Gravity Wave Diagnostics and Characteristics in Mesoscale Fields

Christopher G. Kruse and Ronald B. Smith

Yale University, New Haven, Connecticut

(Manuscript received 11 March 2015, in final form 18 June 2015)

ABSTRACT

As numerical models of complex atmospheric flows increase their quality and resolution, it becomes valuable to isolate and quantify the embedded resolved gravity waves. The authors propose a spatial filtering method combined with a selection of quadratic diagnostic quantities such as heat, momentum, and energy fluxes to do this. These covariant quantities were found to be insensitive to filter cutoff length scales between 300 and 700 km, suggesting the existence of a "cospectral gap." The gravity waves identified with the proposed method display known properties from idealized studies, including vertical propagation, upwind propagation, the relationship between momentum and energy flux, and agreement with fluxes derived from an alternative method involving simulations with and without terrain. The proposed method is applied to 2- and 6-km-resolution realistic WRF simulations of orographic and nonorographic gravity waves over and around New Zealand within complex frontal cyclones. Deep mountain wave, shallow mountain wave, jet-generated gravity wave, and convection-generated gravity wave events were chosen for analysis. The four wave events shared the characteristics of positive vertical energy flux, negative zonal momentum flux, and upwind horizontal energy flux. Two of the gravity wave events were dissipated nonlinearly.

1. Introduction

Gravity waves are atmospheric buoyancy oscillations that transport energy and horizontal momentum vertically throughout the atmosphere (McLandress 1998). The vertical propagation and dissipation of gravity waves are important as the carried energy and momentum are deposited wherever these waves break, affecting the mean flow. Gravity waves and their dissipation have long been recognized to be important in middle atmosphere dynamics (Fritts 1989). Important gravity wave sources are mountains, convection, unbalanced flow, and wind shear (Fritts and Alexander 2003).

The increasing computational resources available to atmospheric scientists have recently made it practical to resolve a majority of the gravity wave spectrum in high-resolution simulations. Realistic, fully nonlinear, and nonhydrostatic simulations down to 1-km resolution are now relatively common. This recent capability presents a unique opportunity to study gravity waves from a variety of sources embedded within these more physically realistic simulations.

There have been many numerical studies of mountain- and convection-generated gravity waves. Much of this work has been idealized, where terrain and/or background states were specified and rotation neglected (e.g., Klemp and Lilly 1978; Fovell et al. 1992). In these types of simulations, gravity wave perturbations may be easily defined relative to the specified background state, which then may be used to quantitatively study their dynamics. Jet- or imbalance-generated gravity waves have also been studied within idealized numerical simulations (Plougonven and Snyder 2007; Lin and Zhang 2008), which develop in synoptic-scale baroclinic systems. In these domains, synoptic-scale quasigeostrophic variations in fields (e.g., pressure) may obscure gravity wave perturbations. Realistic simulations have also been used to study real mountain wave events (e.g., Doyle et al. 2005; Jiang et al. 2013), which may also have synoptic-scale variations. A major objective here is to construct a method to isolate gravity wave perturbations from these synoptic-scale variations in realistic mesoscale fields.

The simplest way to define gravity wave perturbation fields in realistic output is to subtract the mean (e.g., Doyle et al. 2005). This method may be appropriate for
small domains, but gravity wave perturbations become increasingly dominated by synoptic-scale variations as domain size increases. A more sophisticated approach is to apply a box moving-average filter to the original field (e.g., Jiang et al. 2013). This effectively low passes the original field, allowing perturbations to be defined relative to the moving-averaged background field. The effective cutoff length scale of the box moving-average method is not isotropic, however. Recently, investigators have employed 2D spectral filtering to isolate gravity wave perturbations in mesoscale (Lin and Zhang 2008) and global (Wu and Eckermann 2008, their Fig. 10) domains. This method allows the spectral response and cutoff length scale of the filter to be explicitly specified. Yet another method was used by Shutts and Vosper (2011), where low- and high-resolution domains were subtracted to isolate wave perturbations. This method assumes the coarse domain does not resolve the gravity waves resolved in the fine-resolution domain and that the two solutions have not significantly diverged. The variety of methods makes meaningful quantitative comparisons of fundamental gravity wave quantities impossible.

Here we propose a method to quantify standard gravity wave quantities that does not depend on model, configuration, resolution, domain location, gravity wave orientation, or gravity wave source. The method is similar to that used by Lin and Zhang (2008). It uses isotropic spectral high-pass filtering to isolate gravity wave perturbations. The main difference is that a smooth-edged response function in wavenumber space is used to prevent Gibbs phenomenon oscillations. This study goes beyond previous work by identifying many quadratic gravity wave diagnostic quantities, allowing many characteristics of gravity waves to be diagnosed and robustly quantified.

The diagnostic method is tested in deep (80 Pa, ~45-km top) realistic Weather Research and Forecast (WRF) Model simulations forced by ECMWF grids. High-resolution simulations nested to 2-km resolution were completed for a deep mountain wave event, a shallow mountain wave event, and a convection-generated gravity wave event. Additionally, a 6-km continuous simulation was completed for the 2014 austral winter (24 May–31 July) over the New Zealand region, and a jet-generated gravity wave event was also chosen within this simulation for further analysis. Gravity wave characteristics in these four events are compared and summarized. Further details on the WRF Model setup are provided in appendix A.

The proposed method and quadratic diagnostic quantities are discussed in section 2. A discussion of the physical justification for such a method is provided in section 3. In section 4, the method is tested in two mountain wave events. Further analysis of mountain waves in the 2- and 6-km-resolution simulations is provided in section 5, while jet-generated and convection-generated orographic gravity wave events are presented in section 6. Conclusions are provided in section 7.

2. Diagnostic method description

The proposed four-step method transforms a complex 2D field of dynamic variables into a simpler field of wave diagnostic quantities. It combines several standard signal processing algorithms. The starting point is a set of gridded 2D velocity, pressure, and temperature fields from a numerical simulation or interpolated observational dataset. Ideally, these fields are on a level surface and a uniform square grid as the method assumes constant horizontal resolution. The first two steps (deplaning and high-pass spatial filtering) split the field into a smoothly varying background state and small-scale perturbations. The third step takes pointwise products of the perturbation quantities to form quadratic diagnostic quantities (e.g., momentum flux). The last step involves low-pass spatial filtering to smooth the field and reduce noise, effectively “regionalizing” the diagnostics. These steps are described in the following subsections.

Note that limited area models often make use of map projections leading to nonuniform grids on the earth. In the presented WRF simulations, the grid resolution in earth distance varied by less than 3% in all domains and the grids were taken to be uniform. However, grid resolution may vary significantly across a domain depending on size and projection. Regridding may be necessary over a smaller area of interest to ensure the fields are on a uniform grid or nearly so. This method would have to be revised if working with very large or global domains.

a. Deplaning fields

The original 2D field, \( a(x, y) \), is deplaned by subtracting a least squares best-fit plane, \( \bar{a}(x, y) \), producing a full perturbation field, \( \hat{a} = a - \bar{a} \). This step is analogous to detrending conventionally applied to 1D data series before performing Fourier analysis.

Deplaning is performed because the following step involves the discrete Fourier transform (DFT) and high-pass filtering. Many fields (e.g., pressure in Fig. 1a) have synoptic-scale variations and are aperiodic. Performing the DFT on an aperiodic field introduces high-wavenumber spectral artifacts (Denis et al. 2002). These high-wavenumber artifacts are retained in high-pass filtering, which manifest along domain edges and are proportional to the original aperiodicity. Deplaning...
reduces aperiodicity and edge artifact amplitude. An alternative method for addressing aperiodicity was proposed by Errico (1985) using edge values only, but this method may generate high-wavenumber artifacts in the interior of the field that can be confused with gravity waves [e.g., compare Figs. 1 and 3 of Denis et al. (2002)].

While edge artifact amplitude is reduced via deplaning, edge artifacts are not eliminated (e.g., Fig. 1d). Edge artifacts decay away from the boundaries with a decay length scale proportional to the cutoff length scale, $L$.

By $0.55L$ from the boundaries, edge artifacts are negligible (see appendix C). This decay length scale does not depend on deplaning. If the outer portions of the domain are not of interest, deplaning is optional.

b. High-pass filtering

The deplaned field still contains a broad spectrum of variations, with synoptic-scale variations often dominating. These variations are unrelated to gravity waves. Two dimensional spectral high-pass filtering is employed to remove synoptic-scale variations and isolate the gravity wave–scale perturbations (see appendix B).

---

**Fig. 1.** Examples of (a) full pressure ($p$), (b) deplaned pressure ($\tilde{p}$), (c) low-passed deplaned pressure ($\tilde{p}_{lp}$), and (d) high-passed deplaned pressure or perturbation pressure ($\tilde{p}_{hp} = p'$) from a realistic 2-km WRF simulation. A cutoff length scale of $L = 400$ km was used. These analyses are valid at 1800 UTC 24 Jun 2014 at 4-km MSL.
The deplaning and filtering steps partition the original 2D field into three parts: the planar part $\bar{a}$, the low-passed perturbation part $\bar{a}_l$, and the high-passed perturbation part $\bar{a}_h$:

$$a = (\bar{a} + \bar{a}_l) + \bar{a}_h = \bar{a} + a'. \quad (1)$$

The sum of the planar part and low-pass part make up the smoothly varying background field (i.e., $\bar{a} = \bar{a} + \bar{a}_l$), and the high-pass part is the perturbation part (i.e., $a' = \bar{a}_h$). Overbars indicate background quantities as defined in Eq. (1) unless otherwise stated.

Figure 1 gives an example of partitioning a realistic 2D pressure field from a 2-km-resolution WRF simulation. The original pressure field, $p$, at 4-km MSL is shown in Fig. 1a. Deplaning this 2D field produces the full perturbation field, $\bar{p}$, shown in Fig. 1b. The low-passed $\bar{a}_l$ and high-passed $\bar{a}_h = p'$ perturbation parts with $L = 400$ km are shown in Figs. 1c and 1d, respectively. Orographic pressure perturbations are obvious in the $p'$ field. Edge artifacts also apparent but do not extend significantly into the domain as previously discussed.

c. Quadratic quantities

The perturbation quantities from the high-pass filter may be analyzed directly or used to compute quadratic diagnostic quantities. Table 1 contains many variance (e.g., $w^2$) and covariance (e.g., $w'p'$) quadratic diagnostic quantities. Variance quantities are useful for quantifying the intensity of disturbances but not for identifying disturbance type. Covariances, on the other hand, contain phase information that can be used to distinguish between disturbance type. These quantities may be plotted without manipulation, but it is often useful to smooth or average them over some domain, consistent with the area averaging and integration often performed in theoretical studies.

The simplest diagnostic for gravity waves is probably the perturbation vertical velocity, $w'$. If a smooth positive-definite field showing regions of active $w'$ is desired, one should form its variance $\sigma_w^2 = \langle w'^2 \rangle$, where the angle brackets indicate spatial smoothing of the contained variable in this paper but could also indicate area averaging/integration. Vertical velocity from large-scale quasigeostrophic flow is typically quite small, so patches of $\sigma_w^2$ will identify regions of waves and/or convection.

Another simple diagnostic is the perturbation temperature, $T'$. For gravity waves, patches of large $\sigma_T^2$, signify regions of vertical parcel displacements, $\eta$. In an environment with a well-defined lapse rate, temperature anomalies can be approximated by $T' = (\gamma - \gamma_0)\eta$, where $\gamma$ is the background lapse rate and $\gamma_0$ is the dry adiabatic lapse rate. Gravity wave potential energy, $PE = \bar{\rho}N^2\eta^2/2$, may be defined in terms of the temperature perturbation and stability as

$$PE = \bar{\rho} \left( \frac{\bar{g}}{\bar{N}} \right)^2 \frac{T'}{T}^2. \quad (2)$$

This quantity can also be evaluated regionally by replacing $T'^2$ by $\sigma_T^2$. Gravity wave temperature perturbations have been observed remotely from space (Eckermann and Preusse 1999; Alexander et al. 2009) and from ground-based lidar (Duck et al. 2001) and used to estimate potential energy and momentum flux.

The gravity wave kinetic energy is the pointwise or smoothed quadratic quantity

$$KE = \frac{\bar{\rho}}{2} (u'^2 + v'^2 + w'^2), \quad (3)$$

which for hydrostatic waves can be approximated by $KE \approx \bar{\rho}/2(u'^2 + v'^2)$. In simple vertically propagating waves in a nonrotating flow ($f = 0$), the regional potential and kinetic energy may exhibit equipartition ($PE = KE$). The total wave energy density is $ED = PE + KE$.

Momentum and energy fluxes are commonly used to quantify gravity waves. We define the two components of the vertical flux of horizontal momentum

$$MF_x = \bar{\rho}u'w', \quad MF_y = \bar{\rho}v'w'. \quad (4)$$

The magnitude of the horizontal momentum flux vector, $|\text{MF}| = \sqrt{MF_x^2 + MF_y^2}$, has been used by Geller et al. (2013).
The three components of the energy flux are

\[
\begin{align*}
EF_x &= p'u' \\
EF_y &= p'u' \\
EF_z &= p'w' .
\end{align*}
\] (5)

Energy flux captures a fundamental property of gravity waves—that they move their energy through a fluid via pressure–velocity correlations. In unsheared flow, the group velocity \( c_g = \frac{EF}{ED} \) describes the movement of the wave packet relative to the fluid.

The vertical heat flux, \( HF_z = \bar{p}c_z w'/T' \), is another useful diagnostic. It can identify regions of thermal convection \( (HF_z > 0) \) or regions where shear-induced turbulence in a stably stratified fluid is mixing heat downward \( (HF_z < 0) \). Small-amplitude gravity waves transport no heat \( (HF_z = 0) \).

Other potentially useful quadratic quantities such as pseudomomentum are not considered in this paper [see Durran (1995)].

Not all useful diagnostics are simple products of perturbation quantities. Horizontal divergence computed from the full velocity field is usually an indicator of gravity waves as quasigeostrophic flows are nearly divergence free. Wave dissipation can also generate mesoscale (i.e., \( f = 0 \)) Boussinesq potential vorticity \( [PV = \zeta \cdot \nabla p/\rho_0] \) where none was present before (Smith 1989), making it a useful diagnostic. In numerical models with a prognostic turbulent kinetic energy (TKE) variable, the positive-definite TKE is a useful indicator of subgrid turbulence caused by strong shears or unstable lapse rates, indicating regions of wave dissipation. Eddy diffusivities for heat or momentum may also be useful in this regard. In this paper, we focus on the quadratic diagnostics listed in Table 1.

d. Low-pass filtering

When computed pointwise, the quadratic quantities in Table 1 may be noisy. It may even be difficult to determine the dominant sign of the quantity. Low-pass filtering or areal averaging of diagnostic quantities is usually helpful in this regard. Low-pass filtering objectively simplifies these fields and allows the dominant sign of the diagnostic quantity to be determined. In this paper, low-pass filtering of quadratic quantities uses the same Gaussian spectral filter described in appendix B.

The \( EF_z \) field is an example of such a complex field. Pointwise, this diagnostic field is both positive and negative (Fig. 2a). Low-pass filtering (Fig. 2b) reveals that values of positive energy fluxes are much larger than adjacent negative fluxes, giving a net positive regional value. For mountain waves, if the smoothing is not excessive, patches of large energy flux can still be identified corresponding to regions of high terrain. For this reason, all low passing of quadratic diagnostic quantities was performed with a shorter cutoff length scale \( L = 150 \) km.

The need for smoothing quadratic quantities is not limited to cases with complex terrain or multiple wave
sources. It is needed even in the simplest case. An example is the classic solution for stratified flow over a single smooth hill (Queney 1948) (see appendix D).

3. Physical justification for spatial filtering and quadratic diagnostics

Here we address the question of whether the identification of gravity waves with spatial filtering and quadratic diagnostic quantities has a physical basis.

One justification for a spatial filter would be the existence of a spectral gap between gravity wave and synoptic-scale motions. An early search for a spectral gap in horizontal wind speed (VanderHoven 1957) found a wide gap in the frequency domain between the “weather cycle” with a period of four days and the “turbulent eddy cycle” in the boundary layer with a period of 1 min. This gap is in the frequency domain, however, and provides no support for the spatial filtering proposed here. Galmarini and Thunis (1999) estimated the error in Reynolds averaging associated with using a running mean to separate mean and perturbation quantities. They concluded, however, that “apart from vertical motion, no distinct scale separation appears in the atmospheric frequency distribution of energy” and that such a gap does not exist. The difficulty in isolating gravity waves with simple filtering is characterized by Bretherton (1969): “The decomposition of velocity and temperature structure ... can only be achieved by a very sophisticated analysis (if indeed a truly objective distinction is, in principle, possible).”

A careful attempt to separate gravity waves from quasi-2D turbulent motions in aircraft data was carried out by Cho et al. (1999). Their method used multiple variables such as the full wind vector and temperature. They were handicapped by not having perturbation pressure as we have now (e.g., Smith et al. 2008).

A systematic approach to gravity wave identification in numerical model output is described by Lane and Zhang (2011). They examine the joint frequency and wave-number spectra of disturbances and identified those that satisfied the dispersion relation for gravity waves. To apply such a method to horizontal wavenumber spectra alone requires extra assumptions, as shown below.

Internal gravity waves in the earth’s atmosphere are confined in frequency between the Brunt–Väisälä frequency and the Coriolis parameter:

$$N > |\omega_i| > |f|.$$  \hspace{1cm} (6)

This criterion must be applied to the intrinsic frequency: the frequency seen by an air parcel as it passes through the wave. In one dimension, the intrinsic frequency may be expressed as

$$\omega_i = \omega - \Omega k.$$  \hspace{1cm} (7)

Intrinsic frequencies higher than $N$ cannot be caused by buoyancy forces and frequencies less than $|f|$ are

![Figure 3](image-url)
prevented by the Coriolis force. If perturbations are wavelike (i.e., nearly periodic), the intrinsic frequency can be written

$$\omega_I = (c_p - \pi)k,$$  \hfill (8)

where $c_p$ is the wave phase speed. Thus, if $(c_p - \pi)$ can be estimated or bounded, the frequency criterion [Eq. (6)] becomes an approximate scale criterion.

As an example, let $(\pi - c_p) = 20 \text{ m s}^{-1}$ so that the Coriolis cutoff in the midlatitudes ($f \approx 10^{-4} \text{ s}^{-1}$) gives a threshold wavenumber of $|k| = |f|/(c_p - \pi) = 5 \times 10^{-6} \text{ m}^{-1}$ and a wavelength threshold of 1257 km. With cutoff length scale of $L = 1257 \text{ km}$, high-pass filtering retains only waves with $|\omega_I| > |f|$.

At the other end of the gravity wave spectrum, there is the buoyancy cutoff at $|\omega_I| = N$. If a mesoscale model is run with 2-km resolution, the smallest resolvable wavelength would be about 12 km. With $(U - c_p) = 20 \text{ m s}^{-1}$, the intrinsic frequency is $\omega_I = 0.01 \text{ s}^{-1}$, which matches a typical value for $N$ in the troposphere. Thus, any
well-resolved wave will satisfy $|\omega| = N$. No imposed buoyancy cutoff is necessary.

This discussion makes it clear that high-pass spatial filtering is at best a crude approximation to a Coriolis cutoff and that gravity waves cannot generally be distinguished spectrally from larger-scale phenomena.

Covariance quadratic quantities are key in this method, as they allow the phase between velocity, temperature, and pressure to be exploited. While there might not be a spectral gap for an individual variable, there may be a cospectral gap in covariance quantities between synoptic and mesoscale motions (see section 4a). With a cospectral gap, filtering and covariance quadratic quantities allow mesoscale quantities to be distinguished from synoptic-scale quantities. Further, gravity wave and convective mesoscale motions can be distinguished via covariance quadratic quantities like $\langle HF_z \rangle$ and $\langle EF_z \rangle$. Gravity waves do not transport heat vertically, while convection does; conversely, gravity waves have horizontal energy flux but nonsheared convection does not. These ideas are tested in the following section.

4. Method verification

In this section, the filter-based identification of gravity waves is tested a number of ways. First, gravity wave energy and momentum fluxes are shown to be insensitive to the cutoff length scale over a considerable range. Second, the relationship between wave energy and momentum fluxes is shown to be consistent with
linear theory. Third, we exhibit the robust property of mountain waves to propagate upwind. Finally, fluxes derived from the proposed filtering method are compared with an alternative method.

These tests are applied to 2-km WRF simulations of two mountain wave events on 14 and 24 June 2014. The winds at 4-km MSL, just above the highest terrain of New Zealand, are shown for these events in Fig. 3. The 14 June event (Fig. 3a) was a relatively weak event, with strongest winds near 20 m s\(^{-1}\) traversing the southern South Island. The 24 June event (Fig. 3b) was stronger, with \(\approx 30\) m s\(^{-1}\) winds more broadly affecting the entire South Island.

**a. Sensitivity to cutoff length scale**

The cutoff length scale \(L\) is the key parameter in isolating mesoscale perturbations. Ideally, covariance or flux quantities will be insensitive to \(L\) in some range between mesoscales and synoptic scales. To check for this sensitivity, the South Island area-integrated \(\text{EF}_z\) and \(\text{MF}_x\) were computed varying \(L\) from 10 to 2000 km on 2D fields at a single time and height (12 km) during the 14 June event. This analysis was repeated every 3 h producing the curves shown in Fig. 4. As \(L\) increases from 10 to 300 km, fluxes increase as more of the gravity wave spectrum is included. For wavelengths from 300 to 900 km, the fluxes plateau; covariances of \(p'\), \(w'\), \(u'\), \(w'\) are small and \(\text{EF}_z\) and \(\text{MF}_x\) are insensitive to \(L\).

Within this cospectral gap region, standard deviations of \(\text{EF}_z\) and \(\text{MF}_x\), normalized by the mean were less than 10% in the 14 and 24 June events. Beyond 900 km, many of the flux curves diverge; presumably some aspect of the frontal cyclone is producing correlated perturbation fields over the South Island.

The same analysis was performed for \(\text{EF}_x\) (Fig. 5). The spectral gap for this variable is less evident and more narrow than \(\text{EF}_z\) and \(\text{MF}_x\). This may be due to the fact that \(\text{EF}_z\) and \(\text{MF}_x\) contain \(w'\), which does not have a “red” spectrum (i.e., spectral power increasing with increasing wavelength), whereas \(\text{EF}_x\) is made up of \(p'\) and \(u'\), which both typically have a red spectrum. Based on Figs. 4 and 5, a single cutoff length scale of \(L = 400\) km was used when high-pass filtering to isolate gravity wave–scale perturbations.

While there are clear cospectral gaps for \(\text{EF}_z\) and \(\text{MF}_x\) in the New Zealand region in the presented events, it is unclear at present how general this result is. Further analysis is needed.

**b. Relationship between energy and momentum fluxes**

In steady, nonrotating, linearized flow within a horizontally uniform environment, the horizontally averaged gravity wave energy and momentum fluxes are related by

\[
\text{EF}_x(z) = -\mathbf{u}(z) \cdot \text{MF},
\]

**FIG. 8.** Low-passed (\(L = 150\) km) energy flux vector components (W m\(^{-2}\)) and high-passed (\(L = 400\) km) temperature perturbations (colored shading, K) at 12 km MSL valid at 1800 UTC (a) 14 and (b) 24 Jun 2014. The horizontal energy flux field is visualized with vectors. Vertical energy flux is contoured in black at (a) 1, 3, and 5 W m\(^{-2}\) and (b) 5, 10, and 15 W m\(^{-2}\). Background-state pressure is contoured every 2 hPa in blue, representing approximate background velocity streamlines.
where overbars here indicate spatial averaging over all wave perturbations, \( \mathbf{u} = \mathbf{u}_i + v_j \) is the horizontal wind vector, and \( \mathbf{MF} = \mathbf{MF}_x i + \mathbf{MF}_y j \) is the vertical momentum flux vector (Eliassen and Palm 1960). Equation (9), hereafter referred to as the E–P relation, is also valid pointwise,

\[
\mathbf{EF}_z = -\mathbf{u} \cdot \mathbf{MF},
\]

which follows from the linearized Bernoulli equation \((\rho' = \rho \mathbf{u} \cdot \mathbf{u}')\) assuming steady, linear, and unsheared \((\partial\mathbf{u}/\partial z = 0)\) flow.

The pointwise E–P relation is tested in Fig. 6. The filtering method was used to produce \( \mathbf{u}_0, v_0, w_0, p_0, \) and \( \mathbf{u}, v \) fields as defined in Eq. (1), which were then used to evaluate the left- and right-hand sides of Eq. (10) without the negative sign. The pointwise E–P relation is well satisfied over a wide range of flux values, with scatter presumably associated with unsteadiness, shear, and nonlinear effects. The agreement even extends to the lower-right corner of the diagram, where pointwise negative energy flux and positive momentum flux are seen. Pointwise negative values of \( \mathbf{EF}_z \) locally should be expected even for the simplest flow over a single smooth hill (appendix D).

Another important constraint on steady, linear mountain waves is the conservation of momentum flux with height with background wind shear (Eliassen and Palm 1960). According to Eq. (9), \( \mathbf{EF}_z \) will vary with height proportionately with \( \mathbf{u}(z) \). Physically, this variation in \( \mathbf{EF}_z \) is caused by a conversion from mean-state shear energy to wave energy flux according to [from Eq. (9)]

\[
\frac{d\mathbf{EF}_z}{dz} = -\mathbf{MF} \cdot \frac{d\mathbf{u}}{dz}.
\]

This behavior is clearly seen between \( z = 7 \) and \( z = 15\) km in both events (Figs. 7b and 7f), where strong winds and shear were present (Figs. 7a and 7e). \( \mathbf{MF}_z \) was roughly conserved (Figs. 7c and 7g), and perturbations were linear (Figs. 7d and 7h). In this range, \( \mathbf{EF}_z \) reaches a maximum at the altitude of maximum wind speed. Below \( z = 7\) km and above \( z = 15\) km, the waves are nonlinear and neither Eqs. (9) nor (11) are valid.

**c. Upwind propagation**

A known property of mountain waves is their tendency to propagate upwind to balance the advection of their energy downwind. In the case of 2D terrain and waves, wave ray paths tilt downwind if the waves are nonhydrostatic with an intrinsic frequency just slightly less than \( N(\omega_i = \pi k < N) \). For longer hydrostatic waves, however, with \( \omega_i = \pi k \ll N \), the ray path is vertical indicating that the upwind propagation of the wave energy exactly balances the downwind energy advection (Smith 1979).

In 3D terrain, the situation is less definite as some laterally propagating waves can be launched by the complex terrain (Smith 1980) or refracted by lateral shear (Jiang et al. 2013). The most energetic waves are launched with their horizontal wavenumber vector pointed into the wind, however. This property is used in the design of wave drag parameterization schemes (McLandress 1998).

The proposed method captures the upwind propagation in both events at 12 km (Fig. 8). The background wind direction, approximately parallel to the smoothed isobars, differed between the two events. The vectors in the diagrams represent the horizontal energy flux vector, \( \mathbf{EF}_h = \langle \mathbf{p}' \mathbf{u}' \rangle i + \langle \mathbf{p}' \mathbf{v}' \rangle j \), where angle brackets indicate low passing with \( L = 150\) km. These vectors point into the wind, indicating that these waves are propagating into the flow. The location of the \( \mathbf{EF}_h \) and \( \mathbf{EF}_z \) maxima directly over the terrain confirms that upwind propagation is balanced by the mean-wind advection.

**d. An alternative “difference” method**

An alternative approach to computing mountain wave perturbations and fluxes uses two simulations: one simulation with terrain and one without. That is, the surface elevation is set to zero. The remaining contrast in
surface roughness and thermal properties may generate some small gravity waves. No-terrain fields define the background state. Perturbations are found by subtracting this background state from fields with terrain:

$$d' = a_{Mtn} - a_{No,Mtn}.$$  \hspace{1cm} (12)

“Differenced” fluxes \((EF_{z,\text{diff}})\) are compared to the filtering method \((EF_{z,\text{filt}})\) by calculating the following ratio at a number of vertical levels:

$$R = \frac{EF_{z,\text{filt}}}{EF_{z,\text{diff}}}. \hspace{1cm} (13)$$

These ratio profiles are shown in Fig. 9 every 3 h in the 14 June simulation. Despite variable background wind and, hence, variable flux profiles throughout this event, the two methods typically differ by less than 20%.

The difference method is simpler than the proposed filtering method but has significant disadvantages. For example, \((EF_{z})\) and \((MF_{x})\) fields are visualized in Figs. 10 and 11. In both mountain wave events, energy and momentum fluxes are mainly confined to the South Island orography. The mountain waves propagate deeply, in excess of 40 km MSL, in the 14 June event (Figs. 10a and 11a). In the 24 June event (Figs. 10b and 11b), the flux towers terminate just below 20 km, suggestive of dissipation below. Nonlinearity is also visualized in Fig. 11. Isosurfaces of \((NLR)\) are noted both near the orography and aloft in regions of dissipation, suggesting these mountain waves are both generated and dissipated nonlinearly. Mountain wave propagation, dissipation, and nonlinearity are shown more quantitatively in Figs. 7c, 7d, 7g, and 7h.

Isosurfaces of \((T^2)\) are also shown in Fig. 12. This pattern can be explained using the expression for temperature variance in a monochromatic steady 2D linear hydrostatic gravity wave:

$$\sigma_{T}^{2} = \left( \frac{dT}{dz} + \frac{g}{c_{p}^{2}} \right)^{2} \frac{MF_{x}}{\rho Nk^2}. \hspace{1cm} (14)$$

5. Mountain waves

Mountain waves are easily visualized using the proposed method, illustrating many 3D characteristics. For example, \((EF_{z})\) and \((MF_{x})\) fields are visualized in Figs. 10 and 11. In both mountain wave events, energy and momentum fluxes are mainly confined to the South Island orography. The mountain waves propagate deeply, in excess of 40 km MSL, in the 14 June event (Figs. 10a and 11a). In the 24 June event (Figs. 10b and 11b), the flux towers terminate just below 20 km, suggestive of dissipation below. Nonlinearity is also visualized in Fig. 11. Isosurfaces of \((NLR)\) are noted both near the orography and aloft in regions of dissipation, suggesting these mountain waves are both generated and dissipated nonlinearly. Mountain wave propagation, dissipation, and nonlinearity are shown more quantitatively in Figs. 7c, 7d, 7g, and 7h.

Isosurfaces of \((T^2)\) are also shown in Fig. 12. This pattern can be explained using the expression for temperature variance in a monochromatic steady 2D linear hydrostatic gravity wave:

$$\sigma_{T}^{2} = \left( \frac{dT}{dz} + \frac{g}{c_{p}^{2}} \right)^{2} \frac{MF_{x}}{\rho Nk^2}. \hspace{1cm} (14)$$

FIG. 10. Three-dimensional isosurfaces of \((EF_{z})\) valid at 1200 UTC (a) 14 and (b) 24 Jun 2014. The 1 W m\(^{-2}\) isosurface is visualized. Gray horizontal planes indicate the 10-, 20-, 30-, and 40-km levels. The lowest level visualized is 4 km. All isosurface figures created with VAPOR (Clyne et al. 2007).
In the 14 June event with deep propagation (Fig. 12a), the variance becomes noticeable in the low stratosphere owing to increased lapse rate and decreased wind speed $[\overline{\rho(u)}]$. Aloft, it grows steadily as a result of decreasing air density $[\overline{\rho(z)}]$. In the 24 June event with stronger generation but shallow propagation (Fig. 12b), the variance is evident in the low wind layer at 15 km (Fig. 7e) but diminishes above as wave dissipation reduces $M_F(z)$. Even the small amount of surviving wave amplifies aloft as $\rho(z)$ decreases and the larger variance reappears near 40 km. The density effect in Eq.(14) between 15 and 40 km is a factor of 20. These 3D visualizations of quadratic diagnostics are extremely informative and allow simple conceptual models of mountain wave characteristics to be qualitatively tested.

Note that while these isosurface diagrams in Figs. 10 and 11 are useful conceptually, the structure of these 3D flux isosurfaces and hence their interpretation may be sensitive to the isosurface chosen. As these fields have been smoothed with an isotropic Gaussian filter (section 2d), their horizontal profile may resemble

$$F(r/a) = F_{\text{max}} \exp[-(r/a)^2],$$

with $r = \sqrt{x^2 + y^2}$ the horizontal distance from the center of the flux tower at each altitude $z$. The two parameters $F_{\text{max}}$ and $a$ are functions of $z$. The radius of the $F^*$ isosurface tower is then $a^* = a[\ln(F^*/F_{\text{max}})]^{1/2}$. Assuming a constant, if the chosen isosurface value $F^*$ is small (i.e., $F^* \ll F_{\text{max}}$), the radius of the isosurface tube $a^*$ is quite insensitive to $F_{\text{max}}$. On the other hand, if $F^*$ is slightly less than $F_{\text{max}}$, $a^*$ is very sensitive to $F_{\text{max}}$. An increase in $F_{\text{max}}$ with height will greatly enlarge $a^*$, while a decrease in $F_{\text{max}}$ with height (i.e., below $F^*$) will make the isosurface tube contract and disappear. Note also that when a flux quantity is depicted, its isosurface is not a stream tube of that flux. A change in $a^*$ does not represent a confluence or a diffuence of that flux.

Time series of $EF_z$ and $MF_z$ may also be easily computed to characterize strength and frequency of mountain wave launching events. The $EF_z$ time series were computed in the 6-km continuous WRF simulation for the 2014 austral winter (24 May–31 July). A variety of storms affected New Zealand during this period. Energy flux time series at three different levels are shown in Fig. 13. Individual mountain wave events are easily identified. Deeply propagating events can also be identified by studying the upper- and lower-level fluxes.

An additional useful application of the filter method is to examine how it depicts the role of low tropospheric wind speed on wave generation. Both linear and nonlinear mountain waves may be associated with the
deformation of low-altitude winds by terrain. To test this relationship, the South Island–averaged EF is plotted versus the average wind speed at 4 km in the 6-km WRF simulation (Fig. 14). Each point is a 3-hourly instant during the winter 2014 simulation. As expected, wave energy flux increases with wind speed but with considerable scatter. A least squares quadratic fit (i.e., \( \text{EF}_z \propto u^2 \)) is overlaid, as might be expected from linear theory. The poor fit suggests that the wave generation mechanism may be nonlinear, consistent with Figs. 7d and 7h.

6. Nonorographic gravity waves

The proposed method may also be used to quantitatively investigate nonorographic gravity waves as well. Jet-generated and convection-generated gravity wave events within 6- and 2-km-resolution WRF simulations (appendix A), respectively, are presented in this section.

a. Jet-generated gravity waves

Gravity waves events are quite apparent and frequent in midstratospheric analyses of vertical velocity and temperature over the Southern Ocean south of New Zealand. This is a particularly stormy region, with frequent midlatitude cyclones and high-amplitude upper-tropospheric waves. Within the complex jet structures in the upper troposphere, flow imbalances may act as a source of gravity waves (Lin and Zhang 2008). One such nonorographic event occurred on 11 July 2014 and is illustrated in Fig. 15. A wave packet with northwest–southeast-oriented phase lines is apparent to the southwest of New Zealand. The packet moved northeast with time, with the more intense perturbations evident for at least 9 h.

The \( \text{EF}_h \) components (Fig. 15) reveal a southwest propagation direction of the wave packet relative to the fluid, perpendicular to the phase fronts. The \( \text{EF}_z \) contours indicate vertical propagation of the packet. The 3D extent and vertical propagation of this packet is...
visualized in Fig. 16a. The 1 W m$^{-2}$ EF$_z$ isosurface originates near 20 km and expands upward, reaching at least 40 km MSL.

More details of the vertical structure of the wave packet are illustrated in Fig. 17, which shows an east–west vertical cross section of the wave packet along 52°S. The wave packet was vertically coherent, originated near 15 km, and had a westward phase tilt consistent with the vertical propagation shown in Fig. 15. Subsequent analyses (not shown) show this coherent wave packet translating east, retaining its vertical structure. This suggests a single phase speed for the whole packet, which was found to be 15.7 m s$^{-1}$ toward 52° east of north.

This wave packet is summarized quantitatively in the profiles in Fig. 18, where diagnostics were averaged over the polygon illustrated in Fig. 15. Figure 18a shows the wind speed through the wave packet relative to the phase speed defined by

$$U_{\text{rel}} = U \cdot \frac{|\kappa|}{|\kappa|} - c_p = U_\perp - c_p,$$  \hspace{1cm} (16)

where $c_p$ is the earth-relative phase-speed vector magnitude. Figure 18a suggests air parcels were flowing through the wave from west to east.

Average energy and zonal momentum flux profiles are shown in Figs. 18b and 18c, respectively. The packet had increasing vertical energy flux until the upper sponge layer, which started near 34 km, effectively damped the waves. This increase in energy flux is likely due to wave interaction with the mean shear, similar to mountain waves [Eq. (11)]. The kinks in the profiles of $U_{\text{rel}}$ and EF$_z$ collocated at 30-km altitude in Fig. 18a and 18b provide further evidence for this interaction. The packet contained negative MF$_x$ (Fig. 18c). The increases in negative MF$_x$ between 15 and 25 km suggest gravity wave sources in this layer, consistent with the appearance of the packet in Fig. 17.

While the isotropic NLR defined in Table 1 is generally useful, knowledge of wave orientation and speed allows a better NLR to be defined. This non-linearity ratio was defined perpendicularly to the phase fronts and takes into account the phase speed of the packet:

$$\text{NLR}_{x_r} = \frac{u_x^2}{U_{\text{rel}}^2}.$$  \hspace{1cm} (17)
Here, $u'_x = \mathbf{u}' \cdot \mathbf{k}/|\mathbf{k}|$ is perturbation horizontal wind vector perpendicular to the phase fronts. Where the wave packet is first apparent ($\approx 15$ km in Fig. 17), NLR$_{dr}$ is high (Fig. 18d). Here, both EF$_z$ (Fig. 18b) and negative MF$_x$ (Fig. 18c) increase. An additional spike in NLR$_{dr}$ is noted near 20 km, again associated with increases in EF$_z$ and negative MF$_x$. At 25 km and above, EF$_z$ varied proportionately with the wind speed through the packet, MF$_x$ was approximately constant, and NLR$_{dr}$ was low. In a reference frame moving with the wave packet, this jet-generated packet shared many characteristics with mountain waves: nonlinear generation, wave- mean-shear interaction, upwind propagation, and nearly constant MF$_x$ where flow was linear.

b. Convection-generated gravity waves

Wintertime convection is frequent over the Tasman Sea region to the northwest of New Zealand. Convection may generate gravity waves (Clark et al. 1986; Kuettner et al. 1987; Fovell et al. 1992), which may also be analyzed with the proposed method. A convectively active event over this region occurred on 10 July 2014 associated with a dissipating surface low pressure system moving southeast. This event was simulated with WRF at 2-km resolution and analyzed with the proposed method.

Multiple regions of convection-oriented roughly north–south occurred within the simulation, visualized via HF$_z$ at 4 km in Fig. 19a. In the stratosphere, temperature perturbations and associated energy fluxes were collocated with the convection below (Fig. 19b). The linear convective region near 165$^\circ$E was associated with a cold front, and the perturbations above may be due in part to both convection and the front. The boxed region of convection (Fig. 19b) was chosen for further analysis as it was more than 400 km from the cold front and stratospheric perturbations here are more likely solely due to the convection below.

Area-averaged profiles of quadratic diagnostics are used to quantitatively describe the convection and stratospheric waves within the boxed region in Fig. 19b. The heat flux profile (Fig. 20e) suggests this convection was deep, with positive heat fluxes extending to the tropopause at 9 km. Within the
troposphere, EF\(z\) and MF\(x\) are due to velocity and pressure perturbations within the buoyant convection. The transition from negative to positive convective EF\(z\) with height (Fig. 20b) may be due to negative hydrostatic pressure perturbations below the latent heating and positive dynamic pressure perturbations above collocated with updrafts. In Fig. 16b, \(\langle EF_z \rangle\) is noted in the mid- to upper troposphere in regions of convection. The convective MF\(x\) profile (Fig. 20c) is consistent with the regions of vertical shear in Fig. 20a.

At and just above the tropopause (9–11 km), there was strong vertical shear. The convective region along 160°E moved east at \(\approx 9.5 \text{ m s}^{-1}\), similar to the mean tropospheric eastward wind speed. The shear at and above the top of the convection produced increasing wind speed relative to the convection. The tops of the convection may act as obstacle to the faster flow just above, producing upstream propagating gravity waves via the “moving obstacle” mechanism (e.g., Beres et al. 2002). In this region, both EF\(z\) and negative MF\(x\) increase, while HF\(z\) is nearly zero (Fig. 20). Since EF\(z\) and negative MF\(x\) are not convective here, their increase is interpreted as wave generation. The horizontal energy fluxes at 15 km are oriented primarily upstream (Fig. 19b), consistent with waves produced via the moving obstacle mechanism.

The quadratic diagnostics in the stratosphere are consistent with vertically propagating gravity waves. Between 11 and 16 km, EF\(z\) varied proportionately with \(u\), suggesting wave-/mean-shear interaction. The quantity MF\(x\) was relatively constant where NLR was low, and HF\(z\) was nearly zero (Fig. 20). The EF\(z\) in Figs. 20b and 16b extend well into the stratosphere and cannot be associated with buoyant convection there. The location of the wave fields and fluxes very near the convection below (Fig. 19) suggest a dominant phase

---

**Fig. 17.** East–west vertical cross section along 52°S valid at 0600 UTC 11 Jul 2014. Vertical velocity, multiplied by \((\rho_0)^{1/2}\) to account for amplitude growth over multiple density scale heights, is color shaded (kg m\(^{-1/2}\) s\(^{-1}\)). East wind component (m s\(^{-1}\)) is shown in blue contours.

---

**Fig. 18.** Profiles of quantities averaged over the area within the polygon in Fig. 15. In (a), the wind speed (m s\(^{-1}\)) perpendicular to phase fronts relative to the phase speed [Eq. (16)] is shown. The vertical profiles of average (b) EF\(z\), and (c) MF\(x\) are shown (W m\(^{-2}\)). (d) The maximum low-passed NLR\(d\), defined in Eq. (17) is shown. The phase speed was found to be 15.7 m s\(^{-1}\) at 52° from true north.
speed near the speed of the convection or $\approx 10\,\text{m s}^{-1}$. With this phase speed, a critical level was present near 18 km (Fig. 20a). Just below this level, negative MF$_x$ and NLR decrease and increase sharply, suggestive of nonlinear dissipation.

The stratospheric gravity waves in this event, again, had many similarities with the steady mountain wave events; these waves contained positive EF$_z$ and negative MF$_x$, exhibited wave-/mean-shear interaction, and were dissipated nonlinearly.

7. Conclusions

A method for quantifying gravity waves in mesoscale fields was proposed in this paper. This method combines 2D spectral filtering with a selection of quadratic diagnostic quantities. Covariant diagnostic quantities (e.g., vertical energy flux) allow differentiation between synoptic and mesoscale quantities with the existence of a cospectral gap. Diagnostics also allow different types of mesoscale motion to be distinguished. This method may
be used to quantify both orographic and nonorographic gravity waves and to compare them between models.

These ideas were tested in realistic 2-km mesoscale WRF simulations. Filter-derived mountain wave fluxes were relatively insensitive to the high-pass filter cutoff length scale in the range of 300–700 km, providing evidence for a cospectral gap and robust mesoscale flux estimates (Figs. 4 and 5). The resulting wave fields quantitatively satisfied relations predicted from idealized linear gravity wave theory (e.g., Fig. 6) and reproduced many known gravity wave characteristics. Filter-derived fluxes agreed within 20% of those from an alternative difference method where perturbations were defined relative to a no-mountain simulation.

The filtering method was applied to realistic 2-km event simulations and a 6-km winter-long WRF simulation over the New Zealand region to illustrate how this method might be used. Vertical energy flux time series (Fig. 13) revealed the episodic nature of the mountain wave events throughout the 2014 winter and identified event simulations and a 6-km winter-long WRF simulation over New Zealand. Filter-derived mountain wave flux diagnostics in COAMPS by Alex Reinecke and in NIWA’s version of the UKMO Unified Model by Michael Uddstrom and Stuart Moore. Jeffrey Park’s time series course was helpful in developing the proposed method. Finally, we would like to acknowledge David Fritts, James Doyle, Steve Eckermann, and Michael Taylor for their help in organizing the DEEPWAVE field project in New Zealand, which motivated this work.

APPENDIX A

WRF Model Setup

Two types of WRF (v3.6.0) simulations were completed: high-resolution simulations with a 6-km outer domain and 2-km inner nest and a 6-km-resolution continuous simulation over Austral winter (24 May–31 July). Initial and boundary conditions were provided by ECMWF analysis (valid at 0000, 0600, 1200, and 1800 UTC) and 3-h forecast (valid at 0300, 0900, 1500, and 2100 UTC) grids. The high-resolution simulations were completed with 150 vertical levels extending from the surface to 80 Pa, with the highest vertical resolution near the surface decreasing to Δz = 470 m at the domain top. The vertical resolution in the winter-long simulation also decreased with height, with the coarsest vertical resolution of Δz = 1115 m. A 10-km upper damping layer was specified in all simulations. A time step of 25 s was chosen for the 6-km domains, scaled appropriately for the 2-km nests. Parameterization selections are given in Table A1.

APPENDIX B

2D Spatial Filtering

Two dimensional spatial filtering is performed in three steps. The discrete Fourier transform (DFT) is performed on the original field, a(x, y), to produce a 2D field of Fourier coefficients, ā(k, l). These coefficients
are then multiplied with a response function, $\hat{r}(k, l)$, to produce filtered coefficients. Then, the inverse DFT is performed to produce the filtered field in physical space. Performing the forward and inverse 2D DFT is trivial in most programming languages and is not discussed here.

The response function here takes a Gaussian form, which provides a reasonably sharp cutoff in wavenumber space and does not produce Gibbs phenomenon oscillations. The low-pass response function is defined as

$$\hat{r}_{lp}(\kappa) = e^{-|\kappa|^2 L^2},$$  \hspace{1cm} \text{(B1)}$$

where $\kappa = ki + j$ is the spatial wavenumber vector, $k = 1/\Lambda_x$ is the east spatial wavenumber, $l = 1/\Lambda_y$ is the north spatial wavenumber, $|\kappa|^2 = \kappa \cdot \kappa$ is the squared magnitude of the spatial wavenumber vector, and $L$ is the cutoff wavelength. This Gaussian low-pass response function is isotropic; it is maximized (i.e., unity) at $|\kappa| = 0$ and quickly decays exponentially and identically in all directions in wavenumber space as $\kappa$ increases.

The low-passed field in wavenumber space is

$$\hat{a}_{lp} = \hat{a}_{lp} \cdot$$  \hspace{1cm} \text{(B2)}$$

The high-pass response function is defined by

$$\hat{r}_{hp}(\kappa) = 1 - \hat{r}_{lp}.$$ \hspace{1cm} \text{(B3)}$$

The three filtering steps may be repeated to produce both high- and low-passed parts of some 2D field; however, since $\hat{r}_{lp} + \hat{r}_{hp} = 1$, $\hat{a} = \hat{a}_{lp} + \hat{a}_{hp}$.

**APPENDIX C**

**Edge Artifact Extent**

Aperiodicity in 2D fields produces edge artifacts after high-pass filtering. Periodic Fourier basis functions reproduce aperiodic edge discontinuities by soliciting primarily high-wavenumber Fourier components (Denis et al. 2002). These components are retained in high-pass filtering, producing edge artifacts (e.g., Fig. 1d). The objective here is to find a length scale beyond which edge artifacts are negligible.

Low-pass filtering (appendix B) is equivalent to convolving the inverse DFT of the low-pass response function with the original data series. In 1D, the continuous inverse Fourier transform of the low-pass response function, $\hat{r}_{lp}(\kappa) = e^{-|\kappa|^2 L^2}$, is $r_{lp}(x) = \mathcal{F}^{-1}(\hat{r}_{lp}) = Ae^{-x^2/(2L^2)}$, with $A$ some constant. The high-pass response function in physical space is $r_{hp}(x) = \mathcal{F}^{-1}(1 - \hat{r}_{lp}) = \delta(x) - r_{lp}(x)$, where $\delta(x)$ is the Dirac delta function. Convolving $r_{lp}(x)$ with some field (i.e., high-pass filtering) is equivalent to subtracting the low-passed field from the original field. Only low-pass filtering need be considered to understand how edge discontinuities extend into the domain.

The width of $r_{lp}(x)$ is linearly proportional to $L$. The quantity $r_{lp}$ decreases to 5% of its maximum at a distance $L_{5\%} = \pm \sqrt{\ln 20/\pi} L = \pm 0.55 L$ away from the maximum. We consider convolving some field containing a discontinuity at $x_0$ with $r_{lp}$. The discontinuity’s influence on the filtered field decays away from $x_0$ since less of $r_{lp}$ straddles $x_0$. By $|x - x_0| = L_{5\%}$, the influence of the discontinuity on the filtered field is minimal.

To test this artifact length scale, high-pass filtering of 1D single saw tooth data series (linear series beginning at $\pm A$ and ending at $\mp A$) was performed. The $L$ was varied between $5\Delta x$ and $D/2$, where $\Delta x$ was the domain length.
resolution and $D$ was the domain width. At a distance of 0.55$L$ from the domain edges, edge artifacts magnitude decreased to 1.4% of the value at the edge. Beyond 0.55$L$, edge artifacts are negligible. Within 0.55$L$ of domain edges, filtered quantities are suspect, except for filtered quantities like $w'$ that are often nearly periodic to begin with.

**APPENDIX D**

**Energy and Momentum Fluxes in the Queney Solution**

The motivation for smoothing quadratic quantities such as the wave energy and momentum flux with a low-pass filter (i.e., section 2) does not arise solely from wanting to simplify the waves from complex terrain. To pass filter (i.e., section 2) does not arise solely from such as the wave energy and momentum flux with a low-pass filter or area integration to identify the primary upward nature of the energy flux and downward nature of the momentum flux. When regions of reversed flux are found, low-pass filtering assists in distinguishing between chance local superposition of Fourier components as in the Queney solution and significant beams of wave energy from reflection or secondary generation are negative (if $U > 0$) and positive, respectively, and independent of height.

To display the complexity of the point-by-point flux fields, we start with the Queney vertical displacement field

$$
\eta(x, z) = h_m \left( \frac{a \cos \hat{z} - x \sin \hat{z}}{a^2 + x^2} \right),
$$

where the ridge height and half-width are $h_m$ and $a$, and $\hat{z} = Nz / U$. From Eq. (D3), the perturbation velocity and pressure fields can be derived, and from those fields, the wave energy density and energy flux are obtained. The energy density is

$$
ED(x, z) = \frac{\rho}{2} \left( u'^2 + N^2 \eta^2 \right) = \frac{\rho (Nh_m a)^2}{2(a^2 + x^2)}
$$

The energy flux vector in earth-relative coordinates is

$$
\bf{EF}(x, z) = (EF_x + UED)\hat{i} + EF_z \hat{k},
$$

where

$$
EF_x(x, z) = p' u' = -\rho U^2 h_m^2 a^2 \left( \frac{a \sin \hat{z} + x \cos \hat{z}}{a^2 + x^2} \right)^2,
$$

which is negative definite (if $U > 0$) and

$$
EF_z(x, z) = p' w' = -\rho U^2 h_m^2 a^2 \left\{ \frac{(a \sin \hat{z} + x \cos \hat{z}) (x^2 - a^2) \sin \hat{z} - 2ax \cos \hat{z}}{(a^2 + x^2)^3} \right\},
$$

which is mostly positive but with weak negative regions at altitudes near $\hat{z} = \pi/2, 3\pi/2, \ldots$. This vector field is plotted in Fig. D1, illustrating the meandering and even locally downward direction of the energy flux vector field. The fact that Eq. (D5) is non-divergent ($\nabla \cdot \bf{EF} = 0$) indicates that energy does not accumulate or dissipate anywhere in the field. The pointwise formulas [Eqs. (6) and (7)] Eqs. (D6) and (D7) satisfy the Eliassen–Palm relation $\bf{EF} = -U \cdot \bf{MF}$ as do the integrals in Eqs. (D1) and (D2). The spatial variation in the $\bf{EF_z}$ and $\bf{MF}$ fields motivates the use of a low-pass filter or area integration to identify the primarily upward nature of the energy flux and downward nature of the momentum flux. When regions of reversed flux are found, low-pass filtering assists in distinguishing between chance local superposition of Fourier components as in the Queney solution and significant beams of wave energy from reflection or secondary generation aloft. The Queney solution also has significant pointwise heat fluxes ($w' T$), but these average to zero.

**REFERENCES**


