A Formulation of Three-Dimensional Residual Mean Flow Applicable Both to Inertia–Gravity Waves and to Rossby Waves

TAKENARI KINOSHITA* AND KAORU SATO
Department of Earth and Planetary Science, University of Tokyo, Tokyo, Japan

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ABSTRACT

The three-dimensional (3D) residual mean flow is expressed as the sum of the Eulerian-mean flow and the Stokes drift. The present study derives formulas that are approximately equal to the 3D Stokes drift for the primitive equation (PRSD) and for the quasigeostrophic equation (QGSD) using small-amplitude theory for a slowly varying time-mean flow. The PRSD has a broad utility that is applicable to both Rossby waves and inertia–gravity waves. The 3D wave activity flux whose divergence corresponds to the wave forcing is also derived using PRSD. The PRSD agrees with QGSD under the small-Rossby-number assumption, and it agrees with the 3D Stokes drift derived by S. Miyahara and by T. Kinoshita et al. for inertia–gravity waves under the constant-Coriolis-parameter assumption. Moreover, a phase-independent 3D Stokes drift is derived under the OG approximation.

The 3D residual mean flow in the upper troposphere in April is investigated by applying the new formulas to the European Centre for Medium-Range Weather Forecasts (ECMWF) Interim Re-Analysis (ERA-Interim) data. It is observed that the PRSD is strongly poleward (weakly equatorward) upstream (downstream) of the storm track. A case study was also made for dominant gravity waves around the southern Andes in the simulation by a gravity wave–resolving general circulation model. The 3D residual mean flow associated with the gravity waves is poleward (equatorward) in the western (eastern) region of the southern Andes. This flow is due to the horizontal structure of the variance in the zonal component of the mountain waves, which do not change much while they propagate upward.

1. Introduction

The residual mean flow and Eliassen–Palm (EP) flux in the transformed Eulerian-mean (TEM) equations derived by Andrews and McIntyre (1976, 1978) are powerful tools for diagnosing the meridional circulation and wave–mean interaction in the troposphere and/or middle atmosphere. The residual mean flow is expressed as the sum of the Eulerian-mean flow and the Stokes drift under the small-amplitude assumption, and it is approximately equal to the zonal-mean Lagrangian-mean flow when the wave is linear, steady, and adiabatic and when no dissipation occurs. The EP flux is equal to the product of the group velocity and the wave activity density under the Wentzel–Kramers–Brillouin (WKB) approximation and is a useful physical quantity for describing the wave propagation (Edmon et al. 1980). The residual mean flow and the zonal-mean zonal wind acceleration are related to the divergence of EP flux in the zonal momentum equation. When there are no critical levels, the divergence of the EP flux is zero for linear, steady, and conservative waves. Under such conditions, the waves neither drive the residual mean flow nor accelerate the zonal-mean zonal wind. This is called the nonacceleration theorem (Eliassen and Palm 1961; Charney and Drazin 1961). Haynes et al. (1991) proposed “the downward control principle,” which is an important theory describing the driving mechanism of the residual mean flow. According to this mechanism, the residual meridional flow in the middle atmosphere is driven by the wave forcing associated with breaking and/or dissipation of the waves propagating from the troposphere. In the lower boundary of the atmosphere,
the residual meridional flow is driven by frictional forcing. The residual vertical flow appears because of mass conservation of the residual meridional flow. Thus, the wave forcing can drive the residual mean flow below. The TEM equations have been widely used to examine the dynamical transport, wave activity, and wave propagation in the meridional cross section of the atmosphere.

On the other hand, there are some studies suggesting the necessity of examining three-dimensional (3D) transport and local wave activity. Recently, Sato et al. (2009a) showed that the increase in ozone after a mature stage of the Antarctic ozone hole significantly depends on the longitude (even in a dynamically stable polar vortex in spring) based on the ozonesonde observations at Syowa Station and the observations by the satellite-onboard Improved Limb Atmospheric Spectrometer II (ILAS-II) in 2003. They indicated the importance of lateral transport based on the longitude from the backward trajectory and tracer–tracer correlation analyses. Sato et al. (2009b) and Sato et al. (2012) showed that the dominant sources of mesospheric gravity waves are steep mountains such as the Andes, tropospheric jet–front systems in winter, and vigorous monsoon convection in summer by analyzing hourly data from a 3-yr simulation using a gravity wave–resolving general circulation model. However, in order to examine the 3D residual mean flow and wave activity flux, it is necessary to generalize the TEM equations to three dimensions.

Hoskins et al. (1983), Trenberth (1986), and Plumb (1986) extended the TEM equations to 3D using the time mean instead of the zonal mean under the quasigeostrophic (QG) approximation. Hoskins et al. (1983) and Trenberth (1986) defined the residual mean flows that can satisfy the continuity equation and eliminate the eddy forcing from the thermodynamic equation. Plumb (1986) defined the residual mean flows as literally “residual” terms that are derived by including the 3D wave activity flux into the horizontal momentum equation. However, they did not examine how their residual flows are related to the 3D Stokes drift like the residual mean flow in the TEM equations. Although Takaya and Nakamura (1997, 2001) derived a phase-independent 3D wave activity flux without using the time mean, the relation between the 3D residual flow and the 3D Stokes drift was not examined. Thus, it is necessary to investigate how the 3D Stokes drift in the QG equation is described.

On the primitive equation, Miyahara (2006) and Kinoshita et al. (2010) extended the TEM equations to 3D using the time mean under the assumption that the Coriolis parameter is constant. The dispersion relation for the inertia–gravity wave is used for the terms included in the horizontal component of the 3D Stokes drift. Thus, their Stokes drift is applicable only to inertia–gravity waves. It should be noted that their wave activity flux agrees with the 3D wave activity flux describing the propagation of Rossby waves, which was derived by Plumb (1986) under the small-Rossby-number assumption.

Noda (2010) formulated a generalized 3D TEM equation for a plane wave under the WKB approximation. However, since the covariance of perturbations is included in the denominator of the formulas for the Stokes drift, his formulas can only be used for a purely monochromatic wave.

Except for Noda (2010), these previous studies assumed the quasigeostrophic approximation or a constant Coriolis parameter. Thus, their formulas are applicable to either Rossby waves or inertia–gravity waves. The purpose of this study is to formulate the 3D Stokes drift and residual mean flow applicable to both Rossby waves and inertia–gravity waves. The obtained residual mean flow is applied to the upper troposphere and to the region of the southern Andes to demonstrate its usefulness. Moreover, from the standpoint of the mean state, we formulate the 3D wave activity flux whose divergence corresponds to the wave forcing to the mean flow. From the standpoint of waves, the relation between the 3D wave activity flux and the group velocity needs to be discussed. A companion paper, Kinoshita and Sato (2013), derives the unified dispersion relation (including inertia–gravity waves and Rossby waves), examines the relation, and formulates a modified 3D wave activity flux describing the wave propagation. It is discussed that this modified 3D wave activity flux is slightly different from the 3D wave activity flux defined in the present article, which is closely related to pseudomomentum flux.

This paper is arranged as follows. In section 2, the 3D Stokes drift is derived from the definition of the primitive equation (PRSD). The 3D wave activity flux is also derived by substituting PRSD into the time-mean horizontal momentum equation. We show that PRSD agrees with the Stokes drift for inertia–gravity waves (IGSD) derived by Kinoshita et al. (2010) when the Coriolis parameter is constant. Section 3 derives the 3D Stokes drift for the QG equation (QGSD) and shows that PRSD agrees with QGSD when the Rossby number is small. A phase-independent 3D Stokes drift is also derived using an assumption of the sinusoidal wave for geopotential under the QG approximation, which is applicable to stationary Rossby waves. The 3D Stokes drift is applied to a statistical study of the 3D residual mean flow around the storm-track region in the upper troposphere using the ERA-Interim data in section 4. Section 5 shows another case study by applying PRSD.
to examine the 3D residual mean flow associated with gravity waves around the southern Andes simulated by a gravity wave–resolving general circulation model. Finally, concluding remarks are given in section 6.

2. The time-mean 3D Stokes drift (PRSD) in the primitive equation system

a. A formulation of the PRSD and transformed time-mean equations

The primitive equations in the log-pressure coordinate system consist of the equations of motion, hydrostatic relation, and continuity and thermodynamic equations as follows:

\[
Du - f v + \Phi_x = X, \quad (2.1a)
\]
\[
Dv - fu + \Phi_y = Y, \quad (2.1b)
\]
\[
\Phi_z = H^{-1} R \theta e^{-\kappa z/H}, \quad (2.1c)
\]
\[
u_x + vy + \rho_0^{-1}(\partial w)_z = 0, \quad (2.1d)
\]
\[
D\theta = Q, \quad \text{and} \quad (2.1e)
\]
\[
D = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \quad (2.1f)
\]

where \( z \) is the log-pressure height; \( u, v, \) and \( w \) are the zonal, meridional, and vertical velocities, respectively; \( \rho_0 \) is the basic density; \( \Phi \) is the geopotential; \( f \) is the Coriolis parameter; \( \theta \) is the potential temperature; \( H \) is the scale height; \( R \) is the gas constant; \( \kappa \) expresses the ratio of \( R \) to the specific heat at a constant pressure; \( X \) and \( Y \) are unspecified horizontal frictions and other nonconservative mechanical forcings; and \( Q \) is the diabatic heating. The suffixes \( x, y, \) and \( z \) denote the partial derivatives.

Each variable is decomposed into its time mean (overbar) and instantaneous deviation (prime). The present study uses \( \bar{U}, \bar{L}, \bar{H}, \bar{f}, \) and \( \bar{N} \) as the scales or typical magnitudes for the time-mean horizontal velocity, the background horizontal scale, the background vertical scale, the Coriolis parameter, and the buoyancy frequency. The inverse square root of the Richardson number is written as

\[
\bar{r}_z / N \sim U/NH. \quad (2.2)
\]

The ratio \( \mu = h/H \) is defined, where \( h \) is the scale for \( z \) in the perturbation equations, and the following relations are assumed:

\[
U/NH \ll \mu \ll 1, \quad \text{and} \quad (2.3a)
\]
\[
1/T \ll U/L = f, \quad (2.3b)
\]

where \( T \) is the typical time scale for the background flow. The assumption \( (2.3a) \) is valid for tall mean flow scaling and large Richardson number and was used by Andrews and McIntyre (1976). The second assumption \( (2.3b) \) is valid when the scales of advection and the Coriolis parameter are larger than that of the time derivative, respectively. For the perturbation, the following assumption is further used:

\[
\mu \sim u'/U \sim x/L, \quad (2.4)
\]

where \( x \) is the scale for the horizontal perturbation. The time-mean and perturbation velocities are expanded asymptotically in powers of \( \mu \):

\[
\pi = \pi_0 + \mu \pi_1 + O(\mu^2 U), \quad (2.5a)
\]
\[
v = \nu_0 + \mu \nu_1 + O(\mu^2 U), \quad (2.5b)
\]
\[
\bar{w} = \mu \bar{w}_1 + O(\mu^2 U H/L), \quad \text{and} \quad (2.5c)
\]
\[
u' = u'_0 + \mu u'_1 + O(\mu^2 u'), \quad (2.5d)
\]

where the first order of the time-mean vertical velocity is assumed to be negligible. For the time-mean state, the balanced flow relation in the zonal momentum equation, whose magnitude is \( O(1) \), is given as follows:

\[
\pi_0 \bar{u}_x + (\pi_{0y} - f) \nu_0 + \bar{\Phi}_x = 0. \quad (2.6)
\]

Hereafter, \( \bar{u}_0 \) is regarded as the balanced flow, and the remaining terms are regarded as the unbalanced flow. For the perturbation, the above mentioned assumptions are summarized as follows:

- Wave amplitudes are small and constant.
- The differentiations of the time-mean variables are neglected.
- The time-mean vertical velocity is neglected.

Note that these assumptions are used to derive the Stokes drift in the order of the waves. In other words, the differentiations of the time-mean variables are included in the background order equations as shown later [e.g., the time-mean zonal momentum equation (2.19a)].

The perturbation equations whose magnitudes are \( O(\mu) \) are given in the following equations:

\[
\bar{D}u' - f v' + \Phi_x' = 0, \quad (2.7a)
\]

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\[ \mathcal{D} \Phi' + \mathcal{D} \phi' + \Phi' = 0, \quad (2.7b) \]
\[ \phi'_z = H^{-1} R \theta' e^{-\kappa z/H}, \quad (2.7c) \]
\[ u' + v' + \rho_0^{-1} (\rho_0 w')_z = 0, \quad (2.7d) \]
\[ \mathcal{D} \theta' + w \mathcal{D} z = 0, \quad (2.7e) \]

where the nonconservative and diabatic terms are negligible. By using the hydrostatic relation (2.7c), the thermodynamic equation (2.7e) is expressed as
\[ \mathcal{D} \Phi' + w \mathcal{D} \frac{N^2}{\partial t} = 0, \quad (2.8) \]
where \( N^2 = H^{-1} R \theta e^{-\kappa z/H} \) is the buoyancy frequency squared, which expresses the static stability. For the perturbation, a form of plane wave is considered:
\[ A' = A_0 e^{i(kx + ly + m z - \omega t)}, \quad (2.9a) \]

where \( A' \) is the arbitrary perturbation; \( k, l, \) and \( m \) are the zonal, meridional, and vertical wavenumbers, respectively; and \( \omega \) is the ground-based angular frequency. Note that assuming the meridional structure of the perturbation to be \( e^{i(ly)} \) is valid when the meridional variation of the Coriolis parameter is small. The basic density is expressed as
\[ \rho_0 = \rho_s \exp(-z/H), \quad (2.9b) \]

where \( \rho_s \) is the surface density. Substituting (2.9) into (2.7) and (2.8), the wind perturbations are written in terms of \( \Phi' \) as
\[ u' = \frac{k \omega + i l \phi'}{\omega^2 - f^2} \Phi', \quad v' = \frac{l \omega - i k \phi'}{\omega^2 - f^2} \Phi', \quad (2.10) \]
\[ w' = -\frac{\omega (m - i 2H)}{N^2} \Phi', \]
where \( \omega = \omega - kn - lh \) is the intrinsic frequency. Parcel displacements (\( \xi', \eta', \zeta' \)) are obtained as follows by using (2.10) and \( \mathcal{D} \xi' = u', \mathcal{D} \eta' = v', \mathcal{D} \zeta' = w' \):
\[ \xi' = \frac{-il + k \omega}{\omega (\omega^2 - f^2)} \Phi', \quad \eta' = \frac{kl + i \omega}{\omega (\omega^2 - f^2)} \Phi', \quad (2.11) \]
\[ \zeta' = -\frac{(im + 1/2H)}{N^2} \Phi'. \]

Note that the relation between the parcel displacement and the perturbation wind (\( \mathcal{D} \xi' = u', \mathcal{D} \eta' = v', \mathcal{D} \zeta' = w' \)) neglects the terms that include time-mean wind shear. The time-mean Stokes drift is given in the following using the same assumption (i.e., the amplitude of perturbation is small, and the time-mean wind shear is negligible):
\[ \pi^S = (\xi' u')_x + (\eta' v')_y + \rho_0^{-1} (\rho_0 \xi' w')_z \]
\[ = (\eta' u')_y + \rho_0^{-1} (\rho_0 \xi' w')_z, \quad (2.12a) \]
\[ \pi^S = -(\xi' v')_x + (\eta' u')_y + \rho_0^{-1} (\rho_0 \xi' w')_z \]
\[ = -(\eta' u')_y + \rho_0^{-1} (\rho_0 \xi' w')_z, \quad (2.12b) \]
\[ \pi^S = (\xi' w')_x + (\eta' v')_y + \rho_0^{-1} (\rho_0 \xi' w')_z \]
\[ = -(\eta' u')_y - (\xi' v')_y. \quad (2.12c) \]

Here, it should be noted that the deformations in the second equality of each equation are made by using the relations (\( \xi' u' = (\xi' w') = (\xi' v') = 0 \), (\( \xi' v' = (\eta' u') = (\eta' v') \), (\( \xi' w' = (\xi' v') \), and (\( \eta' v') = (\eta' w') \) under the assumption that the time-mean wind shear is small. By using (2.10) and (2.11), \( \xi' \eta' \) is expressed in terms of \( \Phi' \) as follows:
\[ \xi' \eta' = \frac{f (k^2 + l^2)}{(\omega^2 - f^2)^2} \Phi^2. \quad (2.13) \]

Similarly, the kinetic energy, \( u' \Phi'_y/2f \), and \( v' \Phi'_x/2f \) are written as follows:
\[ \frac{1}{2} (u'^2 + v'^2) = \frac{1}{2} \frac{(\omega^2 + f^2)(k^2 + l^2)}{(\omega^2 - f^2)^2} \Phi^2 \quad (2.14a) \]
\[ \frac{u' \Phi'_y}{2f} = \frac{1}{2} \frac{(\omega^2 - f^2)^2 k^2}{(\omega^2 - f^2)^2} \Phi^2 \quad (2.14b) \]
\[ \frac{v' \Phi'_x}{2f} = -\frac{1}{2} \frac{(\omega^2 - f^2)^2 l^2}{(\omega^2 - f^2)^2} \Phi^2 \quad (2.14c) \]

Therefore,
\[ \xi' \eta' = \frac{1}{2f} \left( \frac{u'^2 + v'^2 - u' \Phi'_y}{f} + \frac{v' \Phi'_x}{f} \right). \quad (2.15) \]

It should be noted that (2.14b), (2.14c), and (2.15) cannot be used in the equatorial region since the Coriolis parameter is included in their denominator. Next, \( \xi' u' \) and \( \xi' v' \) are written in terms of \( \Phi' \) in a similar way in the following:
Kinoshita et al. (2010) is expressed as follows:  
\[ \tilde{u}' = \frac{-l_{m}f + k\omega/2H}{N^2(\omega^2 - f^2)} \Phi' \]  
\[ \tilde{v}' = \frac{k_{m}f - l\omega/2H}{N^2(\omega^2 - f^2)} \Phi' \]  
(2.16)

Thus, the 3D Stokes drift (PRSD) in the primitive equation is formulated as follows:

\[ \mathbf{u}^S = \left( \frac{S_{(p)}}{f} \right)_y - \frac{1}{\rho_0} \left( \frac{\rho \tilde{u}' \Phi'}{N^2} \right)_z, \]  
(2.17a)

\[ \mathbf{v}^S = -\left( \frac{S_{(p)}}{f} \right)_x - \frac{1}{\rho_0} \left( \frac{\rho \tilde{v}' \Phi'}{N^2} \right)_z, \]  
(2.17b)

\[ \mathbf{w}^S = \frac{\tilde{u}' \Phi'}{N^2} + \frac{\tilde{v}' \Phi'}{N^2}, \]  
(2.17c)

where \( S_{(p)} = (1/2)(\tilde{u}'^2 + \tilde{v}'^2 - \tilde{u}' \Phi' / f + \tilde{v}' \Phi' / f) \). The subscript \( (p) \) is used to distinguish from the notation \( \overline{S} = (1/2)(\tilde{u}'^2 + \tilde{v}'^2) / N^2 \) used by Miyahara (2006) and Kinoshita et al. (2010). The IGSD derived by Kinoshita et al. (2010) is expressed as follows:

\[ \overline{\mathbf{u}}^S_{(IG)} = \left( \frac{\overline{S}}{f} \right)_y - \frac{1}{\rho_0} \left( \frac{\rho \overline{u}' \Phi'}{N^2} \right)_z, \]  
(2.18a)

\[ \overline{\mathbf{v}}^S_{(IG)} = -\left( \frac{\overline{S}}{f} \right)_x - \frac{1}{\rho_0} \left( \frac{\rho \overline{v}' \Phi'}{N^2} \right)_z, \]  
(2.18b)

\[ \overline{\mathbf{w}}^S_{(IG)} = \frac{\overline{u}' \Phi'}{N^2} + \frac{\overline{v}' \Phi'}{N^2}. \]  
(2.18c)

IGSD (2.18) is equal to PRSD except for the term \( \overline{S}_{(p)} / f \). This term is approximately equal to \( \overline{S} / f \) when using the assumptions shown in section 2b. PRSD (2.17) is applicable not only to inertia–gravity waves but also to other types of waves, including Rossby waves, since a dispersion relation for these particular kinds of waves is not used for the derivation, unlike the Stokes drift derived by Miyahara (2006) and Kinoshita et al. (2010).

Finally, the horizontal momentum equation and thermodynamic equation for the time-mean field are given by

\[ \overline{u}_x + \overline{u}_y \overline{u}_x + (\overline{u}_y - f) \overline{u}_y + \overline{u}_z \overline{w}_x = -\overline{F}_x - (\overline{u}' \overline{v}')_x - \rho_0^{-1}(\rho_0 \overline{w}' \overline{w}')_z + \overline{F}_1, \]  
(2.19a)

\[ \overline{u}_t + (\overline{u}_x + f) \overline{u}_x + \overline{u}_y \overline{v}_y + \overline{u}_z \overline{w}_z = -\overline{F}_y - (\overline{u}' \overline{v}')_y - \rho_0^{-1}(\rho_0 \overline{w}' \overline{w}')_z + \overline{F}_2, \]  
(2.19b)

\[ \overline{F}_{z1} + \overline{F}_{z2} \overline{u}_x + \overline{F}_{z2} \overline{v}_y + \overline{N}^2 \overline{w}_y = \overline{F}_{z2} \left[ \frac{\overline{S}_{(p)}}{f} \right]_y - \kappa \overline{u}' \Phi'_z \left[ \frac{\overline{S}_{(p)}}{f} \right]_x + \frac{\kappa \overline{v}' \Phi'_z}{N^2} \right] \]  

\[ - \rho_0^{-1} \left\{ \rho_0 \overline{u}' \Phi'_z N^2 + \overline{u}' \Phi'_z z_x \right\} + (\rho_0 \overline{w}' \overline{w}' e^{-\kappa z / H})_z e^{-\kappa z / H} + \overline{F}_{z2} \left[ \frac{\overline{S}_{(p)}}{f} \right]_x + \frac{\kappa \overline{v}' \Phi'_z}{N^2} \right] \]  

(2.19c)

where \( \overline{u}_x, \overline{v}_y, \) and \( \overline{w}_z \) are the zonal, meridional, and vertical components of the 3D residual mean flow (which is equal to the sum of the time-mean velocity and the 3D Stokes drift), respectively, and \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) are the 3D wave activity fluxes, which are written as
Thus,
$$\frac{1}{2f} \left( -\frac{u'\Phi'_{y}}{f} + \frac{v'\Phi'_{z}}{f} \right) = -\frac{1}{2} \left( \Phi'$^2 \right) \frac{N^2}{\omega^2 - f^2}. \quad (2.24)$$

This result indicates that PRSD is equal to IGSD (2.18) when the Coriolis parameter is constant. The relation between PRSD and the Stokes drift (QGSD) for the quasigeostrophic (QG) equation is shown after QGSD is derived in section 3.

3. A formulation of the 3D Stokes drift in the QG equation system

a. The time-mean QGSD

The perturbation equations for the QG equation system are given as follows under the same assumptions as for (2.7):

$$\overline{\mathbf{D}_g} u'_g - f_0 v'_g - \beta y u'_g = 0, \quad (3.1a)$$
$$\overline{\mathbf{D}_g} v'_g + f_0 u'_g + \beta y u'_g = 0, \quad (3.1b)$$
$$\Phi'_z = H^{-1} R\theta'_e e^{-\kappa_2 H}, \quad (3.1c)$$
$$u'_{ax} + u'_{ay} + \rho_0^{-1}(\rho_0 w'_a)_z = 0, \quad (3.1d)$$
$$\overline{\mathbf{D}_g} \theta'_e + w'_a \overline{\theta}_0 z = 0, \quad \text{and} \quad (3.1e)$$
$$\overline{\mathbf{D}_g} = \frac{\partial}{\partial t} + \pi \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}. \quad (3.1f)$$

where $f_0$ is the Coriolis parameter at the reference latitude, $\beta = f_0$ is the meridional gradient of the Coriolis parameter at the reference latitude, $\theta_0(z)$ is the reference potential temperature, $\theta_0$ is the departure from $y_0$, and suffixes $g$ and $a$ denote the geostrophic and ageostrophic components, respectively. For the perturbation of the geopotential, a form of plane wave in (2.9a) is considered. The perturbations of the geostrophic flows, ageostrophic flows, and potential temperature are written in terms of $\Phi'$ as

$$u'_g = -\frac{i \tilde{t}}{f_0} \Phi', \quad v'_g = \frac{f_0}{f_0} \Phi', \quad (3.2a)$$
$$u'_a = -\frac{\tilde{\omega}}{f_0^2} \tilde{\beta} y \Phi', \quad v'_a = -\frac{\tilde{\omega}}{f_0^2} \frac{f_0}{f_0} \Phi', \quad (3.2b)$$
$$w'_a = -\frac{\tilde{\omega}(m - i2H)}{N^2} \Phi', \quad \theta'_e = \frac{\overline{\theta}_0 (im + 1/2H)}{N^2} \Phi', \quad \text{and} \quad (3.2c)$$
\[ N^2 = H^{-1} R \Phi_0 e^{-\kappa z/H}, \] (3.2d)

where (3.2a) expresses the geostrophic balance. The geostrophic and ageostrophic components of the parcel displacements are obtained as follows after some manipulation:

\[ \xi'_g = \frac{l}{\omega f_0} \Phi', \quad \eta'_g = -\frac{k}{\omega f_0} \Phi', \] (3.3a)

\[ \xi'_a = -\frac{\beta y + ik \omega}{\omega f_0^2} \Phi', \quad \eta'_a = \frac{k \beta y - il \omega}{\omega f_0^2} \Phi', \] and (3.3b)

\[ \xi'_a = -\left( \frac{im + 1/2H}{N^2} \right) \Phi'. \] (3.3c)

Each term of \( \eta'_w = \eta'_g u'_g + \eta'_a u'_a + \eta'_g u'_a + \eta'_a u'_a \) is expressed as follows:

\[ \overline{\eta'_g u'_g} = 0, \] (3.4a)

\[ \overline{\eta'_g u'_a} + \overline{\eta'_a u'_g} + \frac{f_0(k^2 + l^2)}{f_0^2} \overline{\Phi'^2} = \frac{u'_g + u'_a}{f_0}, \] (3.4b)

\[ \overline{\eta'_a u'_a} = -\frac{\beta y(k^2 + l^2)}{f_0^2} \overline{\Phi'^2} = -\beta y \overline{(u'_g + u'_a)^2} . \] (3.4c)

Next, \( \overline{\zeta'_w} \) and \( \overline{\zeta'_v} \) are written in terms of \( \Phi' \) in a similar way

\[ \overline{\zeta'_g u'_g} \overline{\Phi'^2} = \frac{u'_g \Phi'_z}{N^2}, \]

\[ \overline{\zeta'_a u'_a} \overline{\Phi'^2} = -\frac{\overline{v'_a \Phi'_z}}{N^2} \quad \text{and} \quad (3.5a) \]

\[ \overline{\Phi'_z} \overline{\Phi'_z} = \frac{1}{2f} \left( u'_g^2 + v'_g^2 - \frac{u'_g \Phi'_z}{f} + \frac{v'_g \Phi'_z}{f} \right) = \frac{(u'_g + v'_a)^2 + (u'_g + v'_a)^2}{2(f_0 + \beta y)^2} - \frac{(u'_g + v'_g) \Phi'_x}{f_0} + \frac{(v'_g + v'_a) \Phi'_x}{f_0} - 2 \frac{\beta y u'_g v'_a}{f_0} + \frac{v'_g + v'_a}{2f_0} + \frac{v'_g + v'_a}{2f_0} + \frac{2 \beta y u'_g v'_a}{2f_0} \approx \frac{u'_g + v'_a}{f_0} . \] (3.7)

Next, \( (u'_g \Phi'_z/N^2) \) and \( (u'_a \Phi'_z/N^2) \) are deformed in a similar way:

\[ \frac{u'_g \Phi'_z}{N^2} \approx \frac{u'_g \Phi'_z}{N^2}, \quad \frac{u'_a \Phi'_z}{N^2} \approx \frac{u'_a \Phi'_z}{N^2} . \] (3.8)

Substituting (3.7) and (3.8) into (2.17), it is shown that PRSD agrees with QGSD. This result and the result in section 2b indicate that PRSD (2.17) is applicable to both Rossby-type waves and inertia–gravity waves.

b. A phase-independent 3D Stokes drift

Stationary planetary waves play an important role in driving the meridional circulation in the polar winter stratosphere. However, QGSD cannot be used for the
study of the 3D Stokes drift associated with stationary waves (unlike the Stokes drift in the TEM equations) since QGSD is derived for the time mean instead of the zonal mean. Thus, in this section, referring to the method used by Plumb (1985) and Takaya and Nakamura (1997, 2001) to eliminate twice phase oscillation, a phase-independent 3D Stokes drift in the QG equation is derived from its original definition, which has not been done in previous studies. Phase independence means that twice phase oscillation of waves is eliminated without using the zonal or time mean in the formulas.

First, for the perturbation of the geopotential, a sinusoidal wave is considered:

\[ \Phi' = P_0 e^{izH} \sin \theta, \quad \Theta = kx + ly + mz - \omega t + \alpha, \]

\[ \Psi' = P_0 e^{izH}, \quad (3.9) \]

where \( \alpha \) is an arbitrary constant. Substituting (3.9) into (3.1) yields

\[ u'_{y} = -\frac{\cos \Theta}{f_0} \psi', \quad \psi' = \frac{k \cos \Theta}{f_0} \psi', \quad (3.10a) \]

\[ u'_{x} = -\frac{k \omega \sin \Theta + l \beta y \cos \Theta}{f_0} \psi', \]

\[ u'_{z} = -\frac{l \omega \sin \Theta + k \beta y \cos \Theta}{f_0} \psi', \quad (3.10b) \]

\[ w' = -\frac{m \omega + \omega^2 H \cos \Theta}{N^2} \psi'. \quad (3.10c) \]

Moreover, the geostrophic and ageostrophic components of the parcel displacements are written from (3.10) as

\[ \xi'_{y} = \frac{l \sin \Theta}{\omega f_0} \psi', \quad \xi'_{x} = -\frac{k \sin \Theta}{\omega f_0} \psi', \quad (3.11a) \]

\[ \xi'_{z} = -\frac{k \omega \cos \Theta + l \beta y \sin \Theta}{\omega f_0} \psi', \]

\[ \eta'_{y} = -\frac{l \omega \cos \Theta - k \beta y \sin \Theta}{\omega f_0} \psi', \quad (3.11b) \]

\[ \eta'_{z} = -\frac{m \cos \Theta + 1/2H \sin \Theta}{\omega N^2} \psi'. \quad (3.11c) \]

By using (3.10) and (3.11), \( u'_{g} e'_{a} \) in the zonal and vertical components of the Stokes drift is written in terms of \( \Psi' \) as

\[ u'_{g} e'_{a} = -\frac{l \cos \Theta}{f_0} \left( \frac{m \cos \Theta + 1/2H \sin \Theta}{\omega N^2} \psi' \right) \]

\[ = \frac{lm \cos^2 \Theta + 1/2H \sin \Theta \cos \Theta}{f_0 N^2} \psi', \quad (3.12) \]

which includes the phase \( \Theta \). To eliminate the phase-dependent terms, we use the antisymmetric component of the diffusion tensor \( K \), which is defined as follows:

\[ K = \begin{bmatrix} 0 & \frac{1}{2}(\eta'_{u} - \xi'_{u}) & \frac{1}{2}(\xi'_{u} - \xi'_{w}) \\ -\frac{1}{2}(\eta'_{u} - \xi'_{u}) & 0 & \frac{1}{2}(\xi'_{u} - \eta'_{w}) \\ -\frac{1}{2}(\xi'_{u} - \xi'_{w}) & -\frac{1}{2}(\xi'_{u} - \eta'_{w}) & 0 \end{bmatrix}. \quad (3.13) \]

The divergence of the time-mean antisymmetric component of the diffusion tensor \( \rho_0^{-1} v \cdot (\rho_0 K) \) is equal to the time-mean Stokes drift \( \vec{u} \) because \( (\xi'_{u}, \xi'_{w}) = -\frac{1}{2}(\eta'_{u}, \eta'_{w}) \) under the assumption that the time-mean wind shear is small. The term \( \xi'_{w} w' \) is written in terms of \( \Psi' \) as

\[ \xi'_{w} w' = -\frac{lm \sin^2 \Theta + 1/2H \sin \Theta \cos \Theta}{f_0 N^2} \psi'^2. \quad (3.14) \]

Thus,

\[ \frac{1}{2}(u'_{g} e'_{a} - \xi'_{w} w') = \frac{lm}{2f_0 N^2} \psi'^2. \quad (3.15) \]

This is phase independent and corresponds to \( u'_{g} \Psi'/N^2 \) in QGSDs (3.5a) and (3.6a). Similarly, the geostrophic components of \( \eta'_{u}, \xi'_{v}, \) and \( (1/2)(\eta'_{u} - \xi'_{v}) \) are written respectively as follows:

\[ \eta'_{g} u'_{g} = -\frac{m \cos \Theta}{\omega f_0} \left( -\frac{l \cos \Theta}{f_0} \right) \psi' = \frac{kl \sin \Theta \cos \Theta}{\omega f_0^2} \psi'^2, \quad (3.16a) \]

\[ \xi'_{g} \psi' = \frac{l \sin \Theta}{\omega f_0} \left( \frac{k \cos \Theta}{f_0} \right) \psi' = \frac{kl \sin \Theta \cos \Theta}{\omega f_0^2} \psi'^2, \quad (3.16b) \]

and hence,

\[ \frac{1}{2}(\eta'_{g} u'_{g} - \xi'_{g} \psi') = 0. \quad (3.16c) \]

The terms of the next order are expressed as
\[
\eta_k'u'_a + \eta_a'u'_g = \left[ -\kappa \sin \Theta \left( -\kappa \omega \sin \Theta + \ell \beta_y \cos \Theta \right) + \frac{-\kappa \omega \cos \Theta + \ell \beta_y \sin \Theta \left( -\ell \cos \Theta \right)}{f_0^2} \right] \Psi^2 \\
= \frac{-\kappa \omega (k^2 \sin^2 \Theta + \ell^2 \cos^2 \Theta) - 2k \beta_y \sin \Theta \cos \Theta \Psi^2}{f_0^3}, \tag{3.17a}
\]

\[
\xi_k'u'_a + \xi_a'u'_g = \left[ \frac{\ell \sin \Theta \left( -\kappa \omega \sin \Theta + \ell \beta_y \cos \Theta \right)}{f_0^2} - \frac{k \omega \cos \Theta + \ell \beta_y \sin \Theta \left( k \cos \Theta \right)}{f_0^2} \right] \Psi^2 \\
= \frac{-\omega (k^2 \sin^2 \Theta + \ell^2 \cos^2 \Theta) - 2k \beta_y \sin \Theta \cos \Theta \Psi^2}{f_0^3}, \tag{3.17b}
\]

\[
\frac{1}{2} (\eta_k'u'_a + \eta_a'u'_g - \xi_k'u'_a - \xi_a'u'_g) = \frac{1}{2} \frac{k^2 + \ell^2}{f_0^3} \Psi^2. \tag{3.17c}
\]

The deformation of \( \xi_k'u'_a \) and \( \eta_k'u'_a \) is omitted since this is obtained in the same way as for \( \xi_a'u'_g \). Thus, the phase-independent 3D Stokes drift is expressed as follows:

\[
u^S = \frac{1}{2} (\eta_k'u'_a + \eta_a'u'_g - \xi_k'u'_a - \xi_a'u'_g) y \\
+ \frac{1}{2\rho_0} \left[ \rho_0 \xi_k' u'_a - \xi_k' w'_a \right] z, \tag{3.18a}
\]

\[
u^S = \frac{1}{2} (\eta_k'u'_a + \eta_a'u'_g - \xi_k'u'_a - \xi_a'u'_g) x \\
+ \frac{1}{2\rho_0} \left[ \rho_0 \xi_k' u'_a - \xi_k' w'_a \right] z, \tag{3.18b}
\]

\[
\omega^S = \frac{1}{2} \left( \xi_k' u'_a - \xi_a' w'_a \right) x - \frac{1}{2} \left( \xi_k' w'_a - \xi_a' u'_a \right) y. \tag{3.18c}
\]

Next, the phase-independent 3D Stokes drift is deformed without using parcel displacements. In the same way as in the previous subsection, \( \xi_a'u'_g \) and \( \xi_a'u'_g \) are written as follows:

\[
\xi_a'u'_g = -\frac{u_g' \Phi'_z}{N^2}, \xi_a'u'_g = -\frac{u'_g \Phi'_z}{N^2}. \tag{3.19}
\]

Then, \( \xi_k'u'_a \) and \( \eta_k'u'_a \) are expressed with the perturbations of the horizontal velocities and geopotentials as follows:

\[
\xi_k'u'_a = -\frac{lm \sin^2 \Theta + \ell/2H \sin \Theta \cos \Theta \Psi^2}{f_0^2} \\
= \frac{-\ell m \sin \Theta - \ell/2H \cos \Theta \left( \sin \Theta \right)}{f_0^2} \left( \sin \Theta \right) \Psi^2 \tag{3.20a}
\]

\[
\eta_k'u'_a = \left[ \frac{-\ell m \cos \Theta}{f_0^2} \left( \sin \Theta \right) \Psi^2 \right] \tag{3.20b}
\]

Therefore, the phase-independent 3D Stokes drift is expressed without using parcel displacements as follows:

\[
u^S = \left[ \frac{1}{2f_0} \left( u_g'^2 + v_g'^2 - \frac{\Phi'_x \Phi'_{xx} + \Phi'_y \Phi'_{yy}}{2f_0^2} \right) \right] y \\
- \frac{1}{\rho_0} \left[ \rho_0 \frac{u'_g \Phi'_{zz} - u'_g \Phi'_{zz}}{2N^2} \right] z, \tag{3.22a}
\]

\[
u^S = \left[ \frac{1}{2f_0} \left( u_g'^2 + v_g'^2 - \frac{\Phi'_x \Phi'_{xx} + \Phi'_y \Phi'_{yy}}{2f_0^2} \right) \right] x \\
- \frac{1}{\rho_0} \left[ \rho_0 \frac{u'_g \Phi'_{zz} - u'_g \Phi'_{zz}}{2N^2} \right] z, \tag{3.22b}
\]

\[
\omega^S = \left[ \frac{1}{2N^2} (u_g' \Phi'_z - u'_g \Phi'_z) \right] x + \left[ \frac{1}{2N^2} (v'_g \Phi'_z - v'_g \Phi'_z) \right] y. \tag{3.22c}
\]
These forms are equal to the terms included in the residual mean flow derived by Takaya and Nakamura (2001). Note that Takaya and Nakamura derived these terms by using the horizontal momentum equation and the phase-independent wave activity flux and did not show that their terms agree with the Stokes drift. It is emphasized again that the phase-independent 3D Stokes drift in (3.22) is derived from its original definition in this section.

4. Example 1: 3D residual mean flow in the upper troposphere

The time-mean 3D Stokes drifts are formulated in sections 2 and 3 for transient perturbations. As an application of these formulas, we focus on the upper troposphere (250-hPa level) where transient waves are dominant. In the zonal-mean fields, there is the Ferrel cell in the middle latitudes, which is expressed by the equatorward flow in the upper troposphere. The zonal-mean residual mean flow is poleward there since the zonal-mean Stokes drift is poleward and stronger than the Ferrel cell (e.g., Edmon et al. 1980). Though their 3D structures are not clear, there is a possibility that they have organized structures because there are longitudinal regions having strong transient waves, which are called storm tracks.

In this section, PRSD, QGSD, and IGSD are compared in terms of the magnitudes of their horizontal

![Graph showing the difference in magnitude of IGSD and QGSD from PRSD at 250 hPa on 15 Apr.](image-url)
components. Next, the distribution of the meridional component of the 3D residual mean flow including PRSD (2.17) is examined.

a. Data description

The ERA-Interim data gathered over 19 yr (from 1990 to 2008) at a time interval of 6 h are used for the analysis. The data are distributed on $1.5^\circ \times 1.5^\circ$ horizontal latitude and longitude grids at 37 vertical levels. The time-mean field is obtained by applying a low-pass filter to the data with a cutoff period of 60 days. The disturbance field is defined as the deviation from the time-mean field. We calculate the climatological time-mean quantities over the 19 yr. The results are shown for 15 April, when the transient disturbances are strong in the Northern Hemisphere (Nakamura 1992; Sato et al. 2000). Note that the low-pass-filtered data of 15 April roughly correspond to the data averaged over 30 days with the center at 15 April. The residual mean flow, momentum flux, and Stokes drift are obtained by smoothing using the same low-pass filter. It should be noted that the analyses in this section and in section 5 use equations in spherical coordinates, not Cartesian coordinates, and they are introduced in the appendix.

b. Comparison of three kinds of Stokes drift (PRSD, QGSD, and IGSD)

Figure 1 shows the horizontal distribution of the difference in the magnitudes of IGSD (Fig. 1a) and QGSD (Fig. 1b) from PRSD and the absolute values of PRSD at 250 hPa on 15 April. The three line plots on the right show the absolute values of the zonal-mean PRSD (left), the zonal-mean difference of IGSD (Fig. 1a) and QGSD (Fig. 1b) from PRSD (middle), and the magnitude of the zonal-mean difference relative to PRSD (right). Since PRSD, IGSD, and QGSD have terms that include the Coriolis parameter in the denominator, Fig. 2 and other horizontal maps show the results only in latitudes higher than $3^\circ$. Note that QGSD is calculated using the constant Coriolis parameter defined for each of the latitudinal bands.

Around the tropical region, where the QG assumption is not relevant, QGSD is larger than PRSD, and the
zonal mean of their difference is twice as large as the zonal-mean PRSD. On the other hand, the zonal mean of their difference in IGSD relative to PRSD is small, though IGSD tends to be smaller than PRSD. The zonal mean of the difference in QGSD relative to PRSD is relatively small in the middle latitudes, where the QG disturbances are dominant (it is almost zero from 35° to 68°N). However, the difference in QGSD is somewhat large in the high latitudes, which seems to be partly due to a small PRSD. Another possible explanation is that there are few QG disturbances in the high latitudes because $\beta$ is small. On the other hand, the difference in IGSD relative to PRSD is generally small except for the midlatitudes of 50°–70°S.

These results suggest that PRSD roughly agrees with IGSD from the tropical to subtropical regions and roughly agrees with QGSD in the extratropical region. Since PRSD theoretically includes the QG waves and inertia–gravity waves, PRSD seems to be valid in all regions except for the equatorial region, where $f$ is almost zero.

c. The meridional component of the 3D residual mean flow

The 3D residual mean flow is described by the sum of the time-mean wind velocity and the Stokes drift. The centrifugal force cannot be neglected where the time-mean wind velocity is strong. The time-mean balanced flow (not geostrophic wind) is appropriate in such regions. In this study, the time-mean balanced flow is regarded as the background wind, which is obtained by using the method of Randel (1987). The sum of the unbalanced flow and the Stokes drift is regarded as the residual mean flow associated with disturbances, which is expressed as $\mathbf{v}$ in this study. Note that the 2D residual mean flow in the TEM equations is also the sum of the ageostrophic flow and the Stokes drift. Figure 2 shows the horizontal distributions of the meridional component of the 3D residual mean flow, the time-mean balanced flow, the unbalanced flow, and the Stokes drift (PRSD) at 250 hPa in the Northern Hemisphere on 15 April. The zonal average is shown on the right.
contours show the variance of the geopotential height disturbances. Two remarkable storm tracks are located from 150°E to 120°W and from 60°W to 0° in the middle latitudes, which is similar to the results of Nakamura (1992) and Sato et al. (2000). The distribution of the residual mean flow is not zonally uniform. A comparison between Figs. 2a and 2b indicates that the time-mean balanced flow is the main component of the residual mean flow. However, the zonal-mean balanced flow has almost vanished. Thus, the zonal-mean meridional transport is roughly equal to the sum of the unbalanced flows and the Stokes drift (Figs. 2a, 2c, and 2d). Hereafter, though the time-mean balanced flow may affect the 3D transport, we focus on the residual mean flow associated with disturbances $\Psi$. From a comparison between Figs. 2c and 2d, it is found that the distributions of the unbalanced flow and the Stokes drift are related in the location of the storm tracks. The unbalanced flow is weakly poleward (strongly equatorward) in the upstream (downstream) region, while the Stokes drift is strongly poleward (weakly equatorward) in the upstream (downstream) region. In the extratropical region, the zonal-mean unbalanced flow is equatorward, which corresponds to an upper branch of the Ferrel cell. The zonal-mean Stokes drift is poleward and stronger than the unbalanced flow. Thus, zonal-mean transport is weakly poleward, which is in agreement with previous studies (e.g., Edmon et al. 1980).

Figure 3 is the same as Fig. 2, but for the longitude–pressure cross section of the average in the latitudes from 30° to 60°N. The unbalanced flow is equatorward in the region from the center to downstream of the storm tracks. The Stokes drift is strongly poleward (weakly equatorward) in the upstream (downstream) region, and it is especially strong at 400–200 hPa. In the next subsection, the structure of the unbalanced flow and the Stokes drift around the storm tracks are examined.

d. The unbalanced flow around the storm tracks

The meridional component of the balanced flow satisfies the following relation, which is the difference between the time-mean zonal momentum equation and the balanced wind equation:

$$
\nabla (U_{UB}) = -(\Pi_y - f)^{-1}[\Pi_x + (\Pi_{BA})_x v_{UB} + (\Pi_{UB})_x v_{BA}] + \Pi_{UB} v_{BA} + \Pi_y w + (u'^2)_x + (u'^2)_y + \rho_0^{-1}(\rho_0 u'w')_z,
$$

where the suffixes (BA) and (UB) denote the balanced and unbalanced flows, respectively. Figure 4 shows the
horizontal distribution of the unbalanced flow $\overline{\mathbf{u}}_{\text{UB}}$ (Fig. 4a), the time-mean zonal wind acceleration $\overline{\mathbf{u}}_t$ (Fig. 4b), the zonal advection terms $\overline{\mathbf{u}}_{(BA)} \cdot \overline{\mathbf{u}}_{\text{UB}} + \overline{\mathbf{u}}_{\text{UB}} \cdot \overline{\mathbf{u}}_{(BA)}$ (Fig. 4c), the meridional advection term $\overline{\mathbf{u}}_{(BA)} \cdot \overline{\mathbf{v}}_{\text{UB}}$ (Fig. 4d), the vertical advection term $\overline{\mathbf{u}}_{\text{UB}} \cdot \overline{\mathbf{w}}$ (Fig. 4e), and the sum of the terms including the momentum flux $(\overline{\mathbf{u}}^2) + (\overline{\mathbf{u}^2})_y + \rho_0^{-1}(\overline{\mathbf{u}} \overline{\mathbf{w}})^2$, (Fig. 4f) at 250 hPa on 15 April. These terms (except for the unbalanced flow) are divided by $- (\overline{\mathbf{u}}^2)$. The unbalanced flow (Fig. 4a) around the storm tracks is mainly due to the terms that include the momentum flux (Fig. 4f), though the advection terms are not negligible in the tropical region. The time-mean zonal wind acceleration is negligibly small in all regions. Figure 5 shows the zonal, meridional, and vertical derivatives of the momentum flux at 250 hPa on 15 April. The contour lines express the variances of the geopotential height disturbances. It is found that the zonal and meridional derivatives of the momentum flux $- (\overline{\mathbf{u}^2})$ and $- (\overline{\mathbf{u}^2})_y$ are dominant around the storm tracks. The zonal derivative of the momentum flux $- (\overline{\mathbf{u}^2})$, corresponds to the poleward (equatorward) unbalanced flows in the upstream (downstream) region of the storm tracks. Since the zonal wind is proportional to the meridional derivative of the geopotential under the QG approximation, $- (\overline{\mathbf{u}^2})$ is related to a longitudinal gradient of the variance in the geopotential height disturbances. The meridional derivative of the momentum flux $- (\overline{\mathbf{u}^2})_y$, which is equal to the meridional derivative of the meridional component of the wave activity flux in the zonal momentum equation $\rho_0^{-1}(F_{12})_y$, under the QG approximation, almost corresponds to the equatorward unbalanced flow from the center to the equatorward regions of the storm tracks. Thus, the unbalanced flow around the storm tracks is related to the longitudinal distribution of the geopotential height disturbances and meridional component of the wave activity flux since the QG approximation is valid there.

e. The Stokes drift around the storm tracks

The meridional component of the Stokes drift is strongly poleward (weakly equatorward) in the upstream (downstream) region of the storm tracks (Fig. 2). This characteristic structure around the storm tracks is observed in January, July, and December (not shown). To examine this structure, the longitudinal–pressure cross sections of the first and second terms of the meridional component of the Stokes drift in (2.17b) averaged over the latitudes from 30° to 60°N are shown in Figs. 6a and 6b. It is found that the Stokes drift around the storm tracks is mainly due to the vertical derivative of the heat flux. The longitudinal–height cross section of the heat flux is displayed in Fig. 6c. It should be noted

![Fig. 5. Horizontal maps of (top) the term including the zonal derivative of momentum flux $-(\overline{\mathbf{u}^2})/(\overline{\mathbf{u}_t} - f)$, (middle) the term including the meridional derivative of momentum flux $-(\overline{\mathbf{u}^2})_y/(\overline{\mathbf{u}_t} - f)$, and (bottom) the term including the vertical derivative of momentum flux $-\rho_0^{-1}(\overline{\mathbf{u} \overline{\mathbf{w}}})^2/(\overline{\mathbf{u}_t} - f)$ at 250 hPa on 15 Apr. The solid curves in all maps show the variance of the geopotential height. Contour interval is $4 \times 10^3$ m².](image-url)
that the heat flux is obtained for \( \nu' \) and \( \theta' \) over a period of 2–8 days and corresponding to the synoptic-scale disturbances, which are the dominant components of the disturbances in this region. It is suggested that the equatorward Stokes drift is caused by the existence of negative heat flux at 500–300 hPa.

Next, transient disturbances affecting the distribution of the heat flux are examined. Figure 7 shows (Fig. 7a) a Hovmöller diagram of geopotential disturbances with periods of 2–8 days at 250 hPa during April 1999 and (Fig. 7b) the longitudinal–pressure cross section of the geopotential (contours) and heat flux (shading) of the disturbance on 21 April 1999. These are averaged over the latitudes from 30° to 60°N. In the upstream region of the storm track (−150°E−180°), the disturbance has a baroclinic structure and an almost positive heat flux. On the other hand, in the downstream region (−135°−105°W), the disturbance has a barotropic structure, and a positive heat flux (negative heat flux) at 100–200 (300–600) hPa is observed in front of the disturbance. This characteristic structure can be explained by using the geostrophic balance \( \nu' \sim \Phi_x' \) and hydrostatic balance \( \theta' \sim \Phi'_z \).

To confirm whether the characteristic structure of the disturbances in the downstream region was observed in other years, a scatter diagram is made of the heat flux...
averaged from 30° to 60°N at 400–300 hPa versus at 200–150 hPa for a longitude range of 150°E–180° (upstream of the storm tracks) (Fig. 8, left) and 135°–105°W (downstream of the storm tracks) (Fig. 8, right) in March–April from 1990 to 2008. A number of events with a positive heat flux at both 400–300 hPa and 200–150 hPa (P–P events) are seen in the upstream region, while many events having a negative heat flux at 400–300 hPa and a positive heat flux at 200–150 hPa (N–P events) are seen in the downstream region.

To examine the N–P events in more detail, a time series of a composite of the longitude and pressure section is made (Fig. 9) using 210 cases when the heat flux in more than 66% of the region (at 400–300 hPa in the longitude of 105°–135°W) and the average heat flux in this region were negative and when the average geopotential disturbance in this region was negative in March–April from 1990 to 2008. A composite was made for the time lags \( t \) from -2 to +2 days around the reference day. The color shows the heat flux and the contour lines express the geopotential disturbance. It is observed that the disturbance propagating from the upstream region is stalled and dissipated in the longitude region of 105°–135°W. This is the same result as the case in April 1999 (Fig. 7a). A similar feature is obtained for the case of positive geopotential disturbances (not shown). The barotropic geopotential structure makes a positive (negative) heat flux at the upper (lower)-height region to the east of the geopotential center. Therefore, the weak equatorward Stokes drift seen in the downstream region of the storm tracks is caused by the process in which the disturbances generated in the upstream region become barotropic and dissipate during the propagation to the downstream region. The longitude–height cross section of this process is shown schematically in Fig. 10.

5. Example 2: The meridional component of the 3D residual mean flow associated with gravity waves around the southern Andes

Gravity waves propagate momentum primarily in the vertical direction and play an important role in driving the general circulation in the middle atmosphere. Although the residual mean flow associated with gravity waves in the meridional-height section has been examined using the TEM equations, the zonal transport and longitudinally dependent meridional circulation associated with gravity waves have not been investigated. Using a gravity wave–resolving general circulation model (GCM), Sato et al. (2009b, 2012) shows that the origin of the gravity waves propagating into the mesosphere is not zonally uniform but is distributed over the mountain regions and jet–front systems in winter and the monsoon regions in summer. In this section, PRSD is applied to the gravity wave–resolving GCM data in order to examine the 3D residual mean flow associated with the strong gravity waves around the southern Andes.

a. Description of gravity wave–resolving general circulation model

Simulation data from a high-resolution global spectral climate model [Center for Climate System Research (CCSR)/National Institute for Environmental Studies]
analyzed as disturbances, and gravity waves are defined as the components having total horizontal wavenumbers larger than 21 (wavelengths of 180–1800 km) in this study. To clarify the contribution of the gravity waves to the total monthly mean transport, the monthly mean field was used for estimating the wave activity flux associated with the gravity waves, although gravity waves may interact with the mean field averaged over a shorter period. The analysis is mainly made for the month of July, when the energy of the gravity waves is large around the southern Andes. Note that the mountain waves are extracted as the transient components. This extraction method works when the flows generating the mountain waves are transient (Sato 1990). The validity is confirmed later.

b. The 3D residual mean flow around the southern Andes

As empirically shown in sections 4d and 4e, the sum of the time-mean agradient flow and the Stokes drift (the residual mean flow associated with disturbances) is roughly balanced with the wave activity flux divergence

$$\frac{2}{r_0^2 - 1} \left( \frac{\pi_y - f}{\nabla \cdot \mathbf{F}} + \mathbf{v} \mathbf{v} \right) \approx -\rho_0^{-1} (\mathbf{v} \cdot \mathbf{F}).$$  (5.1)

To examine whether the relation in (5.1) is also held for gravity waves, we made a longitude–pressure section of the meridional component for the 3D residual mean flow (Fig. 11a), that for the 3D residual mean flow associated with disturbances (Fig. 11b), $-\rho_0^{-1} (\mathbf{v} \cdot \mathbf{F})/(\pi_y - f)$ associated with all disturbances (Fig. 11c), and $-\rho_0^{-1} (\mathbf{v} \cdot \mathbf{F})/(\pi_y - f)$ associated with gravity waves (Fig. 11d). The distribution of $-\rho_0^{-1} (\mathbf{v} \cdot \mathbf{F})/(\pi_y - f)$ is in good agreement with the 3D residual mean flow associated with disturbances (Figs. 11b and 11c). From a comparison between Figs. 11a and 11b, it is found that the time-mean balanced flow, which is the difference between the 3D residual mean flow and that associated with disturbances, is dominant. This is similar to the results of section 4. The residual mean flow associated with the gravity waves is poleward at levels higher than 1 hPa, suggesting that this is likely a part of the meridional circulation from the summer hemisphere to the winter hemisphere in the mesosphere. Moreover, it is found that the residual mean flow associated with disturbances is poleward (equatorward) to the west (east) of the Andes at levels lower than 1 hPa. This structure is mainly due to gravity waves (Figs. 11b and 11d). These results suggest that the mountain waves generated over the Andes are modulated by slowly varying flows and can be extracted as the
transient components. A strongly equatorward residual mean flow is observed at 0.2–0.05 hPa, which corresponds to the disturbances, except for gravity waves (Figs. 11c and 11d).

Next, the wave activity flux divergence associated with gravity waves over the Andes is examined by dividing it into three components: $-\rho_0^{-1}(F_{11})_x$, $-\rho_0^{-1}(F_{12})_y$, and $\rho_0^{-1}(F_{13})_z$. Figure 12 shows the longitude–pressure section of the respective terms, $-(\vec{u}^2)_x$ and $\rho_0^{-1}(\rho_0\vec{u}'\vec{w}')_x$. It is clear that for $-\rho_0^{-1}(F_{13})_z$, $-\rho_0^{-1}(\rho_0\vec{u}'\vec{w}')$, is especially dominant at levels higher than 1 hPa. Moreover, for $-\rho_0^{-1}(F_{11})_x$, $-(\vec{u}^2)_x$ is especially dominant at levels lower than 1 hPa. Thus, these dominant components of the wave activity flux divergence mainly correspond to the residual mean flow associated with the gravity waves in their respective regions.

Finally, we examine the wave activity flux associated with the stationary gravity waves in order to confirm the validity of extracting the mountain waves as the transient components. Here, the disturbance components having total horizontal wavenumbers larger than 21, which include the stationary components (unlike in the previous analysis), are regarded as gravity waves. We calculate the divergence of the wave activity flux associated with the gravity waves by using the low-pass-filtered value of horizontal wavenumber 15 as the spatial mean instead of the time mean. Figure 13 shows $-\rho_0^{-1}(\vec{V} \cdot \vec{F}_1)/(\vec{u}_r - f)$ (Fig. 13a) associated with the gravity waves that include stationary components, $-\rho_0^{-1}(F_{11})_x$ (Fig. 13b), $-\rho_0^{-1}(F_{12})_y$ (Fig. 13c), and $-\rho_0^{-1}(F_{13})_z$ (Fig. 13d). It is found that $-\rho_0^{-1}(\vec{V} \cdot \vec{F}_1)/(\vec{u}_r - f)$ corresponding to the equatorward (poleward) flow exists in the western (eastern) region of the southern Andes at levels lower than 1 hPa and that this equatorward flow is mainly due to the zonal derivative component $-\rho_0^{-1}(F_{11})_x$ (Figs. 13a and 13b). There is strong wave activity flux convergence, which corresponds to poleward flow at levels higher than 1 hPa. This is mainly due to the vertical derivative component $-\rho_0^{-1}(F_{13})_z$ (Figs. 13a and 13d). These results are similar to those of the previous analysis using the time mean (Figs. 11 and 12). Thus, it is found that the mountain waves are extracted as the transient components. Moreover, it is suggested that the mountain waves provide the pseudomomentum to back-ground states through their dissipation, particularly at levels higher than 1 hPa, which corresponds to the poleward residual mean flow. It is also interesting that the poleward (equatorward) residual mean flow in the western (eastern) region of the southern Andes at levels lower than 1 hPa is due to characteristic mountain waves whose zonal flow variances are localized over the mountains because of small horizontal group velocities (Figs. 12 and 13). It should be noted that the divergence $\mathbf{F}_1$ [the right-hand side of (5.1a)] is nonzero even though no dissipation exists in the case of 3D time-mean flow, in contrast to the 2D zonal-mean case, in which the 2D-EP flux theory and noninteraction theorem are justified (Plumb 1990).

6. Summary and concluding remarks

In this study, the 3D Stokes drift has been newly formulated from its definition for the primitive equations (PRSD) and quasigeostrophic equations (QGSD). It was confirmed that PRSD is applicable to both gravity waves and Rossby waves. The phase-independent Stokes drift in the quasigeostrophic equations has also been formulated. Moreover, the 3D wave activity flux whose divergence expresses the wave forcing in the momentum equation of the time-mean flow has been derived using PRSD.

Next, PRSD, QGSD, and IGSD were compared in terms of the magnitude of their horizontal components using the ERA-Interim data. It was indicated that...
PRSD is valid in all regions except for the equatorial region.

We examined the meridional components of the time-mean agradient flow and the Stokes drift included in the 3D residual mean flow in the upper troposphere in April. The results are summarized as follows:

- The time-mean unbalanced flow and the Stokes drift had characteristic structures around the storm tracks.
- The distribution of unbalanced flow is mainly due to the zonal and meridional fluxes of the zonal momentum.
- The zonal flux of the zonal momentum corresponds to the poleward (equatorward) unbalanced flow in the upstream (downstream) region.
- The meridional flux of the zonal momentum almost corresponds to the equatorward unbalanced flow in all regions of the storm tracks.
- The Stokes drift is strongly poleward (weakly equatorward) in the upstream (downstream) region.
- From a comparison between $(\nabla \cdot F_1) / (U_y \cdot f)$ and $(1/r_0)(\rho_0(\nu^2 \Psi' / N^2))_z$ included in the Stokes drift, the vertical structure of the eddy heat flux is found to differ between the upstream and downstream regions of the storm tracks because of the corresponding

![Figure 11](image-url)
difference in the disturbance structure and the dissipating disturbance in the downstream region.

In addition, we have examined the meridional component of the 3D residual mean flow associated with gravity waves around the southern Andes using the gravity wave–resolving GCM data. The results are summarized as follows:

- This residual mean flow is almost equatorward at levels higher than 1 hPa, likely because of the mountain wave dissipation and/or breaking during upward propagation.
- At levels lower than 1 hPa, poleward (equatorward) flows are observed west (east) of the Andes, which are likely to correspond to the variances in the zonal wind components of the quasi-stationary mountain waves with small horizontal group velocities that are generated intermittently around the Andes and then propagate upward.

Finally, the formulas of Hoskins et al. (1983), Trenberth (1986), Plumb (1986), Miyahara (2006), Kinoshita et al. (2010), Noda (2010), and this study are summarized and compared in Table 1.

PRSD has been derived for small-amplitude perturbations embedded in a slowly varying time-mean flow. To verify the validity of this assumption, the difference is evaluated between $3D-flux-M = \frac{\partial F}{\partial x}$ and that from which the time-mean wind shear terms has been eliminated $\frac{\partial F}{\partial x}$. Figure 14 shows longitude–pressure sections of the wave activity flux divergence for $2r^2_0(F_1)$ (Fig. 14a), $2r^2_1(F_0)$ (Fig. 14b), the absolute value of the difference $|F_1 - F_0|$ (Fig. 14c), $F_1$ (Fig. 14d), $F_0$ (Fig. 14e), and the absolute value of the difference $|F_1 - F_0|$ normalized by $F_1$ (Fig. 14f). Note that the difference in Fig. 14c is shown only in the region where the absolute value of $-r_0^2(V \cdot F_1)/(\overline{\tau}_v - f)$ is larger than 0.2 m s$^{-1}$ and that the difference in Fig. 14f is shown only in the
The region where the absolute value of $F_1$ is larger than 10 kg m$^{-2}$ s$^{-2}$. The difference in the wave activity flux divergence, which corresponds to the residual meridional flow, is found to be less than 30% in most of the regions (Figs. 14a,b,c). The difference in the wave activity flux itself is also found to be small (Figs. 14d,e,f). Thus, for most of the region shown in Fig. 14, the effects of the terms having time-mean wind shear are small, and overall, the small-amplitude assumption is valid.

In our formulation, there is a remaining issue. The term including $f$ in the denominator in the Stokes drift would diverge around the equator, as was the case in Miyahara (2006) and Kinoshita et al. (2010). In the definition of the Stokes drift, assuming a small-amplitude plane wave and $f \to 0$ yields $\vec{u}\cdot\vec{\eta} = (1/2)[(k^2 + f^2)\eta f/(\omega^2 - f^2)]\nabla^2 \to 0$. Thus, another term that includes $f$, which goes to zero around the equatorial region, will be needed. The phase-independent Stokes drift in the primitive equation should be formulated to examine the residual mean flow associated with all perturbations that include stationary components. Nevertheless, the 3D residual mean flow derived in this study is applicable to both Rossby waves and inertia–gravity waves and can be used for all regions except for the equatorial region. These formulas will help to improve
TABLE 1. The 3D residual mean flows and wave activity fluxes (corresponding to $F_z$) derived by Hoskins et al. (1983), Trenberth (1986), Plumb (1986), Miyahara (2006), Kinoshita et al. (2010), Noda (2010), and the present study and Kinoshita and Sato (2013).

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<tr>
<td>Residual Mean Flow</td>
<td>$\pi^* = \pi + \frac{\langle \overline{\nu^2} \rangle}{f} - \frac{1}{\rho_0} \left( \frac{\nu' \Phi'}{N^2} \right)_{z}$</td>
<td>$\pi^* = \pi + \frac{\langle \overline{\nu^2} + \nu'^2 \rangle}{2f} - \frac{1}{\rho_0} \left( \frac{\nu' \Phi'}{N^2} \right)_{z}$</td>
<td>$\pi^* = \pi + \frac{\langle z \rangle}{f} - \frac{1}{\rho_0} \left( \frac{\nu' \Phi'}{N^2} \right)_{z}$</td>
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<tr>
<td>Wave Activity Flux</td>
<td>$F_{11} = \rho_0 (\langle \overline{\nu^2} \rangle - \langle \overline{u^2} \rangle)$</td>
<td>$F_{11} = \frac{\rho_0}{2} (\langle \overline{\nu^2} \rangle - \langle \overline{u^2} \rangle)$</td>
<td>$F_{11} = \rho_0 \left( \langle \overline{\nu^2} \rangle - \langle \overline{u^2} \rangle \right)$</td>
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<tr>
<td>$\mathbf{F} \times \zeta \mathbf{A}$?</td>
<td>No</td>
<td>No</td>
<td>No</td>
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<tr>
<td>$\mathbf{F} \times \zeta \mathbf{A}$?</td>
<td>Yes (for constant Coriolis parameter and gravity waves)</td>
<td>Yes (for gravity waves)</td>
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<tr>
<td>Residual Mean Flow</td>
<td>$\tau^* = \tau + \left( \frac{S(N)}{f} \right)_y - \frac{1}{\rho_0} \left( \frac{\bar{u}' \Phi'_N}{N^2} + \bar{v}' \Phi'_N + \bar{w}' \Phi'_N \right)_z$</td>
<td>$\tau^* = \tau + \left( \frac{S(p)}{f} \right)_y - \frac{1}{\rho_0} \left( \frac{\bar{u}' \Phi'_N}{N^2} \right)_z$</td>
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<td>$\bar{v}' = \bar{v} + \left( \frac{S(N)}{f} \right)_x - \frac{1}{\rho_0} \left[ \bar{v}' \Phi'_N + \bar{w}' \Phi'_N \right)_x$</td>
<td>$\bar{v}' = \bar{v} + \left( \frac{S(p)}{f} \right)_x - \frac{1}{\rho_0} \left( \frac{\bar{u}' \Phi'_N}{N^2} \right)_x$</td>
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<tr>
<td>$\bar{w}' = \bar{w} + \frac{\bar{u}' \Phi'_N}{N^2} + \left( \frac{\bar{u}' \Phi'_N}{N^2} \right)_x$</td>
<td>$\bar{w}' = \bar{w} + \frac{\bar{u}' \Phi'_N}{N^2} + \left( \frac{\bar{u}' \Phi'_N}{N^2} \right)_x$</td>
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<tr>
<td>$\bar{S}(N) = \frac{\bar{u}'^2 + \bar{v}'^2 - \bar{u} \bar{v} - (\bar{u} \bar{w} \Phi_N + \bar{u}' \bar{w}' \Phi_N + \bar{w}'^2 \Phi_N^2)N^2}{(\bar{v}_x - \bar{u}_x + 2f)N^2 - \bar{v}_y \Phi_N + \bar{u}_y \Phi_N}$</td>
<td>$\bar{S}(p) = \bar{u} \bar{v} + \left( \frac{\bar{u}' \Phi'_N}{N^2} \right)_x$</td>
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<td>$\bar{v}'(QG) = \bar{v} + \frac{\bar{u}'^2 + \bar{v}'^2}{f} + \frac{1}{\rho_0} \frac{\bar{u}' \Phi'_N}{N^2}$</td>
<td>$\bar{v}'(QG) = \bar{v} + \frac{\bar{u}'^2 + \bar{v}'^2}{f} + \frac{1}{\rho_0} \frac{\bar{u}' \Phi'_N}{N^2}$</td>
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<tr>
<td>$\bar{w}'(QG) = \bar{v} + \frac{\bar{u}'^2 + \bar{v}'^2}{f} + \frac{1}{\rho_0} \frac{\bar{u}' \Phi'_N}{N^2}$</td>
<td>$\bar{w}'(QG) = \bar{w} + \frac{\bar{u}' \Phi'_N}{N^2} + \frac{\bar{u}' \Phi'_N}{N^2}$</td>
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<tr>
<td>$\pi^* = \bar{u} + \bar{v}'^2$</td>
<td>Yes (for monochromatic waves)</td>
<td>Yes</td>
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<tr>
<td>Wave Activity Flux</td>
<td>$F_{11} = \rho_0 \left[ \bar{u}'^2 - \bar{S}(N) + \bar{u}' \left( \frac{\bar{u}' \Phi'_N}{N^2} - \bar{u}' \Phi'_N + \bar{S}(N) \right) \right]$</td>
<td>3D-flux-M</td>
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<tr>
<td>$F_{12} = \rho_0 \left[ \bar{u}' \bar{v}' - \bar{u}' \left( \frac{\bar{v}' \Phi'_N}{N^2} - \bar{u}' \Phi'_N + \bar{S}(N) \right) \right]$</td>
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<tr>
<td>$F_{13} = \rho_0 \left[ \bar{u}' \bar{w}' - \bar{u}' \left( \frac{\bar{w}' \Phi'_N}{N^2} - \bar{u}' \Phi'_N + \bar{S}(N) \right) + \bar{u}_x - f \right] \left( \frac{\bar{v}' \Phi'_N}{N^2} + \bar{v}_x \Phi'_N \right)$</td>
<td>3D-flux-W</td>
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<tr>
<td>$F_{W11} = \rho_0 \left( \bar{u}'^2 - \bar{S} \right)$</td>
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<tr>
<td>$F_{W12} = \rho_0 \left( \bar{u}' \bar{v}' \right)$</td>
<td>$F_{W13} = \rho_0 \left( \bar{u}' \bar{w}' - f \frac{\bar{v}' \Phi'_N}{N^2} \right)$</td>
<td>Yes ($F_{W1}$)</td>
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<tr>
<td>$F \propto \hat{C}_x A^2$</td>
<td>Yes (for monochromatic waves)</td>
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our knowledge of the 3D structure of mass transport in the atmosphere.

Acknowledgments. We thank Yoshihiro Tomikawa and Kazue Suzuki for helping us with the treatment of ERA-Interim data. We also thank Shingo Watanabe for his providing a high-resolution global spectral climate model data. Thanks are due to Toshiyuki Hibiya, Makoto Koike, Masahiro Takagi, and Koutarou Takaya for their fruitful discussions. Deep appreciation goes to Rolando R. Garcia, Saburo Miyahara, and an anonymous reviewer for providing constructive comments. We thank the proofreading/editing assistance from the GCOE program for polishing our document. The GFD-DENNOU library was used for drawing figures. This study is supported by Grand-Aid for Research Fellow (22-7125) of the JSPS and by Grant-in-Aid for Scientific Research (B) 22340134 of the Ministry of Education, Culture, Sports and Technology, Japan.

APPENDIX

Formulas in the Spherical Coordinates

The 3D residual mean flow on the sphere is used and is written as

\[ \vec{u}^* = \vec{u} + \frac{1}{a} \frac{\partial}{\partial \phi} \left( \frac{\nabla (\rho \Phi)}{f} \right) - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0 u' \Phi'_z}{N^2} \right), \]  
(A.1a)

\[ \vec{v}^* = \vec{v} - \frac{1}{a \cos \phi} \frac{\partial}{\partial \lambda} \left( \frac{\nabla (\rho \Phi)}{f} \right) - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0 u' \Phi'_z}{N^2} \right), \]  
(A.1b)

\[ \vec{w}^* = \vec{w} + \frac{1}{a \cos \phi} \frac{\partial}{\partial \lambda} \left( \frac{u' \Phi'_z}{N^2} \right) + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \frac{u' \Phi'_z \cos \phi}{N^2} \right), \]  
(A.1c)
where \( \lambda \) and \( \phi \) are longitude and latitude, respectively, and \( a \) is Earth’s radius. Similarly, IGSD and QGSD on the sphere are expressed as follows:

\[
\overline{\nu}_S^{(IG)} = \frac{1}{a} \frac{\partial}{\partial \phi} \left( \frac{\overline{S}}{f} \right) - \frac{1}{a} \frac{\partial}{\partial \zeta} \left( \frac{\rho_0 \overline{u} \overline{\phi}_z}{N^2} \right), \tag{A.2a}
\]

\[
\overline{\nu}_S^{(IG)} = -\frac{1}{a \cos \phi \partial \lambda} \left( \frac{\overline{S}}{f} \right) - \frac{1}{a} \frac{\partial}{\partial \zeta} \left( \frac{\rho_0 \overline{u} \overline{\phi}_z}{N^2} \right), \tag{A.2b}
\]

\[
\overline{\nu}_S^{(QG)} = -\frac{1}{a \cos \phi \partial \lambda} \left( \frac{\overline{u} \overline{\phi}_z^*}{N^2} \right) + \frac{1}{a \cos \phi \partial \phi} \left( \frac{\rho_0 \overline{u} \overline{\phi}_z^* \cos \phi}{N^2} \right), \tag{A.2c}
\]

\[
\overline{\nu}_{(UB)} = \frac{\left( \overline{u} \cos \phi \right)_{\phi} - f}{a \cos \phi} \left[ \frac{\overline{S}}{f} \right] = -\frac{\left( \overline{V} \cdot \mathbf{F}_1 \right)}{a \cos \phi} \overline{\nu}_S^{(UB)}, \tag{A.4a}
\]

\[
F_{11} = \left[ \overline{u} \overline{\phi}_z - \frac{\pi}{a \cos \phi} \left( \frac{\overline{S}}{f} \right) \right] - \frac{\overline{u} \overline{\phi}_z^*}{N^2} \rho_0 a \cos \phi, \tag{A.4b}
\]

\[
F_{12} = \left[ \overline{u} \overline{\phi}_z - \frac{\pi}{a \cos \phi} \left( \frac{\overline{S}}{f} \right) \right] - \frac{\overline{u} \overline{\phi}_z^*}{N^2} \rho_0 a \cos \phi, \tag{A.4c}
\]

\[
F_{13} = \left[ \overline{u} \overline{\phi}_z - \frac{\pi}{a \cos \phi} \left( \frac{\overline{S}}{f} \right) \right] - \frac{\overline{u} \overline{\phi}_z^*}{N^2} \rho_0 a \cos \phi, \quad \text{and} \tag{A.4d}
\]

\[
\mathbf{V} = \frac{1}{a \cos \phi \partial \lambda} \frac{1}{a \cos \phi \partial \phi} \frac{\partial}{\partial \zeta} \cos \phi \frac{\partial}{\partial \zeta}, \tag{A.4e}
\]

The equation to the meridional component of the balanced flow [see (4.1)] on the sphere is written as

\[
\overline{\nu}_S^{(QG)} = \frac{1}{a \cos \phi \partial \lambda} \left( \frac{\overline{u} \overline{\phi}_z^*}{N^2} \right) + \frac{1}{a \cos \phi \partial \phi} \left( \frac{\rho_0 \overline{u} \overline{\phi}_z^* \cos \phi}{N^2} \right), \tag{A.2f}
\]

The relationship between the residual mean flow associated with disturbances and the 3D wave activity flux divergence [see (5.1)] on the sphere is expressed as follows:

\[
\sum_{\text{disturbances}} \left[ \frac{\left( \frac{\pi}{a \cos \phi} \right)_{\phi}}{f} - \left( \frac{\overline{V} \cdot \mathbf{F}_1}{a \cos \phi} \right) \right] = -\frac{\mathbf{V} \cdot \mathbf{F}_1}{a \cos \phi}, \tag{A.4a}
\]

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