How Does Rain Affect Surface Pressure in a One-Dimensional Framework?

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ABSTRACT

The process of hydrostatic adjustment in a vertical column is discussed in the context of rain formation and sedimentation. The authors assume an event of instantaneous condensation in a midatmospheric layer that removes mass from the gas phase and produces latent heating. It is shown that the rain formation leads to a change of the surface pressure after a short period of acoustic wave activity. There is, however, no hydrostatic surface effect once the particles reach terminal velocity. It is not until the rain reaches the ground that the surface pressure decreases consistently with the mass removed by the phase change.

Only the mass removal introduces perturbations below the layer of rain formation, where it acts to stretch the lower levels, reducing pressure and temperature. Above the layer of rain formation, the effects of latent heating dominate over the effects of mass removal by an order of magnitude.

The hydrostatic adjustment time is found to be approximately equal to $e^2 N_a^{-1}$ (340 s, where $N_a$ is the acoustic cutoff frequency and $e$ is the Euler constant) and is proportional to the temperature of the isothermal basic state. The energy distribution is found to be dominated by the latent heating. However, the mass removal significantly alters the amount of energy lost due to work done by the pressure perturbations. The implications for numerical modeling are discussed.

1. Introduction

Water vapor affects the hydrostatic pressure distribution in the atmosphere the same way as dry air. In particular, the surface pressure $p_s$ is

$$ p_s = \int_{\infty}^{0} \rho g dz, \quad (1) $$

where $\rho = \rho_d + \rho_v$ is the total density (subscripts $d$ and $v$ denote dry and vapor, respectively) and $g$ is the gravitational acceleration. How does the surface pressure react to the formation of rain drops and their downward motion? Drop formation subtracts mass from the moist air and adds it to the water phase while the surrounding air experiences latent heating. Thus, the total density of the gas phase is clearly reduced, which immediately results in a lower hydrostatic surface pressure. However, the effect of the drops and the adjustment process is not represented in Eq. (1).

The drops experience downward acceleration during a brief, initial state of “weightlessness” and are subsequently accelerated until they reach their terminal velocity. It is the purpose of this study to show how diabatic warming, mass removal, and the acceleration of the drops affect the hydrostatic adjustment process and the surface pressure tendency.

Bannon (1995) investigated the one-dimensional hydrostatic adjustment to a pressure perturbation caused by heating in an atmospheric layer of finite depth. He showed that the hydrostatic pressure increases within and above the layer, yielding a net upward displacement. The area below the initially heated air is, however, unaffected.
implying no surface pressure changes. The effects of mass changes were included in a later study (Bannon et al. 2006). The study, however, neglects rain sedimentation and considers the final equilibrium states only.

More recent studies address the adjustment processes due to heating and mass removal in a two-dimensional context including geostrophic adjustment in a rotating frame (Chagnon and Bannon 2001, 2005a,b). Despite the useful generalization to higher dimensionality, we believe that much can still be gleaned from a one-dimensional setting and will thus restrict the focus here to a single atmospheric column.

Comprehensive numerical models have been used to quantify the impact of the mass adjustment process. In a simulation of Hurricane Lili, Lackmann and Yablonsky (2004) found that the vortex deepened by around 3 hPa when the adjustment was explicit in the model equations. The precise mechanisms involved in altering the surface pressure in simulations such as this are difficult to identify because of the complexity of the model. However, because a hurricane is an extreme example of condensation in nature, the quantitative result in the case study can be seen as an upper bound on the strength of the effect.

Section 2 introduces the linear model including the effects of rain sedimentation. In section 3, the initial conditions for the event of rain formation in a finite layer are derived, followed by a brief discussion of the surface pressure when the rain drops reach their terminal velocity. The numerical simulations showing the transient adjustment and for the final state in hydrostatic equilibrium. The diagnostic equations are presented in section 6. Section 7 summarizes the distribution of energies in the final state dependent on the depths of the layer of rain formation as well as the altitude of the layer. Our concluding remarks are given in section 8.

2. The model

The model atmosphere is a vertical column of a stably stratified, inviscid, ideal gas. The basic state comprises an isothermal atmosphere at rest and is denoted by an overbar. As in Bannon (1995), the linearized one-dimensional equations are

\[
\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - g \frac{\rho'}{\bar{\rho}} + F, \tag{2}
\]

\[
\frac{\partial p'}{\partial t} = -\bar{\rho} \frac{\partial w'}{\partial z} - w' \frac{\partial \bar{\rho}}{\partial z}, \tag{3}
\]

\[
\frac{\partial \rho'}{\partial t} = \bar{\rho} g w' - \bar{\rho} \gamma \frac{\partial w'}{\partial z}. \tag{4}
\]

Notation is standard; \( \gamma = c_p/c_v \) is the ratio of the specific heats at constant pressure and volume, and \( F \) describes the acceleration due to the drag of the raindrops exerted in the precipitation layer. We choose an isothermal basic state as in Bannon (1995), with \( \bar{T} = 255 \text{ K} \), which represents a mean tropospheric value. The choice of temperature has no qualitative affect on the solution because it mainly determines the scale height of the basic state and the \( e \)-folding scale of the perturbations after adjustment.

The diagnostic equations are

\[
\frac{\partial \bar{p}}{\partial t} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{T}, \quad \text{and} \tag{5}
\]

\[
\frac{\partial \theta'}{\partial t} = \frac{\rho'}{\gamma \bar{\rho}} - \frac{\rho'}{\bar{\rho}}. \tag{6}
\]

with \( \bar{p} = \rho_0 e^{-z/H_0} \), \( \bar{p} = \bar{p} g H_0 \), \( \rho_0 = 1.36 \text{ kg m}^{-3} \), and \( H_0 = RT/g \).

Here we regard the moist part of the perturbation as a tracer and can thus separate it from the total density field and pertinent equations. We further assumed that the dry potential temperature is conserved, which follows from neglecting the contribution of water substance to the heat capacity of the air. This leads to Eqs. (4) and (6) in which gamma is entirely determined by the heat capacities of dry air. Bannon et al. (2006) used the equations involving the virtual potential temperature; however, conservation of virtual potential temperature depends on the same approximation.

At the initial time \( t = 0 \) we assume instantaneous conversion of water vapor into condensate in the layer \( z_1 \leq z \leq z_1 + h \). The total mass of the drops per unit area is thus given by \( m_w = \int_{z_1}^{z_1+h} \rho_w dz \). We ignore further growth or breakup of the raindrops. In the sequel, the effects of mass removal will be treated in combination with latent heat release as well as separately. The key concerns addressed here are the effects of mass removal and gravitational accelerations of the rain drops on the pressure field.

We define \( F \) as

\[
F = \mu w_w \frac{\rho_w}{\bar{\rho}} [H(z - z_L) - H(z - z_L - h)]. \tag{7}
\]

with

\[
\frac{\partial w_w}{\partial t} = -g - \mu w_w. \tag{8}
\]

where \( w_w \) is the velocity of the drops, \( \mu \) (s\(^{-1}\)) is a retardation coefficient, \( \rho_w \) is the liquid water density (which is identical to the vapor density \( \rho_v \) being condensed), and \( H \) is the Heaviside function with \( z_L = z_1 + w_w t \). The drops
are found in the layer $z_L < z < z_L + h$. One can solve for the evolution and terminal velocity of the rain drops, obtaining $w_0(t) = (1 - e^{-q t})$ with the terminal fall speed $w_0 = -g\mu^{-1}$ for $t \gg \mu^{-1}$, respectively.

Although we use a fairly simple model of rain (e.g., we neglect variations in $w_0$ due to drop size variations), we expect no qualitative change in behavior if condensation occurs slowly and a spectrum of drops forms featuring a wide range of fall speeds.

The standard setup for the solutions discussed below includes $m_w = 0.2$ kg (corresponding to approximately 0.2 mm of precipitation), $z_1 = 4$ km, $h = 2$ km, and $\mu = 0.65$, yielding a terminal velocity of approximately $-15$ m s$^{-1}$.

3. Initial conditions

The initial fields are determined by the instantaneous formation of the condensate in the layer $z_1 \leq z \leq z_1 + h$ at $t = 0$. First, we will address the effects of the mass removal alone, assuming this process to be isothermal, and ignoring diabatic heating. The effects of latent heating will be added in the subsequent step.

Without loss of generality, we consider the unper-urbed hydrostatic atmosphere to be dry except in the layer to be dried out (i.e., $z_1 \leq z \leq z_1 + h$). Then $p_0$ is converted into condensate of mass $m_w$ at $t = 0$. Focusing on moisture at one level does not imply the atmosphere is dry. It simply means the mass removal perturbs a balanced profile associated with only that part of the moisture.

Postulating that the total pressure for $t < 0$ is in hydrostatic balance, $dp/dz = -\rho g$, one can solve for the total pressure and density profiles $p = p_d + p_v$, and $\rho = \rho_d + \rho_v$, where $p_v = \rho_v R T$ and $\rho_v$ solely represent the amount of moisture that is converted into condensate in the layer $z_1 \leq z \leq z_1 + h$. Hydrostatic balance [Eq. (2)] with $w_0' = F = 0$ can be expressed by

$$e^{-\varepsilon H_0} \frac{d}{dz} (p_0 e^{\varepsilon H_0}) = -e^{-\varepsilon H_0} \frac{d}{dz} (p_0 e^{\varepsilon H_0}).$$

Multiplying Eq. (9) by $e^{\varepsilon H_0}$ and integrating by parts with appropriate boundary conditions yields

$$p = p_0 e^{-\varepsilon H_0} - (e - 1) \frac{R}{H_0} \int_0^{\varepsilon H_0} e^{\varepsilon H_0} \frac{d}{dz} \rho_v e^{\varepsilon H_0} \frac{d}{dz} \rho_v e^{\varepsilon H_0} \frac{d}{dz}',$n

$$p_0 = p_s + \left(e(z_1 + h)H_0 - e^{\varepsilon H_0} \rho_0 R T (e - 1) \right), \quad \text{and}$$

$$\rho = \frac{p_d}{RT} + \rho_0 [H(z - z_1) - H(z - z_1 - h)],$$

where $\varepsilon = R \varepsilon / R$ is the ratio of the gas constants for water vapor and dry air and $p_s$ is the surface pressure including the loading by water vapor with density $\rho_v$ from the layer $z_1 \leq z \leq z_1 + h$. Here $p_0$ can be interpreted as the hypothetical surface pressure of an entirely dry atmosphere with the same pressure profile above the moist layer ($z > z_1 + h$). The difference between $p$ and $p_0 e^{-\varepsilon H_0}$ is illustrated in Fig. 1 for the standard setup. The surface pressure with the vapor loading is reduced compared to the purely dry profile $p_0 e^{-\varepsilon H_0}$ by an amount $\Delta p_s = \left[e(z_1 + h)H_0 - e^{\varepsilon H_0} \rho_0 R T (e - 1) \right]$. For $h \to 0$, $\Delta p_s \to -g\rho_0 (e - 1)$, where the factor $(e - 1)$ stems from the replacement of the dry pressure by the water vapor pressure as required for pressure continuity. This yields a jump in dry density of $\varepsilon \rho_0$.

We define the initial perturbations $p_0' = p - \bar{p}_m$ and $\rho_0' = \rho - \bar{\rho}_m$, where

$$\bar{p}_m = p + \rho_0 [H(z - z_1) - H(z - z_1 - h)],$$

$$\bar{\rho}_m = \bar{\rho}_v [H(z - z_1) - H(z - z_1 - h)],$$

$\bar{p} = p_0 e^{-\varepsilon H_0}$, $\bar{p} = p_0 e^{-\varepsilon H_0}$, and $\rho_0 = p_0 R T$, and linearize with respect to $\bar{p}$ and $\bar{\rho}$. Note that the initial perturbation is zero above $z = z_1 + h$.

Furthermore, we set $w_0 = 0$. All contributions proportional to $e^{-\varepsilon H_0}$ are in hydrostatic balance. Thus, the initial imbalance is limited to the layer $z_1 \leq z \leq z_1 + h$ with $-\rho_0 R T$ and $-\rho_0$ in pressure and density, respectively.
If latent heating is considered for the initial fields as well we have to adjust the pressure perturbation at \( t = 50 \), yielding

\[ T_{90} = \frac{L_{\text{v}}}{c_{\text{v}}} \left[ \frac{p}{C_0} \right] \left[ \frac{r}{C_{90}} \right] \left[ \frac{c_{\text{v}}}{C_{18}} \right], \quad (15) \]

where we assumed equal temperature increase of liquid water and dry air by latent heat uptake and neglected values quadratic in perturbations. The latent heat for condensation and specific heat for water vapor are \( L_{\text{v}} = 2.5 \times 10^6 \text{ J kg}^{-1} \) and \( c_{\text{v}} = 4 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} \), respectively.

The heating together with the mass removal yields a total initial pressure perturbation of

\[ p_{90} = p - \frac{L_{\text{v}}}{c_{\text{v}}} \left[ \frac{p}{C_0} \right] \left[ \frac{r}{C_{90}} \right] \left[ \frac{c_{\text{v}}}{C_{18}} \right] H(z - z_L) - H(z - z_L - h), \]

which is depicted in Fig. 2. Inspecting the nondimensional ratio between the effects of mass removal and diabatic heating \( eT_{\text{v}}L_{\text{v}} \), we see that the pressure perturbation due to the latent heating dominates by an order of magnitude.

4. Rain at terminal velocity

It is straightforward to assess the hydrostatic state of the surface pressure once the rain drops fall with terminal velocity \( w_0 \) while still not having reached the ground (i.e., while \( z_L > 0 \)). In the absence of vertical accelerations, the perturbation pressure will balance the gravitational acceleration of the perturbation density and the frictional acceleration due to the rain drops; hence Eq. (2) reduces to

\[ \frac{\partial p'}{\partial z} = -g \left( \rho_0 - g \rho_{\text{v}} H(z - z_L) \right), \quad (17) \]

When integrating from \( z = 0 \) to \( z = \infty \) there is obviously no contribution of the rain (at terminal velocity) to the hydrostatic surface pressure because the mass perturbations in the liquid phase must equal the amount of water vapor removed by condensation. Hence a deviation of the surface pressure remains only as long as vertical accelerations are important. Of course, once the drops reach the ground there is no more drag due to the rain drops, and so the surface pressure tendency equation in a three-dimensional atmosphere

\[ \frac{\partial p_s}{\partial t} = -g \int_s^\infty \nabla_h \cdot \left[ (\rho_d + \rho_{\text{v}}) \mathbf{w}_h \right] dz + g(E - P), \quad (18) \]

must include a precipitation \( P \) (kg m\(^{-2}\) s\(^{-1}\)) and evaporation \( E \) (kg m\(^{-2}\) s\(^{-1}\)) term prescribing the moisture fluxes into the atmospheric column at the lower boundary.

Latent heat release does not remove mass from the atmosphere and therefore does not affect the hydrostatic surface pressure. Of course, if more dimensions are added to the problem, horizontal circulations produce local changes of surface pressure. However, such effects are beyond the scope of this study.

It is interesting to note that hydrometeors in hydrostatic numerical models instantaneously reach terminal velocity or are immediately removed from the atmospheric column. Thus, the hydrostatic pressure in such a model can be obtained as

\[ p_s = g \int_s^\infty (\rho_d + \rho_{\text{v}} + \rho_{\text{c}}) dz, \quad (19) \]

where \( \rho_{\text{c}} \) represents the condensate hydrometeors at terminal velocity contained in the atmospheric column.
The direct impact of mass removal on the surface pressure is avoided, of course, if all condensate reevaporates in the air. This scenario is complicated by the transfer of momentum from the falling condensate to the moist air during reevaporation, which is beyond the scope of this study. However, because the kinetic energy would ultimately be radiated away by acoustic waves or dissipated as heat, there is no reason to expect even an indirect effect on surface pressure.

5. Analytic solutions

a. Transients

To further characterize the transient solutions, one can solve Eqs. (2)–(4) without the rain. The full solution for heating only was presented by Bannon (1995). We follow his derivation but include the effects of mass removal. An equation for \( \omega^2 \) can be derived,

\[
\frac{\partial^2 \omega'}{\partial t^2} = -\frac{c^2}{H_0} \frac{\partial \omega'}{\partial z} + c^2 \frac{\partial^2 \omega'}{\partial z^2},
\]

which can be rewritten by letting \( \omega' = e^{\gamma z_0 H_0} w(z, t) \) as

\[
\frac{\partial^2 w}{\partial t^2} = -N_u^2 w + c^2 \frac{\partial^2 w}{\partial z^2},
\]

where \( N_u = c/2H_0 \) and \( c = \sqrt{\gamma R T} \) are the “acoustic cutoff frequency” and the speed of sound, respectively. The dispersion relation is thus found to be

\[
\omega^2 = N_u^2 + c^2 k^2,
\]

with \( k \) being the vertical wavenumber. The equation for \( w' \) differs from that for \( p' \) and \( p' \), which must be multiplied by \( e^{\gamma z_0 H_0} \) to satisfy the same wave equation. This is, however, in accordance with energy conservation (see section 7).

To solve for the time dependence of \( \omega' \), the initial conditions

\[
w'|_{t=0} = 0 \quad \text{and} \quad \left. \frac{\partial w'}{\partial t} \right|_{t=0} = \left( \frac{T - L_v}{c_e} \right) \frac{R \rho_0}{\overline{\rho}} \left[ \delta \left( z_a - \frac{h}{2} \right) - \delta \left( z_a + \frac{h}{2} \right) \right] + \frac{g}{\overline{\rho}} \rho \left[ H \left( z_a - \frac{h}{2} \right) - H \left( z_a + \frac{h}{2} \right) \right]
\]

have to be met, where \( L_v = 0 \) in the case without latent heating and \( z_a = z - z_1 + h/2 \). Bannon (1995) showed that the solution to the delta functions in the initial perturbations is

\[
w' = \frac{\Delta p}{2\rho_0 c} e^{\gamma z_0 H_0} \left[ e^{\gamma z_0 H_0} W \left( z_a - \frac{h}{2}, t \right) - e^{-\gamma z_0 H_0} W \left( z_a + \frac{h}{2}, t \right) \right],
\]

where \( W(z, t) = H(c^2 t^2 - z_0^2) J_0[N_d(t^2 - z_0^2/c^2)] \), \( \Delta p = \rho \overline{\rho} \overline{R} T \), and \( J_0 \) is the Bessel function of zeroth order. As discussed by Bannon (1995), this solution comprises two rays emitted from each \( \delta \) source propagating upward and downward. It was also pointed out that the net effect of the perturbations initiated by the two \( \delta \) functions is zero vertical displacement for \( t \to \infty \) below the layer of heating. However, the density term in Eq. (24) yields a net upward displacement throughout the entire atmosphere. This can qualitatively be understood by analyzing the limit for \( h \to \infty \) when the density contribution also becomes \( \delta \)-function-like. Thus, two rays will be emitted in opposite directions and both will have upward motion, which yields the net lift for \( t \to \infty \).

A lower boundary condition enforcing \( w = 0 \) is equivalent to the effect of an image source at \( z = -z_1 \) sending up a ray containing downward velocity and reaching the boundary synchronously with the ray propagating downward from \( z = z_1 \). After the two signals reach the boundary, the upward velocity in the physical domain is replaced by downward velocity. The net upward displacement is limited to the time of passage between the incident and reflected waves at a given altitude. This time, and thus the net displacement, increases from \( z = 0 \) to the height \( z_1 + h \), above which it is constant. This amounts to a stretching of the atmospheric column below \( z_1 + h \).

b. Final equilibrium

The hydrostatic balanced state at \( t \to \infty \) can be derived following Bannon (1995), who introduced the conserved quantity

\[
\left[ \left( \frac{p'}{\rho'} - \frac{p}{\rho} \right) \frac{\gamma p}{\gamma p} - \frac{\gamma p}{\gamma p} \right]_{z} = 0.
\]

The hydrostatic pressure \( p' \) is obtained by integrating Eq. (26) from the initial time \( t = 0 \) to \( t \to \infty \):

\[
\frac{\partial^2 p}{\partial z^2} - \frac{1}{4\overline{H}_0} \frac{\Delta p}{\overline{\rho}} e^{\gamma z_0 H_0} \left( \frac{1}{\gamma \overline{H}_0} \frac{\partial p'}{\partial z} - g \frac{\partial p_0'}{\partial z} - \frac{g \overline{\rho} k}{\gamma \overline{H}_0} p_0' \right),
\]

where \( \hat{p}(z) = e^{\gamma z_0 H_0} p' \). The difference with respect to the equation in Bannon (1995) is the existence of an unclear term.
initial perturbation in density \( \rho_0' \) on the right-hand side.\(^1\)

Given initial conditions (14) and (16), Eq. (27) is readily solved using standard techniques, where we assume \( p \) to be continuous throughout the atmospheric column, bounded for \( z \to \infty \), and assume \( \theta_0' = 0 \) at \( z = 0 \), that is, no vertical displacements at the lower boundary. The latter condition is identical to prescribing a drop in surface pressure by the mass equivalent removed from the atmospheric column by the rain (i.e., \( \rho_g h \); see the appendix for details).

Density, temperature, and potential temperature can be derived from the pressure distribution via

\[
\rho' = -g^{-1} \frac{d\rho}{dz}, \quad T' = \left( \frac{p'/\overline{p} - \rho'/\overline{\rho}}{\overline{p}} \right) \overline{T}, \quad \text{and} \quad \theta' = \left( \frac{p'/\gamma\overline{p} - \rho'/\overline{\rho}}{\overline{p}} \right) \overline{\theta}.
\]

The net vertical displacement in conjunction with the adjustment process \( \zeta \) can be calculated using the initial and final distribution of the potential temperature by integrating \( \theta'/\theta' + w\theta'/\theta' = 0 \) over time, yielding

\[
\zeta' = \frac{\theta_0' - \theta_0'}{\theta_z'}, \quad (28)
\]

Using \( \zeta \), the final density, temperature, and potential temperature can also be evaluated via

\[
\rho' = -\rho_0' - \zeta \frac{\overline{\rho}}{\overline{\rho}} \frac{d\zeta}{dz}, \quad (29)
\]

\[
T_0' = -T_0' - T(\gamma - 1) \frac{d\zeta}{dz}, \quad \text{and} \quad (30)
\]

which immediately pinpoint the physical reasons for changes in the pertinent variables. The potential temperature can thus only be changed through advection, the temperature only through compression, and the density through advection and compression.

Because the impact of latent heating and mass removal are separated by an order of magnitude in the initial conditions (see section 3), the effects of mass removal and latent heating are addressed in isolation and combination in order to disentangle the relative contributions to the solution.

1) MASS REMOVAL ONLY

The initial pressure discontinuity is removed, yielding lower pressure above and below the layer in which the mass was removed (Fig. 3). A similar adjustment is evident for the density. However, the initial perturbation is not fully compensated and a significant deficit remains, consistent with an increase in the vertical gradient of hydrostatic pressure. Since the background temperature is constant, the only way to introduce changes in temperature is via compression while the potential temperature is changed through vertical advection. These two processes are evident in the net vertical displacement, which features an upward displacement increasing with height below approximately 6 km and a decrease aloft throughout the layer where the mass was removed. While the net sinking and net rising alter the potential temperature, it is the vertical gradient of \( \zeta \) (i.e., the compression) that causes the changes in the temperature.

\[
\theta_0' = \theta_0' - \zeta \frac{d\theta}{dz}.
\]

\(^1\) The treatment of mass removal reprises part of the work of Bannon et al. (2006). Their corresponding Fig. 2 deviates from the results presented herein, featuring a jump in vertical displacement and hence virtual potential temperature. Here the problem is treated as an initial value problem, whereas Bannon et al. (2006) apply a forcing at \( t = 0 \).
2) Mass Removal and Latent Heating

The solutions including the effects of latent heating bear great resemblance to the solutions presented by Bannon (1995), because the initial conditions are dominated by the effects of heating (Fig. 2). The main features are net atmospheric rising and pressure increase above $z_1$, a density deficit, and temperature indicating a decompression between $z_1 \leq z$ and $z \leq z_1 + h$.

However, the effects of the mass removal are evident in the pressure as well as in the net lifting below $z_1$. There would be no final pressure perturbation in the heating-only case below $z_1$ (Bannon 1995). It follows from the pressure profiles that in an event of rain formation, the most apparent signal is connected to the heating and the pressure increase in the levels aloft.

6. Numerical simulations

To investigate the transience, including the falling rain drops, the coupled set of Eqs. (2)–(4) is integrated numerically using an implicit Euler backward time differencing with a second-order approximation of the vertical derivatives. Because of the finite extent of the domain, appropriate boundary conditions have to be defined at the bottom and top. We have chosen $w = 0$ at $z = 0$ and $z = z_{	ext{top}}$. We have set $z_{	ext{top}} = 200$ km and integrated only up to a final time of 1100 s to ensure that no waves reflected from the upper boundary reenter the domain of interest (i.e., $z \leq 10$ km). The vertical grid spacing and time increment are set to 25 m and 0.025 s, respectively, yielding a Courant number of 0.32 with respect to the speed of sound (here $c = 320$ m s$^{-1}$).

We also tested alternatives to the rigid-lid upper boundary condition such as a sponge layer in the upper part of the domain or an open boundary. However, an effective sponge for sound waves necessitates rather drastic increases in diffusion and one introduces unrealistic reflection of the less rapid modes from the sponge layer. An open boundary condition is only applicable to perturbations with phase speeds directed out of the domain (nongravity modes).

A model validation with respect to the final state of the analytic solutions was carried out, indicating a very good model performance. In fact, all numerical experiments, with and without the falling rain, produce almost identical final-state results compared to the analytical profiles in Figs. 2 and 3 (not shown).

In the following we will discuss four different model experiments. The main aim is to disentangle the effects of mass removal, latent heating, and falling rain drops. We thus compile experiments with and without latent heating, in each case including and excluding falling rain drops. Table 1 summarizes the different model setups and introduces the acronyms used henceforth. Since the model fields for MH and MHR look very similar, we omitted figures for MHR.

The vertical velocity is dominated by a couplet of sound waves in all simulations that disperse in opposite directions away from the initial pressure discontinuities (not shown). For simulations MR and M there is a pressure deficit in the layer of mass removal at $t = 0$ and thus the upper (lower) discontinuity triggers downward (upward) acceleration. Because of a pressure surplus in the layer for MHR and MH, the opposite response is found. The downward-propagating waves are reflected at the surface and then propagate upward with reversed sign in vertical velocity. While M is dominated by the fast sound mode couplet, the other experiments also feature slower modes that are due both to the larger pressure perturbation in MH and MHR and to the falling rain in MR.

It is dynamically instructive to analyze the net effect of the vertical velocity field (i.e., the vertical displacement). The upward-propagating sound waves as well as the sound waves reflected from the surface contribute to the net sinking (rising) in M and MR (MH and MHR) while the initial sound wave propagating downward initiates upward (downward) motion for M and MR (MH and MHR) (Fig. 4). The dominance of the latent heating in the net upward displacement above $z_1$ is clear in MH compared to M and MR.

The fast and slow sound modes are apparent in the pressure fields (Fig. 5) before attaining hydrostatic balance with an increased vertical pressure gradient within the layer of initial rain formation. For M and MR the pressure deficit below $z_1$ is more evident compared to MH, where the increase of pressure above $z_1$ dominates. The pressure front related to the falling rain is clearly distinguishable for MR but less so for MHR (not shown), where the initial perturbations due to the latent heating dominate the entire adjustment process.

The most striking feature in the density (Fig. 6) and potential temperature (not shown) fields is the band in the layer where the mass was removed, indicating a density deficit and a positive perturbation in potential temperature for all experiments. The adjustment within this band is rather fast compared to the rest of the atmospheric column. A close-up of the layer (not shown) reveals that the adjustment period is determined by the time the sound waves from the top and bottom of the layer need to propagate through the layer (i.e., $h/c \approx 6$ s).

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| Table 1. Experiment IDs for numerical simulations. |
|---------------------------------|---------------------------------|
| No latent heating | Latent heating |
| No rain | M | MH |
| Rain | MR | MHR |
FIG. 4. Solutions for vertical displacement for experiments (a) M, (b) MR, and (c) MH.

FIG. 5. As in Fig. 4, but for pressure.
The slower modes during the adjustment process are more visible for density than for potential temperature. Similar to pressure, the rain front is clearly visible in the density for MR but almost indiscernible for MHR (not shown). Again, as for pressure, the density deficit below $z_1$ is clearly visible for M and MR compared to MH.

Figure 7 illustrates the adjustment for $p'$ in MR and MHR at different times during the integration for all experiments. The vertical rain front progressing downward is evident. However, for MHR large-amplitude
oscillations related to slower modes during the adjustment process render the signal less discernible compared to MR.

The evolution of the surface pressure (Fig. 8) features an impulsive signal related to initial sound waves being reflected from the surface at 12.5 s. The sign of this signal is dependent on the sign of the initial pressure perturbation in the layer of mass removal. Thus, experiments M and MR (MH and MHR) yield a negative (positive) pressure perturbation. Once the initial sound waves are reflected from the ground a slower adjustment process takes place with decreasing (increasing) surface pressure for M and MR (MH and MHR). This slower adjustment process is related to modes with smaller vertical wavenumbers where the first term in the dispersion relation (22) becomes dominant (i.e., the phase speed becomes proportional to \(c/2H_0\) divided by the wavenumber). We tested the dependence of the adjustment time scale by increasing the mean temperature \(\mathcal{T}\) (i.e., increasing in the scale height), which indeed yielded a prolonged period of adjustment. The principal behavior of this fast frontlike jump followed by a slower adjustment is depicted in Gill (1982, his Fig. 7.3) and is related to the Bessel-function solution for the given initial pressure perturbations. The first (last) rain drops reach the ground at \(\sim 260 (~ 400)\) s, causing a reduction in surface pressure during the precipitation period. The detection of this effect is easier in MR compared to MHR, where again the amplitude of the slower modes dominate the signal.

7. Energetics

The conservation of energy for the set of Eqs. (2)–(4) becomes (see Bannon 1995)

\[
\frac{\partial}{\partial t}[E_{\text{kin}} + E_{\text{pot}} + E_{\text{elas}}] = -\frac{\partial(p'w')}{\partial z} + \rho w F, \quad (32)
\]

where the kinetic, potential, and elastic energy are defined by \(E_{\text{kin}} = (1/2)\rho w'^2\), \(E_{\text{pot}} = (\rho/2N^2)(g\theta'/\theta)^2\), and \(E_{\text{elas}} = p'^2/2\rho c^2\), respectively. The total energy thus changes in accordance with the work acting on the parcel and the drag exerted by the rain. However, as soon as the rain reaches the ground the last term on the right-hand side vanishes.

During the adjustment period the signal in the kinetic and elastic energy is mainly related to the fast acoustic modes, while the potential energy is changing on slower time scales related to the wake (not shown). For MR and MHR there is also a signal in the elastic energy related to the falling rain, which is most likely linked to the compression of the air due to the rain front (not shown).

The initial and final total energy, with and without latent heating, comprises contributions from elastic and potential energies only. Energies for the case with latent heating are about three orders of magnitude larger than for the case without latent heating. At \(t = 0\) both forms of energy mainly reside in the layer of rain formation. At the final time, however, changes in the potential energy are rather small but almost the entire elastic energy is converted or removed from the system (Fig. 9).

To further investigate the distribution of energies in the final state, we vertically integrate the elastic and potential energy contributions below, within, and above the layer of rain formation. Figure 10a shows that the final
distribution of energies in percentage of the total initial energy. The distribution is significantly altered for an increasing depth of the layer of rain formation $h$ while keeping $z_1$ at its standard value. Qualitatively the result is rather similar to Bannon (1995), with a decrease of potential energy within the layer and an increase of potential and elastic energies in all other areas. In addition to those changes, the area below the layer also contributes to the overall energy budget with increasing $h$. The most striking difference, though, is that the energy loss due to wave propagation is less and increases with $h$. This is due to the mass removal, which becomes increasingly important for larger $h$.

Varying $z_1$ while keeping $h$ constant at its standard value indicates that the ratios of potential energy within the layer as well as the ratios of potential and elastic energy above remain almost constant while elastic energy increases within the layer (Fig. 10b). For large enough $z_1$ the layer below also starts to significantly contribute to the overall budget. The total energy loss decreases with increasing $z_1$.

8. Discussion and concluding remarks

The linear hydrostatic adjustment to rain formation was analyzed utilizing analytic solutions and numerical simulations in the context of a one-dimensional isothermal atmosphere. Although the effects of latent heating dominate the initial and final state, it is only the net mass removal connected to condensation that yields a surface pressure reduction. However, as long as the condensate is falling at terminal velocity, the hydrostatic surface pressure is not changed because of the drag exerted by the falling rain on the atmosphere below.

The time needed for hydrostatic adjustment scales with $N_p$. Since $J(x) = x^{-1/2}$, the $e$-folding time for the dispersion of transients is typically $\tau \approx e^2N_p^{-1} \approx 340$ s, proportional to the basic-state temperature. Hydrostatic numerical models adjust the pressure instantly in one dimension and typically remove or reevaporate hydrometeors instantly as well. Compressible model solutions can diverge from incompressible solutions when the model time step falls below $\tau$. Discrepancies specifically due to mass removal could come from wave interactions and/or forcing updates either above or below the level of the mass perturbation. Most compressible models subcycle the acoustic waves on a small time step and use implicit time differencing in the vertical. As a result, the effective adjustment time scale is not a physical constant but varies in proportion to the small time step. We presume that the slower vertical signal speed associated with longer time steps eventually causes the horizontal adjustment to become important, regardless of the aspect ratio of the initial perturbation. Chagnon and Bannon (2005a,b) considered the effect of a second dimension in the hydrostatic adjustment process, but we have no analytical estimate for the time scale in this case.
The amount of condensation that would directly lower the surface pressure by 1 hPa would also increase the air temperature by about 25 K in the layer of condensation. The numerical case study by Lackmann and Yablonsky (2004) found surface pressure drops of around 3 hPa for a similar midatmospheric temperature perturbation within a hurricane. As they point out, the heating in the three-dimensional case is largely offset by adiabatic cooling, and there would be analogous compensation for the drying. If the heating and drying by both condensation and advection can be lumped together for the purpose of our simple analysis, the hurricane study suggests that the impact on the surface pressure could be largely an indirect result involving a secondary circulation. It is important to recognize that the main role of the secondary circulation is to compensate for the heating and resupply the moisture.

A reduction in hydrostatic surface pressure can only be caused by a net mass removal (precipitation and/or horizontal mass divergence) from the atmospheric column, while both heating and mass removal yield adjusted pressure and density profiles throughout the atmospheric column. This is in contrast to Knippertz and Fink (2008), who suggested a direct response of hydrostatic surface pressure to diabatic heating. This was further discussed by Spengler and Egger (2009) and Knippertz et al. (2009); the latter identified significant contributions to hydrostatic surface pressure tendency at very high altitudes (50–10 hPa). This discrepancy is mainly due to the choice of a pressure coordinate frame for their calculations, which implicitly demands an upper boundary condition for the tendency of geopotential height. The presence of heating anywhere in the column, however, will always induce a net vertical displacement in the atmospheric column at the top of the atmosphere rendering the boundary condition nontrivial.

Despite its simplicity, the material presented could be used for testing the physicality of hydrostatic and non-hydrostatic numerical models. Differences introduced by the choice of different time steps in a hydrostatic model during a rain formation event as well as the hydrostatic adjustment process in a nonhydrostatic model could be used to pinpoint possible model sensitivities and deficiencies when compared to the analytic solutions presented herein.

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**APPENDIX**

**Analytic Solutions**

The equation for the pressure [Eq. (27)] can be solved with an ansatz following:

![Fig. 10. Vertically integrated energy contributions in percentage of the initial total energy for the regions “below,” within (“layer”), and “above” the layer of mass removal for the analytic solutions including latent heat; “waves” indicates the amount of energy lost to acoustic waves. Dependence of the contribution on the depth of the layer \( h \) (a) and on \( z_1 \) with \( h = 2 \) km (b). Dashed lines indicate elastic and solid potential energy for the final state as a percentage of the total initial energy.](image-url)
where \( \lambda = \tfrac{1}{2} H_0 \) and \( G(z) \) is a particular solution to the inhomogeneous differential equation in that layer with

\[
G(z) = \rho_0 g\kappa z e^{\lambda z},
\]

This set of equations can be solved by imposing the lower boundary condition (i.e., a surface pressure reduction equivalent to the mass removal) and requiring that \( \dot{p} \) is continuous at \( z_1 \) and \( z_1 + h \) and finite for \( z \to \infty \). It follows immediately that \( E = 0 \) and we are left with five equations to solve for \( A, B, C, D, \) and \( F \), yielding

\[
p'_e = 2\rho_0 e^{-\frac{h}{H_0}} \sinh \left( \frac{h}{2H_0} \right) H_0 g (1 - \kappa) (1 - \varepsilon)
\]

\[
+ \kappa L_v \left[ -\rho_0 g e^{-\frac{z}{H_0}} (1 - \kappa) \right] \dot{T}_e, \quad z > z_1 + h;
\]

\[
\rho'_e = -\rho_0 \frac{h}{H_0} e^{-\frac{z}{H_0}} (1 - \kappa)
\]

\[
+ 2\rho_0 (1 - \varepsilon) \sinh \left( \frac{h}{2H_0} \right) e^{-\frac{z}{H_0}} \dot{T}_e, \quad z < z_1;
\]

\[
T'_e = 0, \quad z > z_1 + h;
\]

\[
\left[ \rho_v \frac{\kappa L_v}{\rho_0 g H_0} + \rho_v \frac{\kappa h}{\rho_0 e^{-\frac{z}{H_0}}} \right] \dot{T}_e, \quad z_1 \leq z \leq z_1 + h;
\]

\[
T'_e = -\rho_0 \frac{h}{H_0} e^{-\frac{z}{H_0}} (1 - \kappa)
\]

\[
+ 2\rho_0 (1 - \varepsilon) \sinh \left( \frac{h}{2H_0} \right) e^{-\frac{z}{H_0}} \dot{T}_e, \quad z < z_1;
\]

\[
\theta'_e = \left[ \frac{1}{\gamma} e^{\frac{z}{H_0} + \kappa (z + z_1)} H_0 + \kappa e^{\frac{z}{H_0} + \kappa (z + z_1)} H_0 \right] \left[ \rho_v \frac{\kappa L_v}{\rho_0 g H_0} + \rho_v \frac{\kappa h}{\rho_0 e^{-\frac{z}{H_0}}} \right] \dot{T}_e (1 - \varepsilon)
\]

\[
+ \rho_v \left( \frac{z - z_1 - h}{H_0\gamma} + 1 \right) e^{\frac{z}{H_0} + \kappa (z + z_1)} H_0 + \rho_v \frac{h}{H_0} (1 - \kappa) e^{\frac{z}{H_0} + \kappa (z + z_1)} H_0, \quad z_1 \leq z \leq z_1 + h; \quad \text{and}
\]
\[
\theta'_e = \frac{\rho_v h \kappa T}{\rho_0 H_0} e^{\kappa c H_0 (1 - \kappa)} - \frac{\rho_q h \kappa T}{\rho_0 \gamma H_0} e^{c(1+\kappa) H_0} \]
\[-2 \frac{\rho_v}{\rho_0} \kappa T (1 - \varepsilon) \sinh \left( \frac{h}{2H_0} \right) e^{(\kappa + \varepsilon) h^2 H_0}, \quad z < z_t. \]

(A16)

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