of this paper is limited to the introduction and basic demonstration of this concept, the analysis generalizes the specific examples of enhanced forcing efficiency provided by Lin (1987) and Robinson et al. (2008).

In the next section a linear model is introduced that will allow for a computation of the energetic efficiency of a forcing that excites acoustic and gravity waves in a compressible, inviscid, nonhydrostatic atmosphere linearized about a basic state with arbitrary background flow and static stability. We will not assume to know anything about the actual process that generates the waves. Instead, we will examine the dependence of the forcing efficiency and dynamical resistance on the properties of the forcing as well as the properties of the environment into which the forcing is applied. Section 3 will demonstrate several examples of how energetic efficiency varies with forcing geometry against a background flow. Specifically, experiments are designed to test the effect of forcing duration, forcing-relative flow speed, wind shear, and multiple versus isolated forcing elements. Additionally, total energy calculations are performed along gravity wave particle ray paths in order to clarify some of the results obtained in the linear model simulations. Section 4 discusses the results and suggests routes for further study.

2. Forcing efficiency in a linear model

A linear model will be used to examine gravity waves, dynamical resistance, and forcing efficiency throughout this investigation. The eigensolutions of the linear system comprise acoustic and gravity waves. Waves are excited by arbitrary external forcing terms in the momentum and heat equations. These forcing terms may be associated with “fast” and/or diabatic processes taking place within a cumulus cloud. It will be shown that the total energy excited by the time-dependent forcing is dependent on the work performed by the waves upon the forcing.

The compressible, inviscid equations of motion on an $f$ plane, assuming horizontal homogeneity in the $y$ direction, linearized about a hydrostatic basic state (denoted by a subscript $s$) with arbitrary velocity $u_s$ are

\[
\frac{\partial u_s}{\partial t_a} = fu - \frac{du_s}{dz}w - \frac{1}{\rho_s} \frac{\partial p_s}{\partial x} + F_w, \quad (1)
\]

\[
\frac{\partial w}{\partial t_a} = -fu + F_w, \quad (2)
\]

\[
\frac{\partial w}{\partial t_a} = \frac{g}{\theta_s} \frac{\partial}{\partial x} \left( \frac{1}{\gamma H_s} \right) p + F_w + \frac{\gamma p_s}{\theta_s} F_w, \quad (3)
\]

\[
\frac{\partial \theta}{\partial t_a} = -\frac{\partial}{\partial z} w + F_\theta, \quad (4)
\]

and

\[
\frac{\partial p_s}{\partial t_a} = -\gamma \frac{\partial p_s}{\partial x} - \gamma p_s \left( \frac{\partial}{\partial z} + \frac{1}{\gamma H_s} \right) w + \frac{\gamma p_s}{\theta_s} F_w, \quad (5)
\]

where the operator $\partial/\partial t_a$ is defined as

\[
\frac{\partial}{\partial t_a} = \frac{\partial}{\partial t} + u_s \frac{\partial}{\partial x}, \quad (6)
\]

$H_s = RT_s/g$ is the scale height, $R = 287$ J kg$^{-1}$ K$^{-1}$ is the dry ideal gas constant, $g = 9.816$ m s$^{-2}$ is the vertical component of acceleration due to gravity, $\gamma = c_p/c_v = \gamma_s$ is the ratio of specific heat coefficients, and the time-dependent external forcings of momentum and heat are denoted by $F_u$, $F_w$, $F_w$, and $F_\theta$. Although these forcing terms are likely to be interdependent in the real atmosphere (e.g., Song et al. 2003), the present investigation imposes no such constraints (see section 4 for a discussion of this issue).

The energetics of the system (1)–(5) are governed by

\[
\frac{\partial}{\partial t_a} \left( \frac{\rho_s u_s^2}{2} \right) = \rho_s fuw - \rho_s \frac{du_s}{dz}uw - u_s \frac{\partial p_s}{\partial x} + \rho_s u_s F_w, \quad (7)
\]
\[
\frac{\partial}{\partial t_a} \left( \frac{\rho \nu^2}{2} \right) = -\rho_f w + \rho_s F_v, \quad (8)
\]
\[
\frac{\partial}{\partial t_a} \left( \frac{\rho \nu^2}{2} \right) = -\frac{1}{\gamma H_s} w p - w \frac{\partial p}{\partial \theta} + \frac{\rho_p}{\theta_s} w \theta + \rho_s w F_v, \quad (9)
\]
\[
\frac{\partial}{\partial t_a} \left( \frac{\rho s^2 \theta^2}{2N^2 \theta_s^2} \right) = -\frac{\rho_s}{\theta_s} w \theta + \frac{\rho g}{N^2 \theta_s^2} \theta F \theta, \quad (10)
\]
and
\[
\frac{\partial}{\partial t_a} \left( \frac{p^2}{2\gamma \rho_s} \right) = \frac{1}{\gamma H_s} w p - p \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) + \frac{p}{\theta_s} F \theta, \quad (11)
\]
where (7)–(9) describe the evolution of kinetic energy density (KE), (10) the available potential energy density (APE), and (11) the available elastic energy density (AEE). Summing (7)–(11) yields a total energy (TE) equation in which conversion terms are eliminated from the right-hand side:
\[
\frac{\partial \text{TE}}{\partial t_a} = -\mathbf{\nabla} \cdot (p \mathbf{u}) - \rho_s \frac{du}{dz} uw + F_E, \quad (12)
\]
where the total energy generated by the forcing terms is given by
\[
F_E = \rho_s \left( u F_u + v F_v + w F_w + \frac{g^2}{N^2 \theta_s^2} \theta F \theta + \frac{p}{\rho_s \theta_s} F \theta \right). \quad (13)
\]
The first term on the rhs of (12) is a divergence of energy density flux and vanishes when integrated over the full volume of the domain. The second term that is proportional to the background shear represents loss/gain of horizontal KE due to work done by/against the background shear. The final term is the energy generated by the forcing. Note that in general the TE generated by forcing depends on the dynamical response (except for the case in which the time dependence of the forcing is a Dirac delta function). Chagnon and Bannon (2005b) provide a more complete discussion and presentation of the energetics of this system (in the isothermal, unsheared case), including the relative contribution to the total energy provided by gravity waves, acoustic waves, Lamb waves, and the potential-vorticity-conserving steady state.

We now examine the energy generation by the forcing in more detail. For demonstration purposes, the following analysis will consider a forcing of vertical momentum \( F_w \), but the procedure may be generalized to any other type of forcing. Suppose the forcing is turned on at time \( t = 0 \) and shut off at time \( t = \tau \). Let brackets denote a spatial integral over the full domain. The total energy generation is
\[
E_{\text{gen}} = \left[ \int_0^\tau F_E \, dt \right] = \left[ \int_0^\tau \rho_s w F_w \, dt \right]. \quad (14)
\]
Integrating the vertical momentum equation (3), the vertical velocity perturbations may generally be written as
\[
w(t) = D_w(t) + \int_0^t F_w \, dt, \quad (15)
\]
where the function \( D_w \) represents the effects of the internal dynamics on the evolution of the velocity perturbations. Substituting (15) into (14) yields
\[
E_{\text{gen}} = E_{\text{feg}} - E_{\text{dyn}}, \quad (16)
\]
where
\[
E_{\text{dyn}} = -\left[ \int_0^\tau \rho_s F_w D_w(t) \, dt \right]. \quad (17)
\]
and
\[
E_{\text{feg}} = \left[ \int_0^\tau \left( \rho_s F_w \int_0^t F_w \, dt' \right) \, dt \right]. \quad (18)
\]
For the impulsive forcing (Dirac delta function in time), the total energy generation \( E_{\text{gen}} \) can be computed directly from \( E_{\text{feg}} \). For the finite duration forcing \( \tau > 0 \) \( E_{\text{gen}} \) cannot be computed without first considering the dynamical response \( E_{\text{dyn}} \), which may either amplify or impede the total energy generation relative to \( E_{\text{feg}} \). If the dynamical response \( D_w \) is directed opposite to the forcing \( F_w \), then the dynamical resistance \( E_{\text{dyn}} \) will be positive, thus reducing the total energy generated by the forcing. We may also define a forcing efficiency:
\[
\epsilon = \frac{E_{\text{gen}}}{E_{\text{feg}}}. \quad (19)
\]
Note that an efficiency \( \epsilon = 1 \) corresponds to the case in which there is no dynamical response during the application of the forcing. An impulsive forcing has \( \epsilon = 1 \). Under statically stable conditions the forcing efficiency is typically less than one. Nonetheless, configurations of the basic state and forcing exist that minimize the resistance and maximize the efficiency.

The following section demonstrates the forcing efficiency \( \epsilon \) as a function of properties of the forcing and the
basic state. Although a solution is known for (1)–(5) in the special case that the initial state is isothermal and at rest (Chagnon and Bannon 2005a,b), a solution for the general case is unknown. We therefore compute $E_{\text{dyn}}$ by solving the system (1)–(5) numerically in the following way.

1) A discrete Fourier transform is applied to (1)–(5), resulting in system of equations that depend on horizontal wavenumber $k$ as well as height $z$ and time $t$.

2) Vertical derivatives are represented as centered finite differences accurate to fifth order (i.e., using two points on either side of the central level) except near the boundaries where forward and backward representations of second-order accuracy are used.

3) The resulting time-dependent equations are solved using a third-order Adams–Bashforth scheme, except at the first time step when a forward step is used.

4) An inverse fast Fourier transform is finally applied to the horizontal wavenumber solution to obtain the full solution in physical space.

Boundary conditions are periodic in the horizontal and rigid on the vertical boundaries at $z = 0$ and 25 km. Over the range of forcing parameters used in this investigation, the upper boundary is sufficiently far from the tropospheric wave source to ensure that no waves reflected from the upper boundary can interfere with the forcing at a later time. For example, waves generated at $z = 5$ km elevation would have to travel at a vertical speed exceeding 30 m s$^{-1}$ in order to traverse the necessary 40-km distance to arrive back at the forcing elevation within 20 min. Such a speed is larger than the fastest vertical group speeds (approximately 25 m s$^{-1}$) associated with the deepest nonhydrostatic modes (i.e., vertical wavelength 20 km) propagating against a basic state with $N_2 = 0.02$ s$^{-1}$. Note also that acoustic waves contribute insignificantly to solutions when the forcing duration exceeds approximately 3 min (Chagnon and Bannon 2005a). The numerical solution procedure conserves the total perturbation energy to within one-tenth of a percent for simulations of one hour duration in which the shear is zero.

An advantage of using total energy (and the energetic efficiency $\epsilon$) as a basis for analysis is that it provides one single number to characterize an entire experiment evolving in space and time. It is therefore a convenient diagnostic for exploring a vast parameter space, as is done in the following section. Furthermore, total energy is conserved following termination of forcing, except for the case in which there is a background shear. If the environment contains shear, then the total energy of wave perturbations is not conserved due to the work done by/against the mean flow. We proceed with this caveat in mind, noting that the convenience of using the energetic efficiency to investigate dynamical resistance justifies its application here.

3. Forcing efficiency examples

This section demonstrates the basic properties of the dynamical resistance and forcing efficiency. The examples are designed to illustrate the utility of these concepts for analyzing the relationship between waves and forcing over a wide parameter space. We consider the efficiency of forcing centered at coordinates $x_i, z_i$ of the form

$$F = \delta FS(x, z)T(t),$$

where $S(x, z)$ is a function describing the geometry of the forcing, $T(t)$ is a function describing the time dependence of the forcing, and $\delta F$ is the forcing amplitude. The forcing geometry is Gaussian:

$$S(x, z) = \exp\left[-\frac{(x - x_i)^2}{a} - \frac{(z - z_i)^2}{d}\right],$$

where the parameters $a$ and $d$ describe the width and depth of the forcing, respectively. The time dependence is a sine-squared function:

$$T(t) = \sin^2\left(\pi \frac{t}{\tau}\right)\frac{H(t - \tau) - H(t)}{\tau},$$

where $\tau$ is the forcing duration and $H$ is the Heaviside step function. Note that the Gaussian forcing (21) used in the experiments below is consistent with both upper and lower boundary conditions. The heating is centered in the domain such that the value of $S(x, z)$ is the same on the left and right lateral boundaries, thus avoiding problems with the discrete Fourier transform. The vertical velocity and heating are fixed at zero on the lower boundary to avoid spurious wave generation at the boundary; in all experiments, the heating elevation $z_i$ is chosen to be at least twice the heating depth $d$ so as to avoid large heating gradients near the lower boundary. A Gaussian heating geometry has been chosen rather than heating of finite extent in order to avoid the Gibbs phenomenon and the spurious generation of grid-scale waves.

Three sets of experiments are performed in this section. The first is a control experiment in which the forcing is stationary and there is no background wind. The second set of experiments considers the efficiency of a forcing that is moving relative to the background flow, including
cases with and without environmental shear. The third experiment, presented at the end of this section, examines an ensemble of forcing elements (as opposed to a single isolated forcing). In the case of multiple forcing elements, the dynamical response from nearby forcing events may interfere with neighboring forcing elements. All experiments presented below correspond to the case where $N_s = 0.02 \text{ s}^{-1}$ is constant. Experiments in which $N_s$ varied with elevation were performed (e.g., “tropopause-like” doubling of $N_s$; reduced stability layer above forcing) but did not differ in significant qualitative respects from the results shown below. More exotic experiments such as examination of wave ducts in the vicinity of the forcing would constitute an interesting basis for investigation but are deferred to future studies in favor of presenting as simple and basic an analysis as possible for the present.

a. Control experiment: Isolated forcing, no background wind

The demonstration of forcing efficiency begins with the simple experiment in which waves are generated by an isolated heating applied to a resting atmosphere. The control experiment demonstrates the dependence of total energy generation on the duration of forcing. Figure 1 presents the energy efficiency as a function of $\tau$ during two stages of adjustment—one stage lasting only a few minutes during which acoustic waves are generated, followed by a second stage lasting tens of minutes characterized by the emergence of high-frequency gravity waves. A longer duration forcing generates less total energy than a forcing applied rapidly. Similar results have been demonstrated analytically (see Sotack and Bannon 1999; Chagnon and Bannon 2005a). The loss of energy at longer duration is due to the destructive interference of waves whose period is shorter than the time scale of the forcing. Such behavior is typical in statically stable systems for which the dynamical response is normally directed against an imposed forcing. To explain this behavior, Vadas and Fritts (2001) made an analogy to a system comprising a pendulum fixed to a moveable base. If the base is accelerated rapidly from a resting position (i.e., on a time scale shorter than the pendulum period), then an oscillation may be excited. If the forcing time scale is longer than the pendulum period, then higher frequency oscillations will interfere destructively with the forcing, resulting in an oscillation of lesser amplitude. In the following subsection it is demonstrated that these destructive influences can be overcome if the forcing and environment are configured such that the change in wave phase is minimized relative to the forcing.

b. Forcing against a background flow

This subsection considers the effect of a forcing-relative background flow on $\epsilon$ of an isolated forcing. We first examine an unsheared constant $u_0 = U$ and then examine a flow having constant shear $\Lambda$. The unsheared experiment corresponds to the case where the forcing is moving through a static fluid at a speed $U$. Figure 2 presents the forcing efficiency for varying background flow speed as a function of $k$. To produce the field contoured in Fig. 2, the total energy following a prescribed heating was decomposed into its spectral contributions via Fourier transform along $k$. The efficiency (19) was then calculated as the total energy in $k$ normalized by the Fourier transformed integral (18) evaluated at $k$. The parameter $d$ was varied between a shallow case (500 m, Fig. 2a) and a moderate-depth case (2 km, Fig. 2b). The heating width, elevation, and duration were fixed at $a = 5 \text{ km}$, $z_i = 4 \text{ km}$, and $\tau = 20 \text{ min}$, respectively.

Figure 2 demonstrates that a moving forcing is potentially more efficient than a stationary forcing. The enhancement occurs at larger forcing-relative flow speeds $U$ when $k$ is decreased. In the shallow heating case (Fig. 2a) the envelope of enhanced efficiency covers a wide range of speeds. In the moderate depth case (Fig. 2b) the envelope of enhanced efficiency is narrower and confined to higher flow speeds. Because the solution at each wavenumber shown in Fig. 2 contains contributions from both positive and negative vertical wavenumbers and frequencies (and hence wave particles propagating in all quadrants relative to the source), the efficiency contoured in Fig. 2 is symmetric about the $k$ and $U$ axes. Consequently, only one quadrant is shown in Fig. 2.
A wave particle perspective may be adopted in order to facilitate interpretation of the regions of enhanced efficiency and reduced dynamical resistance in Fig. 2. The essence of the wave particle calculation is as follows (see appendix for more detail). We assume that at its point of origin the dynamical perturbation associated with the wave particle is in phase with the forcing. As the wave particle propagates through the forcing region (along the direction of the group velocity), it has a phase that varies in time. The total energy generated by the forcing along the ray path depends on the work performed by the wave particle against the forcing, as in (A4). This work is denoted by $\text{TE}_{\text{ray}}$. If the particle phase varies slowly along the ray path, then the wave particle performs net work in the direction of the forcing. This net work results in a larger quantity of total energy generated along the ray path. If the particle performs net work directed opposite to the forcing, then $\text{TE}_{\text{ray}}$ is negative. Consider gravity waves in an incompressible atmosphere satisfying the dispersion relation:

$$\omega^2 = N^2 \frac{k^2}{m^2 + k^2},$$

(23)

where $\omega$ is the flow-relative wave frequency, and $m$ is the vertical wavenumber. For a constant background $U$, the total change in phase along a ray path (assuming constant $N_z$) is

$$\delta \phi = kd_{\text{fcg}} \left( \frac{-U + c_x}{c_{gz}} \right),$$

(24)

where $d_{\text{fcg}}$ is the vertical path length through the forcing region, $c_{gz}$ is the vertical component of group velocity, and $c_x$ is the horizontal component of the phase velocity. The total work performed by the wave particle along the ray path in the direction of the forcing, found by evaluating (A5), is

$$\text{TE}_{\text{ray}} = A \frac{i c_{gz}}{k(U + c_x)} \left[ 1 - \exp \left( \frac{ik(U + c_x)d_{\text{fcg}}}{c_{gz}} \right) \right],$$

(25)

where $A$ is a constant proportional to the forcing magnitude and initial wave amplitude. Figure 3 contours TE$_{\text{ray}}$ given by (25) as a function of horizontal wavenumber $k$ and $U$ for a range of initial particle vertical wavelengths $\lambda_z = 2\pi / m$ and forcing regions depths $d_{\text{fcg}}$. All curves have been normalized by the coefficient in (25) such that a value of one corresponds to stationary phase along the ray path. Only that quadrant corresponding to wave particles moving into the background flow is shown in Fig. 3. Features common to all of the panels in Fig. 3 include (i) a primary envelope of enhanced efficiency corresponding to minimal total phase change whose width and mean value along the $U$ axis increase for decreasing $k$ and (ii) additional bands of maxima and minima flanking the primary envelope, which correspond to particles that have undergone multiple cycles (e.g., phase given by $n\pi$). Comparing the panels in the columns from right to left reveals that the width of the envelope is larger for a shallower forcing (i.e., shallower particle path). For a shallow forcing, the particle can exit the forcing region at an earlier time in its phase cycle and is therefore less likely to interfere destructively with the forcing. Comparing the panels in the rows from top to bottom reveals that the maximum efficiency occurs at higher $U$ for particles having deeper $\lambda_z$. Waves with deep vertical wavelength have a larger horizontal phase speed and therefore achieve stationary phase at higher forcing relative flow speeds. These features of the analytic wave particle calculation shown in Fig. 3 describe

![Fig. 2. Plot of $\epsilon$ of midtropospheric heating (elevation $d_e = 4$ km) as a function of $U$ and $k$ for forcing $e$-folding depths characteristic of (a) shallow heating ($d = 500$ m) and (b) moderate-depth heating ($d = 2$ km). The heating duration is $\tau = 20$ min.](image-url)
the general characteristics of the efficiency calculated in the linear model simulations (Fig. 2).

The next experiment examines the effect of a background wind shear on forcing efficiency, the results of which will again be clarified by the wave particle perspective. Consider a background wind profile having a constant shear; that is,

$$u_s(z) = \Lambda(z - z_i). \quad (26)$$

Recall that $z_i$ represents the elevation of the heating maximum. Figure 4 presents the forcing efficiency for varying $\Lambda$ as a function of $k$ for a shallow heating ($d = 500$ m, Fig. 4a) and a moderate depth heating ($d = 2$ km, Fig. 4b). In the shallow heating case (Fig. 4a) the
shear has very little impact on the forcing efficiency after 20 minutes. The forcing efficiency is a decreasing function of wavenumber. Note that higher wavenumbers correspond to wave particles having higher frequencies, which undergo a complete cycle in a shorter period of time and may therefore interfere destructively with the forcing at an earlier time than particles of lower wavenumber. In the moderate depth case (Fig. 4b) bands of both maximum and minimum efficiency are evident that slope upward toward larger shear magnitude at lower horizontal wavenumber. As in Fig. 2, the efficiency contoured in Fig. 4 is symmetric about $k$ and $U$; hence, only one quadrant is shown.

To facilitate explanation of the results in Fig. 4, we again adopt a wave particle perspective. For a background wind with constant $\Lambda$, as in (26), the work performed by a wave particle in the direction of the forcing along the ray path, found by evaluating (A5), is

$$T_{E\text{ray}} = A \exp\left(-\frac{i\alpha^2}{4\beta}\right) \sqrt{\frac{i\sigma}{4\beta}} \text{erf}\left[2i\sqrt{i\beta}(d_{\text{fcg}} + \frac{\alpha}{2\beta})\right],$$

(27)

where

$$\alpha = k \frac{U_o + c_a}{c_{gz}}, \quad \beta = \frac{k\Lambda}{2c_{gz}},$$

(28)

$U_o$ is the forcing-relative flow speed at the wave particle’s launch elevation and erf the error function. Figure 5 contours the total energy given by (27) as a function of $\Lambda$ and horizontal wavenumber for a range of $d_{\text{fcg}}$ and $\lambda_z$. Both positive and negative shear values are shown in Fig. 5 because a significant response is evident in both regions of the parameter space (note again that the spectra from the linear model simulations in Fig. 4 contain contributions from both halves). The efficiency along ray paths in the shear case (Fig. 5) shares many properties in common with the constant background flow case (Fig. 3). For example, a deeper heating (larger $d_{\text{fcg}}$) is more destructive than a shallow heating; hence, values of enhanced efficiency occur across a narrower envelope of shear magnitude. Furthermore, a particle having deeper $\lambda_z$ experiences enhanced efficiency at larger values of shear. The differences between the effects of shear and a constant background flow are essentially due to the differences in ray path geometries. In the shear case the ray paths are curved. Particles may therefore spend unequal portions of their lifetime at particular elevations within the forcing. Although the particles are assumed to initially be in phase with the forcing, it is possible for some particles to spend enough time having phase directed against the forcing that the quantity $T_{E\text{ray}}$ may become negative. Note that this does not imply that the total energy is negative; $T_{E\text{ray}}$ is a measure of the work performed by the wave particle in the direction of the forcing and is thus related to $-E_{\text{dyn}}$.

c. Ensemble of multiple forcing elements

Most of what we presently know about gravity waves begins from the examination of the response to an isolated forcing. In nature convection often occurs as a collective or ensemble of elements that may be characterized by a separation length scale in addition to those scales describing the individual elements. This subsection examines the forcing efficiency of an ensemble of heating elements. Three experiments are performed. The first considers an ensemble of warming elements of equal magnitude. The second examines an ensemble of alternating warming and cooling elements of equal magnitude. The third considers an ensemble of elements that consist of a central region of warming flanked by a broader region of cooling, which we shall refer to as the ensemble of hybrid elements. The warming elements in the first two experiments have a geometry given by (21). The geometry of the hybrid elements is given by
where

\[ S(x, z) = A \exp \left[ -\left( \frac{x-x_c}{a} \right)^2 - \left( \frac{z-z_i}{d} \right)^2 \right] - B \exp \left[ -\left( \frac{x-x_c}{a} \right)^2 - \left( \frac{z-z_i}{d} \right)^2 \right], \]

such that the total heating integrated across each element is zero and the width of the cooling region is given by the parameter \( a_c \).

Figure 6 plots the efficiency of an ensemble of elements normalized by the efficiency of an isolated forcing for the three experiments described above. The first experiment (solid curves, Fig. 6a) demonstrates that an ensemble of warming elements is more efficient than an isolated warming. Each warming element generates a...
wave response that is dominated by a horizontally propagating signal of positive potential temperature anomaly (e.g., see Bretherton and Smolarkiewicz 1989). These leading fronts interfere constructively with neighboring warming elements upon arrival. Closely spaced, the elements are conducive to constructive interference and efficiency enhancement. The second experiment (dashed curves, Fig. 6a) demonstrates that the efficiency of an ensemble of alternating cooling and warming elements is reduced relative to the isolated forcing. In this case the leading wave fronts are out of phase and therefore interfere destructively with their nearest neighboring elements upon arrival. The third experiment (Fig. 6b) demonstrates the efficiency of an ensemble of hybrid elements. The response contains both positive and negative anomalies propagating through forcings of both positive and negative sign. The forcing efficiency of the hybrid ensemble can be either enhanced or reduced relative to the isolated forcing, depending on the configuration of the ensemble and the time into the simulation. Figure 6 demonstrates generally that an ensemble of multiple forcing elements can be either more or less efficient than an isolated forcing depending on the phase of the wave response arriving at neighboring elements. The interaction is most pronounced for ensembles characterized by a large degree of symmetry.

4. Discussion

The purpose of this paper has been to demonstrate the concepts of dynamical resistance and forcing efficiency as a means of analyzing the relationship between gravity waves and their sources. If a forcing is applied on a time scale similar to that of the waves it excites, then the dynamical perturbations associated with the waves may interfere constructively or destructively with the forcing itself. It is hypothesized that this effect, identified as the dynamical resistance, may explain some properties of the gravity wave energy spectrum corresponding to a given forcing. If there is little or no resistance to the forcing, then the forcing can generate more energy. Depending on the characteristics of the environment into which the forcing is applied, waves may remain nearly in phase with the forcing. For example, if a forcing is moving through a fluid at a constant speed, then gravity waves having phase speed similar to the forcing-relative flow speed will exert minimal resistance upon the forcing. The significance of this phase locking behavior to gravity waves and convection in nature might be viewed from either one of two possible perspectives. These perspectives, described below, represent two extremes concerning the extent to which waves could interact with their sources.

In the first perspective, we suppose that the evolving characteristics of convection are entirely unaffected by the dynamical resistance imposed by the convectively generated waves. In this scenario, peaks in the wave energy spectrum would correspond to the scales at which gravity waves are optimally phase locked relative to the forcing. This result would challenge the typical assumption that the initial spectrum of waves generated by convection is determined exclusively by the spatial
characteristics of the forcing alone. Similarly, Holton et al. (2002) argued that the dominant vertical scale of waves generated by convection depends not only on the depth of the cloud but also on its duration. Chagnon and Gray (2008) have recently identified the existence of multiple discrete peaks in the gravity wave spectrum above convection both in observations from a wind-profiling Doppler radar and in high-resolution convection-permitting numerical simulations. These peaks could not be explained as having originated due to a parametric instability. As part of a future investigation, we are examining the possibility that these spectral peaks correspond to scales at which gravity wave particles experience minimal phase change relative to the background forcing-relative flow near the tropopause.

In the second perspective, we suppose that an evolving field of convection responds to the dynamical resistance by adjusting toward a more efficient configuration. This is the perspective assumed by Robinson et al. (2008) and by Cohen and Craig (2004) to explain the dominant scales present in ensembles of convection. Figure 7 schematically illustrates one example of how a dynamical resistance might influence the structure of an isolated forcing. Suppose a cloud has formed due to the release of some available energy associated with an instability, denoted $E_{\text{ins}}$ (which might be related to, say, the convective available potential energy in a cloud or the production of subcloud layer entropy). Suppose that the instability induces maximum acceleration along a direction $\alpha_{\text{ins}}$. As this instability produces motion it will excite a dynamical response in the surrounding environment. The dynamical response may in turn do work upon the region containing the instability, denoted $E_{\text{dyn}}$. An interesting question to consider is whether there exists a forcing angle $\alpha$ that allows the forcing to draw energy from the instability while minimizing the resistance from the dynamical response. Could this angle deviate from $\alpha_{\text{ins}}$?

The scope of this investigation has been limited to the introduction and basic demonstration of dynamical resistance, and the context has been highly idealized. Nature does not obey the linear, 2D, inviscid dynamics that have been explored in this paper. A more general formulation of the concepts of dynamical resistance and forcing efficiency are needed before we can assess their relevance to convection and gravity waves in the real world. An attractive candidate is the Eulerian available energy formalism described in Bannon (2005) and applied to the analysis of atmospheric adjustment in Edson and Bannon (2008).

A major shortcoming of the analysis contained in this paper is that no constraints have been placed on the evolution and character of the forcing. In fact, the analysis has made no reference to the actual origin of the forcing at all. An interesting next step might involve the optimization of energetic efficiency subject to a few such constraints. For example, Song et al. (2003) examined the gravity wave forcing in fully nonlinear cloud-resolving numerical simulations. In addition to diabatic heating, the wave forcing was shown to comprise significant contributions from nonlinear heat and momentum fluxes. These nonlinear momentum and heat flux terms were shown to be out of phase with the diabatic heating, thus reducing the total wave forcing. Chun et al. (2008) have proposed that the reduction in total wave forcing could be incorporated into gravity wave drag schemes via a “nonlinearity factor” multiplying the heat and momentum flux terms; a large nonlinearity factor was shown to yield a reduction in wave amplitude above the forcing. In light of these studies, a prudent application of the dynamical resistance calculation would be to explore the efficiency of a forcing comprised of both diabatic and nonlinear heat and momentum flux terms. Furthermore, the nonlinear heat and momentum flux terms may incorporate the effects of waves themselves. Future formulations of the dynamical resistance and forcing efficiency should accommodate the direct effect of waves on the forcing.

Acknowledgments. This work was supported by the UK National Centre for Atmospheric Science under the Weather Directorate. The author is grateful to Michael Reeder and Peter Bannon for helpful discussions. The author wishes to thank the two anonymous reviewers whose feedback led to a significant improvement on the original draft.
APPENDIX

Wave Particle Perspective

Section 3 presents a few examples in which a wave particle perspective is adopted in order to explain the spectral properties of the forcing efficiency calculated using the linear model of section 2. This section describes the wave particle perspective in more detail. The schematic diagram in Fig. A1 depicts a wave particle generated at location $x_0, z_0$ at an initial time. The particle follows a trajectory (i.e., a ray path), defined by the parametric curve $x_p(t), z_p(t)$, traveling through the broader region of forcing $S(x, z)$. Relative to the background flow, the particle has a horizontal phase speed $c_x$.

Let us consider the effect of this wave particle on the energetic efficiency of the forcing $F$. The field perturbations produced by the wave particle along the ray are characterized by a wave function

$$\psi[x_p(t), z_p(t)] = \hat{\psi} \exp(i\phi), \quad (A1)$$

where the phase is given by

$$\phi = k\delta x_p(t) + m\delta z_p(t) - \omega t \quad (A2)$$

in which $k$ and $m$ are horizontal and vertical wavenumbers, $\omega$ is the wave frequency, $\delta x_p(t) = x_p(t) - x_0$, and $\delta z_p(t) = z_p(t) - z_0$. We assume that the medium through which the particle propagates only varies in the $z$ direction. Consequently, $k$ and $\omega$ may be treated as constants that are fixed for the particle, but $m$ may vary in $z$. Because the background is flowing at speed $u_s(z)$, the particle wave frequency may be expressed as

$$\omega = \omega_{\text{intr}} + u_s(z)k, \quad (A3)$$

where $\omega_{\text{intr}}$ is the frequency relative to the moving flow (i.e., the intrinsic frequency). The work performed by the wave particle in the direction of the forcing along the ray path is given by

$$T_{E\text{ray}} = \int_{\text{raypath}} F\psi dt. \quad (A4)$$

The integral (A4) may be thought of as the total work done by the wave particle in the direction of the forcing $F$ as it travels along the ray path through the forcing region. The quantity $T_{E\text{ray}}$ is similar to $-E_{\text{dyn}}$, defined in (17). For convenience, let us assume that the wave perturbation is in phase with the forcing at the initial time and that the spatial structure of $S(x, z)$ is given by a constant value $\hat{S}$ along the ray path. The work performed by the wave particle along the ray path (A4) may then be written as

$$T_{E\text{ray}} = \hat{S}\psi \int_{\text{raypath}} \exp(i\phi) dt. \quad (A5)$$

Fig. A1. Schematic diagram depicting a hypothetical $S(x, z)$ through which a wave particle travels along a ray. The particle induces a perturbation $\phi$ whose phase varies as a function of time along the ray. The flow-relative and forcing-relative phase velocity are drawn at a point along the path. Particles that minimize the total change in phase along the ray (to less than $\pi/2$ radians) interact optimally with the forcing.
where $\hat{S}'$ is a positive real number and the change in phase is given by
\[
\phi = \int_{t_0}^{t} \omega_{\text{intr}} \, dt' = \int_{z_0}^{z} \frac{(u_r + c_s)}{c_g} \, dz'.
\] (A6)

Given a dispersion relation defining $\omega_{\text{intr}}$ and a background flow $u_r$, the integral $\text{TE}_{\text{ray}}$ may be computed by analytical or numerical means. If the background flow is constant, then the integral (A5) is maximized for wave particle having an intrinsic phase velocity $c_s$ equal in magnitude and opposite the background flow, $c_s = -u_r$, for which the phase is stationary along the ray path.

REFERENCES