Sphere-Filling Asymptotics of the Barotropic Potential Vorticity Staircase

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ABSTRACT

For barotropic flow in spherical geometry, the ideal potential vorticity staircase with flat steps and vertical risers exhibits a relationship between prograde jet strength and spacing such that, for regular spacing, the distance between adjacent jets is given by a suitably defined “Rhines scale” multiplied by a positive constant equal to $\sqrt{6}$. This result was obtained previously by the author in the equatorial limit of spherical geometry and by others in periodic beta-plane geometry. An improved asymptotic method has been devised to explain the strength–spacing relationship in sphere-filling solutions. This analysis explains the approximate validity of the equatorial asymptotics and yields new insight on minimum energy states and staircase mode transitions simulated in the presence of random, persistent energy inputs at high horizontal wavenumber.

1. Introduction

Prograde jets are ubiquitous in planetary fluid dynamics (Baldwin et al. 2007). These rotating stratified systems support Rossby waves, quasi-2D turbulent eddies, and their associated lateral transports of angular momentum and potential vorticity (PV), with overturning meridional circulations required to maintain large-scale balance. Various kinds of positive feedbacks exist between the waves, eddies, and mean flow, such that the prograde jets become self-reinforcing. The resulting “PV staircase” (McIntyre and Palmer 1983; Peltier and Stuhne 2002) is a natural end state of the planetary vorticity gradient itself, which may be regarded as an unstable equilibrium in the presence of Rossby waves and instabilities (Dunkerton and Scott 2008, hereafter DS’08).

While the atmospheres and oceans of planetary bodies in our solar system exhibit a rich variety of feedbacks and morphology, all are subject to a simple geometric constraint relating the strength and spacing of prograde jets (Dritschel and McIntyre 2008; DS’08). This constraint is enforced when the mean flow is, at worst, marginally stable to large-scale hydrodynamic instability, the constraint itself being entirely independent of the wave–mean flow phenomenology. Measured in units of a suitably defined “Rhines scale” depending on prograde jet maxima with speed $U$, the marginally stable spacing is given by $\sqrt{6}$.

General staircases constructed entirely from stable segments, as might occur in a model fluid initially at rest, require larger meridional spacing of jets. In this sense, the asymptotic limit is relevant to stable atmospheres. Conversely, a strongly forced system, in which barotropic stabilization cannot keep pace with the forcing, may accommodate more jets than allowed by the asymptotic limit. The first case encompasses a class of numerical experiments in which the input of energy occurs randomly at small scales (Scott and Polvani 2007). With constant energy input the number of jets decreases very slowly in time, and marginal stability is avoided owing to these infrequent “mode transitions” involving lateral transport by transient eddies of larger scale. The second case might be imagined when dry convecting columns in a deep atmosphere attempt to organize jets in the statically stable weather layer (Heimpel et al. 2016).

In general, evaluation of marginal stability requires knowledge of thermal structure as well as of winds (Williams and Kelsall 2015). On Jupiter, for example, the cloud-top drift winds appear supercritical with respect to hydrodynamic instability considering the latitudinal profile of absolute vorticity per se, with significant reversals of zonal-mean meridional vorticity gradient (Ingersoll et al. 2004). But it is conceivable that the actual profile of PV is marginally stable in most places when the required thermal stratification is taken...
into account (Scott and Dunkerton 2017). The same point has been made for different reasons by Thomson and McIntyre (2016). Following DS’08, the barotropic model is reexamined here, but we anticipate that an extension to equivalent barotropic and shallow-water systems will be useful. The latter exhibits equatorial trapping of jets at small equivalent depth (Scott and Polvani 2007).

The jet strength–spacing marginal stability constraint derived by us and others several years ago was based on a beta-plane or equatorial approximation to the PV profile, which gave useful results for sphere-filling solutions, especially for the most likely value of spacing exponent \( p = -1/4 \) having a flat lower envelope of retrograde parabolas instead of the “Roman arch” common to other values of \( p \) (DS’08, their Fig. 21). Further analytical progress on the full sphere was hindered by the complexity of the allowed jet profiles, which, although designed to be regular, led (via the angular momentum constraint) to a sum of various irrational integrals at second order that cannot be evaluated in closed form and a zeroth-order sum critical to the asymptotic analysis, for which the second-order difference between its discrete value and a continuous integral thereof must be known.

Since then, a few key steps have been navigated successfully, and it is the purpose of this paper to describe and exploit them (section 2). Of equal interest is our identification of a minimum kinetic energy state that exists in the parameter space bounded by interior and exterior coalescence points (borrowing from DS’08 terminology). That will be described herein also, as an illustration of the general concept of PV staircase “functional” (section 3).

2. Staircase model: Exact and asymptotic solutions

The ideal potential vorticity staircase of DS’08 assumes a simple relationship between the spacing \( \delta \mu_j \) of westerly jets and their maximum speed \( U \):

\[
\delta \mu_j = \frac{\sqrt{U}}{C}(1 - \mu_j^2)^{-p/2},
\]

(2.1)

where \( \mu_j \) is the sine of latitude \( \theta_j \) of the \( j \)th westerly jet and \( p \) is a small constant assumed to lie in the range \(-0.25 \leq p \leq 0.25 \). All quantities are dimensionless following DS’08. A pictorial representation of the staircase construction is included here in Fig. 1, borrowing from that paper. The lhs of (2.1) may be interpreted loosely as a meridional wavelength of the staircase equal to a latitudinally varying Rhines scale for stationary Rossby waves (represented on the rhs by the square-root dependence on wind speed \( U \) and a function of planetary vorticity gradient \( \beta \)) divided by an arbitrary constant \( C \).

For the regular spacing of westerly jets presupposed by (2.1), this constant is a function of the number of westerly jets spanning the sphere (or “mode index” \( n \) and spacing exponent \( p \)). It was shown asymptotically in DS’08 that \( C^{-1} = \sqrt{6} \) for an angular momentum conserving, neutrally stable staircase constructed from a resting basic state on the sphere. This result was verified numerically [via solution of (2.2) below] and indicates that the neutrally stable staircase spanning the sphere cannot accommodate as many jets as implied by the Rhines scale. Rather, the maximum number of jets is smaller by a factor of approximately \( \sqrt{6} \). Integration of a nonlinear barotropic model on the sphere (DS’08) suggested, albeit vaguely, a preferred value of \( p = -0.25 \) for which both westerly and easterly jets have approximately equal magnitude from pole to pole. The spread of numerical solutions is not surprising since (2.1) represents a subset of possible staircase geometries.

To put into context the discussion that follows, it is helpful to recall the details of the simple asymptotic method outlined in DS’08. This method begins with a sphere-filling modal solution and gradually reduces the latitudinal extent of the staircase to zero. In this “equatorially trapped” limit, the latitudinal spacing of westerly jets and the central latitudes of these jets are reduced systematically to zero, along with \( \mu_p \), the outer boundary of the staircase. The constant of proportionality \( C \) assumes an asymptotic value that is a function of mode number \( n \). By taking a second limit \( n \rightarrow \infty \) after the first limit \( \mu_p \rightarrow 0 \), we obtain the limiting asymptotic value of \( C^{-1} = \sqrt{6} \) indicated above. Dritschel and McIntyre (2008) obtained the same result in periodic
geometry, which is perhaps not surprising since their geometry, and ours in the equatorial limit, share the relevant properties of a beta-plane approximation to the sphere. Somewhat fortuitously, this limiting value provides excellent guidance for sphere-filling modal solutions, particularly those near \( p = -0.25 \), even though the asymptotic method of DS'08 does not apply to such solutions. It is desirable to understand why this is so and to obtain some additional analytical insights for sphere-filling solutions. In particular, it is desirable to establish geometrically a connection between the sphere-filling solutions of DS'08 and the infinite beta-plane result of DM'08.

Here we present a different asymptotic method based on the requirement \( \mu_p = O(1) \), in the limit of large \( n \). Interior and exterior branches of the sphere-filling modal solutions were discussed extensively in DS'08. The so-called interior solutions distribute westerlies and easterlies more or less uniformly across the sphere, with westerlies or weak easterlies near the pole, while exterior solutions have a strong, broad easterly jet in high latitudes and a prevalence of westerlies at low to midlatitudes. The latter solutions require long-range transport of angular momentum in their formation, between low and high latitudes, whereas the former solutions imply a more local redistribution in latitude. The exterior solutions suffer two additional disadvantages making their realization in planetary atmospheres less likely (as noted below) owing to (i) the bimodal phase-speed spectrum of disturbances that might give rise to such solutions and (ii) the high values of kinetic energy required relative to the interior solutions. With these lessons behind us, attention is now directed at a subset of the \((C, U; \mu_p)\) parameter space—for a given combination \((p, n)\)—lying roughly between (i) the upper end of the interior branch along the \((C, U; \mu_p = 1)\) contour, where \( U \) has a local maximum, and (ii) the minimum value of \( U \) along this contour, here referred to as the exterior coalescence point, where the exterior asymptotic branch intersects. [The interior asymptotic branch shown in DS'08 corresponds to a collection of tangent points \((\partial C/\partial U)_{\mu_p} = 0 \) in the range \( \mu_p \in (0, 1] \). The exterior asymptotic branch (not shown) is defined similarly by tangent points \((\partial U/\partial C)_{\mu_p} = 0 \). Here “asymptotic” refers to the equatorially trapped limit of DS'08.] Solutions on this segment of the contour have a realistic and more realizable structure from pole to pole (cf. Fig. 6 of DS'08). A new asymptotic method is developed below, requiring that solutions remain near this contour.

a. Revised strategy

As in DS'08 the angular momentum–conserving staircase assumes the simple form

\[
\sum_{j=0}^{N-1} \frac{1}{2} (m_j + m_{j+1}) \frac{1}{2} (\delta \mu_j + \delta \mu_{j+1}) + \frac{1}{2} (m_N + m_p)(\mu_p - \mu_N) = \sum_{j=0}^{N-1} (1 + U) - \left(1 + \frac{1}{2} U\right) \hat{\mu}_j^2 \frac{\sqrt{U}}{C} (1 + p \hat{\mu}_j^2) + \frac{1}{2} \left[1 - \mu_N^2 + U \left(1 - \frac{1}{2} \mu_N^2\right) + 1 - \mu_p^2\right] (\mu_p - \mu_N)
\]

\[
= \sum_{j=0}^{N-1} (1 + U) - \left(1 + \frac{1}{2} U\right) \hat{\mu}_j^2 \frac{\sqrt{U}}{C} (1 + p \hat{\mu}_j^2) + \frac{1}{2} \left[1 - \mu_N^2 + U \left(1 - \frac{1}{2} \mu_N^2\right) + 1 - \mu_p^2\right] (\mu_p - \mu_N)
\]

\[
\rightarrow (1 + U) \sqrt{U} \sum_{j=0}^{N-1} \left(1 + a_2 \mu_j^2\right) + \left[1 - \mu_N^2 + \frac{1}{2} U \left(1 - \frac{1}{2} \mu_N^2\right)\right] (\mu_p - \mu_N)
\]

\[ (2.3) \]

for small \( \mu_j \), as allowed by the equatorially trapped asymptotics, where

\[
\hat{\mu}_j^2 = \frac{1}{2} (\mu_j^2 + \mu_{j+1}^2) \quad \text{and} \quad (2.4a)
\]

\[
a_2 = p - \frac{1 + U/2}{1 + U}. \quad (2.4b)
\]

In the above equations, Taylor series expansions (in \( \mu_j^2 \)) of (i) planetary angular momentum and (ii) the moment arm for relative flow \( U \) were substituted for their exact functional dependence on \( \mu \). This substitution is appropriate for the asymptotic method of DS'08, wherein \( \mu_p \to 0 \), but it is unlikely to suffice for a revised method that requires a sphere-filling solution \( \mu_p = O(1) \) since at least some of the westerly jets then lie at finite \( \mu \)—more
so as \(n\) increases to infinity. Further discussion highlighting a few differences between equatorially trapped and sphere-filling asymptotics is provided in the appendix.

For sphere-filling asymptotics at large \(N\) we introduce a new definition [distinct from that of DS'08 given by (2.4a)] of midpoint \(\mu_j\):\[
\mu_j = \hat{\mu}_j - \epsilon_j \quad \text{and} \quad \mu_{j+1} = \hat{\mu}_j + \epsilon_{j+1},
\]

such that
\[
\mu_{j+1} - \mu_j = 2\epsilon_j = \frac{1}{2}(\delta\mu_j + \delta\mu_{j+1}). \quad (2.6)
\]
The angular momentum and \(\mu\) spacings associated with the westerly jets at \(\mu_j\) and \(\mu_{j+1}\) are then
\[
m_j = 1 - (\hat{\mu}_j - \epsilon_j)^2 + U\sqrt{1 - (\hat{\mu}_j - \epsilon_j)^2}, \quad (2.7a)
\]
\[
m_{j+1} = 1 - (\hat{\mu}_j + \epsilon_{j+1})^2 + U\sqrt{1 - (\hat{\mu}_j + \epsilon_{j+1})^2} \quad (2.7b)
\]

and
\[
\delta\mu_j = \frac{\sqrt{U}}{C}[1 - (\hat{\mu}_j - \epsilon_j)^2]^{-p}, \quad (2.8a)
\]
\[
\delta\mu_{j+1} = \frac{\sqrt{U}}{C}[1 - (\hat{\mu}_j + \epsilon_{j+1})^2]^{-p} \quad (2.8b)
\]

Although the previous (DS'08) definition of \(\hat{\mu}_j\) was more accurate for the purpose of approximating terms involving \((1 - \mu_j^2)\)—note that \(\epsilon_j\) did not appear in that formulation—this loss of accuracy is inconsequential for the sphere-filling asymptotics because we no longer impose small \(\mu_j\) (as in the equatorially trapped asymptotics) but expect rather that the \(\epsilon_j\) alone become small in the limit of large \(N\), while the \(\mu_j\) [and \(\mu_p = O(1)\)] remain finite. For the sphere-filling asymptotics, local \((\mu)\) Taylor series may be employed, without catastrophic loss of accuracy at any latitude adjacent to the poles, beginning from the exact formulas
\[
m_j = 1 - \mu_j^2 + 2\mu_j\epsilon_j - \epsilon_j^2 + U\sqrt{1 - \mu_j^2 + 2\epsilon_j\mu_j - \epsilon_j^2}, \quad (2.9a)
\]

\[
m_{j+1} = 1 - \mu_{j+1}^2 + 2\mu_{j+1}\epsilon_{j+1} - \epsilon_{j+1}^2 + U\sqrt{1 - \mu_{j+1}^2 + 2\epsilon_{j+1}\mu_{j+1} - \epsilon_{j+1}^2}
\]

and
\[
\delta\mu_j = \frac{\sqrt{U}}{C}(1 - \mu_j^2 + 2\mu_j\epsilon_j - \epsilon_j^2)^{-p}, \quad (2.10a)
\]
\[
\delta\mu_{j+1} = \frac{\sqrt{U}}{C}(1 - \mu_{j+1}^2 + 2\mu_{j+1}\epsilon_{j+1} - \epsilon_{j+1}^2)^{-p}, \quad (2.10b)
\]

which may be combined and written (without approximation) as
\[
\frac{1}{2}(m_j + m_{j+1}) = 1 - \mu_j^2 - \epsilon_j^2
\]

\[
+ \frac{1}{2}U\sqrt{1 - \mu_j^2}[\sqrt{1 - x_-} + \sqrt{1 - x_+}]
\]

\[
(2.11a)
\]

and
\[
\frac{1}{2}(\delta\mu_j + \delta\mu_{j+1}) = \frac{1}{2}\frac{\sqrt{U}}{C}(1 - \mu_j^2)^{-p}[(1 - x_-)^{-p}
\]

\[
+ (1 - x_+)^{-p}], \quad (2.11b)
\]

where
\[
x_\pm = \mp \frac{2\epsilon_j\hat{\mu}_j + \epsilon^2_j}{1 - \mu_j^2}. \quad (2.12)
\]

Terms involving \((1 - \mu_j^2)\), whose values span the semi-open interval [0, 1], are retained exactly (without approximation of small \(\mu_j\)) while a local Taylor series expansion is applied; namely,

\[
(1 - x)^{-p} = 1 + px + \frac{1}{2}p(p + 1)x^2
\]

\[
+ \frac{1}{6}p(p + 1)(p + 2)x^3 + \cdots. \quad (2.13)
\]

The efficiency of the local expansion is apparent, noting that
\[
x_- + x_+ = \frac{2\epsilon_j^2}{1 - \mu_j^2} = O(\epsilon_j^2), \quad (2.14a)
\]

the lhs of (2.14a) being generated by the local expansion and subsequent combination of terms inside the square brackets in (2.11a) and (2.11b). The sum of higher powers is
\[
x_n^\pm + x_n^\pm = \frac{2}{(1 - \mu_j^2)^p} \left\{ \begin{array}{ll}
(2\mu_j\epsilon_j)^p + O(\epsilon_j^{p+2}) & \text{for } n \text{ even} \\
(2\mu_j\epsilon_j)^{p-1}n\epsilon_j^2 + O(\epsilon_j^{p+3}) & \text{for } n \text{ odd} \end{array} \right\}. \quad (2.14b)
As in (2.14a), the sum of odd powers is relegated to one higher order in \( \epsilon_j \).

Some care is warranted evaluating the staircase model and its Taylor series representation approaching the poles \(|\mu| = 1\). The model is the discrete sum of a product of two parts: an \( m \) part, representing piecewise linear segments of angular momentum, and a \( \delta\mu \) part, representing jet spacing. Convergence of the model does not guarantee convergence of a Taylor series representation of it. Regarding the model parts, the relative \( m \) part at the most polar jet is

\[
\frac{1}{2} U \sqrt{1 - \mu_N^2} = \frac{1}{2} U (2\delta_p)^{1/2} \rightarrow O(N^{-5/2}), \tag{2.15}
\]

where \( \delta_p = \mu_p - \mu_N \rightarrow O(N^{-1}) \) is a small gap lying between the formal end of the staircase and its most polar prograde jet (Fig. 4 of DS'08). The jet spacing assigned to this last jet is

\[
\delta\mu_N = \frac{\sqrt{U}}{C} (1 - \mu_N^2)^{-p} = \frac{\sqrt{U}}{C} (2\delta_p)^{-p}, \tag{2.16a}
\]

which may be written

\[
\sum_{j=0}^{N-1} \left[ (1 - \mu_j^2) \left\{ 1 + \frac{\epsilon_j^2}{1 - \mu_j^2} + \frac{1}{2} p(p + 1) \frac{4\mu_j^2 \epsilon_j^2}{(1 - \mu_j^2)^3} \right\} + U \sqrt{1 - \mu_N^2 + 1 - \mu_j^2} (\mu_p - \delta\mu_j) + O(\epsilon_j^2) \right] \sqrt{\frac{U}{C}} (1 - \mu_j^2)^{-p} \tag{2.18}
\]

for conservation of angular momentum and

\[
\mu_N = \sum_{j=0}^{N-1} \frac{\sqrt{U}}{C} (1 - \mu_j^2)^{-p} + O(\epsilon_j^2) \tag{2.19}
\]

for staircase extent. Aside from small variations of spacing owing to a nonzero exponent \( p \), the spacing between westerly jets is \( O(\epsilon_j) \), and, to be consistent, the strength of westerly jets is \( O(\epsilon_j^2) \), requiring that the \( O(\epsilon_j^2) \) correction to the average angular momentum between jets be retained in the \( j \)-discretized conservation law in (2.18). It is necessary also to retain the \( O(\epsilon_j^2) \) corrections to jet spacing multiplying the resting \( m \) profile (leftmost term in sum). The second collection of terms on the lhs (in brackets) contains only one term at second order in \( \epsilon_j \), which is needed nevertheless to balance the planetary contribution to angular momentum between \( \mu_N \) and \( \mu_p \). Equivalently, it can be

\[
\frac{\sqrt{U}}{C} \left( \frac{2}{\xi_N^p} \right)^{-p} \rightarrow O(N^{p-1}), \tag{2.16b}
\]

where \( \epsilon_N = \xi \delta_p \) and \( \xi \in (0, 1) \). So the model itself is well posed in the limit \( N \rightarrow \infty \) for the range of \( p \) considered in DS'08. As for the Taylor series: without the factor \( \xi \) (or with \( \xi = 1 \)) the series (2.13) diverges (converges) for positive (negative) \( p \in [-0.25, 0.25] \). However, the series converges in either case when \( \xi < 1 \), since

\[
\left( \frac{2\epsilon_j}{1 - \mu_j^2} \right)^n < \left( \frac{\epsilon_j}{\delta_p} \right)^n = \xi^n. \tag{2.17}
\]

The expression on the lhs arises from the leading term on the upper rhs of (2.14b) and ensures that the polar singularity of inverse powers of \( 1 - \mu_j^2 \) is kept under control by the jet spacing \( 2\epsilon_j \) with their ratio placed in a geometric series via (2.17). In this way, our small gap formally outside the staircase ensures convergence of the Taylor series in the range of \( p \) investigated and, similarly, the analogous expansion of the \( m \) part for which the equivalent value of \( p = -1/2 \).

With these considerations in mind, the sphere-filling asymptotics become

\[
\mu_N = \frac{1}{3} \mu_N^3, \tag{2.19}
\]

shown that the bracketed quantity may be ignored and the rhs replaced by

simplifying the asymptotic analysis and ensuring it is independent (to second order) of the model gap outside \( \mu_N \).

b. Jet strength–spacing relationship, \( n = 2–23 \)

Exact solutions contouring \( \mu_p = 1 \) in the \((C, U)\) plane are shown in Fig. 2, extending the results of DS'08 to higher-mode indices \( n \). The magnitude of westerly jets decreases with increasing mode number, approximately as \( n^{-1.8} \) over the range shown as noted previously by DS'08. Values of \( p \) extending from \(-0.25 \) (red) to \(+0.25 \) (blue) are shown, with red curves overlapped last in order to highlight what is ostensibly the most realistic value of \( p \) obtained in our previous numerical results.
Filled circles in Fig. 2 indicate (i) a few examples of the interior coalescence point for \( p = 0.25 \), where the equatorial asymptotic branch of interior solution \( \frac{\partial C}{\partial U} \) intersects the \( \mu_p = 1 \) contour, and (ii) a single example of the exterior coalescence point for \( p = 0 \), where the equatorial asymptotic branch of exterior solution \( \frac{\partial U}{\partial C} \) intersects the \( \mu_p = 1 \) contour. The former intersection, by definition, occurs where \( C \) is minimum and the latter, by definition, where \( U \) is minimum. It can be seen that the interior coalescence point converges to approximately \( C^{-1} = \sqrt{6} \) (red vertical line) for large \( n \) while the exterior coalescence point converges to a larger value near \( C^{-1} = \sqrt{3/2} \) (yellow vertical line) predicted by the sphere-filling asymptotics (next subsection).

Figure 3 shows the corresponding values of near-equatorial spacing \( \delta \mu_0 \) in the same format. Unlike U, this quantity continues to decrease with increasing \( C \) beyond the exterior coalescence point. [The geometric constraint therefore does not apply literally to solutions in the right half of the diagram when defined in terms of the global \( U \). Nevertheless, the constraint holds locally, for the same reason noted by Dritschel and McIntyre (2008): a neutral parabola, even if not retrograde absolutely, still requires enough “room” meridionally to fit between adjacent prograde jets when measured relative to a prograde reference line.] As mode index \( n \) become large, the logarithmic spacing between curves asymptotically approaches to a constant, as it must to accommodate the increase of \( n \), noting that \( N\delta \mu_0 \approx 1 \). Detailed diagnosis of each term on the lhs of (2.18) via the numerical method was done to evaluate the magnitude of error in the Taylor series expansion of total absolute
angular momentum in the staircase (rhs). The results (not shown) indicate errors ranging from $O(e^{-2})$ for the smallest $n (=2, 3)$ to $O(e^{-12})$ for the largest $n (=22, 23)$ considered.

c. Asymptotic analysis, $p = 0$

The sphere-filling asymptotic analysis is relatively easiest for $p = 0$ because the spacing of westerly jets is then constant in $m$. Consideration of nonzero $p$ is deferred to the next subsection. For $p = 1$ and $p = 0$, (2.18), (2.19), and their dependencies reduce to

$$
\sum_{j=0}^{N-1} \left[ 1 - \mu_j^2 - \epsilon_j^2 + U \sqrt{1 - \mu_j^2} \right] \mu_0 = 2/3, \quad (2.20a)
$$

$$
\sum_{j=0}^{N-1} \delta \mu_j = \mu_N = N \delta \mu_0 = 1, \quad (2.20b)
$$

$$
\delta \mu_j = \left( j + \frac{1}{2} \right) \delta \mu_0, \quad \text{and} \quad (2.20c)
$$

$$
2 \epsilon_j = \delta \mu_0 = \frac{\sqrt{U}}{C}, \quad (2.20d)
$$

Noting that

$$
\sum_{j=0}^{N-1} \left( j + \frac{1}{2} \right)^2 = \frac{1}{3} N^2 - \frac{1}{12} N \quad (2.21)
$$

implies, using (2.20) and (2.21), that

$$
(N \delta \mu_0) - \frac{1}{3} (N \delta \mu_0)^3 + \left( \frac{1}{12} - \frac{1}{4} \right) (N \delta \mu_0) \delta \mu_0^2
$$

$$
+ \int_0^1 U \sqrt{1 - \mu^2} \, d\mu + O(e^3) = \frac{2}{3} \quad (2.22)
$$

and upon equating powers of $\delta \mu_0$,

$$
1 - \frac{1}{3} = \frac{2}{3} \quad \text{and} \quad (2.23a)
$$

$$
\left( \frac{1}{12} - \frac{1}{4} \right) \delta \mu_0^2 + \left( C \delta \mu_0 \right)^2 \int_0^1 \sqrt{1 - \mu^2} \, d\mu = 0. \quad (2.23b)
$$

Substitution of the continuous integral (equal to one-fourth of the unit circle’s area, $\pi/4$) is permissible since (i) $U$ is $O(e^2)$ already and (ii) small deviations of $\mu_j$, if any, from regular spacing of integrand increments (implicit in the Newtonian calculus $d\mu = \text{constant}$) are inconsequential at second order. When $p = 0$ there are, in fact, no deviations from regular spacing, and one might

Fig. 3. As in Fig. 2, but for the square of near-equatorial jet spacing $\delta \mu_0$.  

ask why the same calculus cannot be applied to the summation of $1 - \mu_p^2$ farther to the left. The answer is that this term is $O(1)$, unlike the integral containing $U$ which is second order in $\epsilon_j$, and it is therefore necessary to include a second-order correction representing the difference between the integral of $1 - \mu^2$ and its discrete sum (this nuance is easily overlooked). The contribution represented by $(1/12)Nd\mu_0^2$ is not small relative to other terms of the same order on the lhs. Zeroth-order terms on the lhs cancel the rhs exactly, while the sum of second-order terms on the lhs (which must equal zero identically when $p = 0$) yields $C^2 = 2/(3\pi)$. It will be noted that this formulation of the matched asymptotics is similar to that of DS’08 but without invoking small $\mu_p$ anywhere in the derivation.

d. Asymptotic analysis, $p \neq 0$

Two complications arise for nonzero $p$: (i) the spacing of jets is nonuniform in $\mu$ and (ii) the summation contains irrational integrals of fractional powers of $1 - \mu_p^2$. Some effort is required to identify these integrals and all must be evaluated numerically (or if discovered in the table of integrals, would involve irrational constants and must be evaluated numerically (or if discovered in the table of integrals, would involve irrational constants and functions evaluated numerically by someone else). It proves convenient to rewrite (2.18) as a second-order summation error:

$$
\sum_{j=0}^{N-1} \delta \mu_0 (1 - \mu_j^2)^{-1-p} - \left( \mu_N - \frac{1}{3}\mu_N^3 \right) + \delta \mu_0^2 \left\{ \frac{1}{4}(p-1)I_2 + \frac{1}{2}p(p+1)I_3 + C^2I_1 \right\} + O(\epsilon_j^2) = 0, \tag{2.24}
$$

discarding the outer part ($\mu_N \leq \mu \leq \mu_p$) and defining

$$
I_1 = \int_0^{\mu_N} (1 - \mu^2)^{1/2-p} d\mu, \tag{2.25a}
$$

$$
I_2 = \int_0^{\mu_N} (1 - \mu^2)^{-3p} d\mu, \quad \text{and} \tag{2.25b}
$$

$$
I_3 = \int_0^{\mu_N} \mu^2 (1 - \mu^2)^{-1-3p} d\mu, \tag{2.25c}
$$

recalling once again that $U = (Cd\mu_0)^2$ as in (2.23b). The lhs is second order because a single power of $\delta \mu_0$ is balanced by $N$ in the limit $N \to \infty$. For the half-width of jet spacing, from (2.19),

$$
\epsilon_j = \frac{1}{2} \sqrt{\frac{U}{C}}(1 - \mu_i^2)^{-p} + O(\epsilon_j^2). \tag{2.26}
$$

The most important insight in (2.24) is that Newtonian calculus is invoked to replace discrete sums with their continuous integral equivalent, as done for one term in the $p = 0$ case [see (2.23b)] because for these integrals only, the difference between the discrete sum and integral is small and contributes to the next (third) order of error. [For the same reason it makes no difference whether the upper bound of continuous integrals in (2.25) is $\mu_p$ or $\mu_N$, although $I_3$ does not exist with the former bound when $\mu_p = 1$ and $p \geq 0$. The cases $I_1$ and $I_2$ exist when $\mu_p = 1$ in the entire range of $p$ considered here, and with the latter bound, all integrals exist when $\mu_N < 1$.] Just as before, this strategy must be avoided for the first term in (2.24), which is $O(1)$ to begin with, because the error of this integral’s discrete evaluation must be known. When $p = 0$ the error is known exactly from the arithmetic sum of $(j + 1/2)^2$ in (2.21); namely, the term $-N/12$ on the rhs. [The arithmetic sum (2.21) can be recovered for nonzero $p$ via a transformed coordinate $\nu = \nu(\mu)$ defined to enforce equal spacing. However, each term in the sum is then weighted by a variable amount.] What is the equivalent term when $p \neq 0$?

With no analytic answer to this question, the alternative (a numerical method) is tricky since (i) the “exact” value of the integral is known only up to machine precision and (ii) its discretized “staircase value” is to be differenced against this exact value, then multiplied by $\delta \mu_0^{-2}$ to be included among the integral terms in (2.24), which are $O(1)$. We are expected, then, to take the difference between two floating point numbers at suitably high precision, the difference becoming smaller with increasing $n$, and then multiply the difference by a large number to arrive at another number in the vicinity of 1/12. As shown next, this method actually works (with double precision) to the highest $n$ considered ($=22, 23$) for all negative $p$ considered and for $p$ slightly above zero. It should be noted that, while the exact value is a geometric entity (viz., the area beneath a curve) independent of its computation, its discrete approximation always depends on the details of the spacing—that is, the definition of integrand increments. In planetary FD terms, it is the piecewise constant PV staircase—a conservative rearrangement of continuous planetary vorticity linear in $\mu$—that discretizes the summation and precludes Newtonian calculus from solving the problem entirely.

Shown in Fig. 4 (left) are estimates of the fraction $f_p$ for nonzero $p$. At $p = 0$ the fraction 1/12 is predicted correctly by the numerical method in the limit $\mu_N \to 1$, while for negative $p$, the values are higher by a much as a factor of 2. The sloping curves are ambiguous, especially for small positive $p$, which bend down sharply to the right. However, the most extreme values (at right) occur at the upper end of the interior branch, where $\mu_N$ comes closest to the pole for any $p$. We are not interested in this
Asymptotic values, albeit with small uncertainties. As expected, errors diminish with increasing
the staircase. The inferred values are shown against their

to the integrals, with increments matching the steps of

and solving for

that “sphere filling” refers to

terior and exterior coalescence points. (Please recall

for negative

values of

the limit

finite span of

alescence points are distinct and remain separated by a
range of high

perpendicular fashion where it is almost flat. For the

(yellow vertical line) intersects the solution trajectory in

coalescence point. All indications suggest so. Indeed,

is exactly vertical in the limit

DS’08, their Figs. 8 and 9), the ascent becoming

dimensional. [For application of the Mercator coordinate to

2D Rossby wave propagation on a spherical surface, see
Hoskins and Karoly (1981).] As defined for zonally av-
eraged systems, the PV staircase is one dimensional. Nor
is latitude useful for equal spacing, as it suffers con-
traction of area approaching the poles; the underlying
conservations of angular momentum, potential vorticity,
etc. are described more naturally in \( \mu \) once the zonal
average is taken. The \( \eta \) coordinate is grossly stretched
whereas when \( p = -0.25 \) the \( \nu \) coordinate is weakly
stretched.

Obviously a true Mercator coordinate is not needed to
explain the isomorphism of the \( p = -0.25 \) sphere-filling
solution to the meridionally periodic beta plane of
point for the purpose of the asymptotic analysis because
the relevant values of \( C \) lie in the neighborhood of in-
terior and exterior coalescence points. (Please recall
that “sphere filling” refers to \( \mu_p = 1 \), while \( \mu_N \to 1 \)
in the limit \( n \to \infty \).) In the right panel of Fig. 4 are shown
values of \( C \) inferred by setting (2.24) identically to zero
and solving for \( C \), but using discretized approximations
to the integrals, with increments matching the steps of
the staircase. The inferred values are shown against their
actual values, and the error is zero along the diagonal
line. As expected, errors diminish with increasing \( n \), but
for negative \( p \), the errors are compact. The two cases of
special interest, \( p = 0 \) and \( p = -0.25 \), agree with their
asymptotic values, albeit with small uncertainties.

Agreement with the equatorial asymptotics (\( \mu_p \to 0 \))
at \( p = -0.25 \) is consistent with the ascent of the equa-
torial branch upward to the interior coalescence point
(see DS’08, their Figs. 8 and 9), the ascent becoming
nearly or exactly vertical at large \( n \). The equatorial limit

m

corresponds to \( (C, U) = (1/\sqrt{6}, 0) \) while the sphere-
filling point of intersection is directly above that (when
\( p = -0.25 \) or displaced to the right (when \( p > -0.25 \)). If
it can be shown that the equatorial branch for \( p = -0.25 \)
is exactly vertical in the limit \( n \to \infty \), the square root of 6
then assumes a new role as the spacing requirement for
the sphere-filling solution \( (p = -0.25) \) at the interior
coalescence point. All indications suggest so. Indeed,
the impression given by Fig. 2 is that (i) the solution
trajectory is tangent to the red vertical line at this special
value of \( C \). By contrast, (ii) the asymptote for \( p = 0 \)
(yellow vertical line) intersects the solution trajectory in
perpendicular fashion where it is almost flat. For the
range of high \( n \) shown, the interior and exterior co-
alescence points are distinct and remain separated by a
finite span of \( C \) even as their respective \( U \) minima

converge logarithmically to zero. The fraction \( f_p \) varies
along the corner of the solution trajectory because it
depends on the jet spacing, which also varies along the
trajectory even when \( p \) and \( \mu_p \) are fixed. Ranges of \( f_p \)
values for which the actual and inferred \( C \) agree best are
indicated by vertical bars in Fig. 4a.

A possible explanation of the former result (i) lies in
the isomorphism between the beta-plane and sphere-
filling solutions, both of which have a flat bottom enve-
lope: that is, identical retrograde parabolas. In a
stretched Mercator-like coordinate, the sphere-filling
solution for \( p = -0.25 \) in the limit \( n \to \infty \) resembles a
beta-plane solution extending progressively to larger \( y \).
To appreciate the subtle variations of spacing in the
DS’08 model it is eye opening to consider two values of
\( p \) outside the range of interest: \( p = -0.5 \) (latitude \( \theta \) such
that \( \mu = \sin \theta \)) and \( p = -1.0 \) (Mercator coordinate
\( \eta \) such that \( \mu = \tanh \eta \)). Both are stretched coordinates
relative to \( \mu \) but neither is germane to the staircase. The
Mercator coordinate was designed for a two-manifold or
“chart” of Earth’s surface so as to preserve horizontal
isotropy. [For application of the Mercator coordinate to
2D Rossby wave propagation on a spherical surface, see
Hoskins and Karoly (1981).] As defined for zonally av-
eraged systems, the PV staircase is one dimensional. Nor
is latitude useful for equal spacing, as it suffers con-
traction of area approaching the poles; the underlying
conservations of angular momentum, potential vorticity,
etc. are described more naturally in \( \mu \) once the zonal
average is taken. The \( \eta \) coordinate is grossly stretched
whereas when \( p = -0.25 \) the \( \nu \) coordinate is weakly
stretched.

Obvious
Dritschel and McIntyre (2008, hereafter DM’08); fortunately, a simpler line of reasoning suffices. This argument is astonishingly direct and agrees with the equatorial asymptotics also. The beta plane, whether at the equator or anywhere else, is a tangent approximation for the Coriolis parameter at a certain latitude. According to (2.1), the special value $p = -0.25$ and conjecture $C = 1/\sqrt{6}$ imply that (dropping subscript $j$)

$$\cos \theta \delta \theta = \delta \mu = \sqrt{6U\beta^*} \rightarrow \sqrt{\frac{6}{\Omega a}} \frac{1}{2} \beta^2 \frac{2}{3} \beta^* = (2b/a)\beta^*$$  

(2.27)

upon substitution of (5.6) of DM’08, setting $y = b$, where $b$ is the half-width of jet spacing, $\beta = (2\Omega/a) \cos \theta$, $\beta^* = \sqrt{1 - \mu^2}$, and $\Omega$ and $a$ are planetary rotation rate and radius in the usual notation. Approaching the pole, with $p = -0.25$, (dimensional) jet spacing $a\delta \theta = 2b$ is allowed to increase slowly in $\theta$ (or decrease slowly in $\mu$) while the zonal wind structure of the staircase (daisy chain of parabolas) remains isomorphic to the beta plane. These slow variations accompany ever-contracting parabolas in the limit $n \rightarrow \infty$, so the daisy chain is endless asymptotically (DM’08) even though the actual distance from equator to pole is finite.

 Needless to say, the “periodic extension” of the local PV-mixing event envisaged by DM’08 is required. Their solution so extended (5.6) contains the square root of 6 (or of 3) implied by the equatorial asymptotics, while their single-event solution in (5.5) does not. Local tangency is the essence of the beta plane and exactly what is implied in the limit $n \rightarrow \infty$. By the same reasoning, the equatorial asymptotic of DS’08 is a special case of beta-plane thinking, the only complication being that $m_j^2$, instead of $m_j$, plays the role of “midpoint,” as might be expected for an equatorial beta plane.

e. Global kinetic energy and mode transition

Global KE is frame relative and requires definition of a resting state, which is problematic on the gas giants. This quantity, as it were, minimizes the moment of inertia of zonal wind about its zero reference line, when such exists. Sharp minima occur in Fig. 5 where prograde and retrograde maxima differ least from their global average. For this reason, owing to its flat bottom envelope, $p = -0.25$ has the least KE of all in the left half of the diagram. Wind profiles in the right half of the $(C, U)$ plane have smaller prograde jet spacing than the asymptotic value but at the expense of much larger KE, attributable not to the local neighborhood, but to the
enormous difference between interior westerlies and exterior easterlies. Such profiles resemble the cartoon character Dilbert, although the “jets” of his hairdo are retrograde. (Caricature aside, gas giants in our solar system do exhibit large differences in tropical versus extratropical jet magnitude, though lacking the contrast of interior and exterior winds at large $C$. Such differences on Jupiter and Saturn likely arise from outcropping tangent cylinders of the respective metallic cores.)

Global kinetic energy is a PV staircase functional: a scalar measure comprising (possibly many) discrete states of local steps/risers in the staircase. A simpler functional is global angular momentum, equivalent to contours of $\mu_p(C, U)$ in the interval $(0, 1]$. Sphere-filling solutions are exclusively at $\mu_p = 1$. Such contours correspond in the above terminology to solution trajectories in the $(C, U)$ plane. States may evolve along such trajectories while conserving global angular momentum but not global kinetic energy. The parameter space between interior and exterior coalescence points is interesting because a sharp minimum of KE exists in this region. In numerical solutions forced by random inputs of energy but not angular momentum at high horizontal wavenumber, a system state with regular-spacing constraint follows a trajectory to higher KE until such a state is no longer realizable or unrealistic for other reasons.

A leftward trajectory in the $(C, U)$ plane is realistic—for example, departing the KE minimum for a particular $n$ and traveling upward through the interior coalescence point toward the end of the interior branch. Two types of trajectories are portrayed in Fig. 6. (i) One preserves hemispheric symmetry by linking up with the next solution trajectory at $(n - 2)$. For $p = 0$, the intersection is exact, occurring at the exterior coalescence point. For $p = -0.25$ an energy impulse is required at the transition (see insert). In either case the transition takes the most poleward prograde jet closer to the pole until the next lower (even or odd) mode is formed. The equatorial state is unaffected. (ii) A second type (bottom panels) requires symmetry flip, from $n$ to $(n - 1)$, implying near-equatorial alteration between super and subrotating states. [Such alternation is the hallmark of the quasi-biennial oscillation (QBO) of the equatorial lower stratosphere; see Dunkerton (2017) and references therein. In the QBO, vertical transports of angular momentum play an important and probably essential role.] According to Fig. 5, this type of trajectory is almost seamless at $p = -0.25$, with a tiny gap in KE values. For other values of $p$, the gap is larger. The symmetry flip is quite obvious at the equator, where a new jet comes or goes. Contrary to our first impression of “waves on a choppy sea”—unlikely realizable—close inspection indicates that midlatitude jets require only a small meridional displacement during the transition.

**Fig. 6.** Mode-$n$ transitions obtained by following solution trajectories from the exterior coalescence point, leftward and upward along the interior branch until coming in close proximity to the next lower mode (a), (b) $(n - 2)$ or (c), (d) $(n - 1)$. The latter require a symmetry flip, most obvious at the equator, where transitions between super- and subrotating states occur. Insert in lower-left corner of each panel shows the evolution of global KE from front to back. The scaling and direction of “time” are completely arbitrary, and the rate may vary.
Spontaneous jet formation in a retrograde parabola was demonstrated by Scott and Tissier (2012), a time reversal of the near-equatorial behavior in Fig. 6d, as it were. Intriguing as such trajectories are, with simple morphology in physical space, they have otherwise not yet been simulated numerically. With energy input, an alternative and seemingly preferred pathway is for adjacent prograde jets to come closer together, albeit very slowly, and eventually to merge (Scott and Polvani 2007). In this way the global staircase configuration accommodates a continuous random input of energy at high horizontal wavenumber. It is not surprising that merger depends on initial proximity, if indeed that is the lesson taught by these experiments. Long-range interaction (e.g., via Rossby waves and instabilities) is implied by the simulated merger (DS’08). The shorter the range, the faster the interaction? Nevertheless, we find the state of understanding unsatisfactory at present. Somewhat bothersome is that, in light of comments regarding global KE, any irregular spacing of jets is not a minimum energy or “ground state” of the system. Should not a natural system find itself in such a ground state eventually, if not initially?

Further numerical study is needed to address this question and to explore the two preferred types of solution trajectory identified in Figs. 5 and 6. Whether or not the examples of Fig. 6 are realizable numerically, they provide a compact depiction of possible regular staircases in the most relevant range of C. In lieu of global KE, a better diagnostic approach is to exploit objective Lagrangian methods (Rutherford and Dunkerton 2017, manuscript submitted to J. Atmos. Sci.). Eulerian kinetic energy is not objective and therefore is not optimal for theoretical analysis of energy–momentum states in fluid continua. Indeed, the physical transitions depicted in Fig. 6 (and their respective inserts showing KE) require conversions between eddy and mean energies, or equivalently, conservation of Lagrangian metrics such as pseudoenergy and Kelvin’s circulation (Andrews and McIntyre 1978a,b; Dunkerton 1980).

3. Conclusions

For barotropic flow in spherical geometry, the ideal potential vorticity staircase with flat steps and vertical risers exhibits a relationship between prograde jet strength and spacing such that, for regular spacing, the distance between adjacent jets is given by a suitably defined “Rhines scale” multiplied by a positive constant equal to $\sqrt{6}$. This result was obtained previously by DS’08 in the equatorial limit of spherical geometry and by DM’08 in periodic beta-plane geometry. An improved asymptotic method has been devised to explain the strength–spacing relationship in sphere-filling solutions.

The staircase model of DS’08 consists of equal prograde (westerly) jets bounding parabolas of variable depth, which may be easterly or westerly at their core. In sphere-filling solutions, a small gap is allowed between the last prograde jet and the pole. Unlike the equatorial asymptotic method of DS’08, which confined the staircase proper to an ever-smaller range of latitudes, the sphere-filling asymptotic method presented herein maintains a small gap near the pole. For the largest negative value of exponent $p = -0.25$ the classic result $C = 1/\sqrt{6}$ is obtained. A simple argument explains why this result agrees with the asymptotics of DM’08 and DS’08. In essence, those beta-plane models are isomorphic to the sphere-filling solution for this special value of $p$.

The concept of PV staircase functional is introduced, featuring global kinetic energy and angular momentum as examples. Global KE exhibits a deep minimum between the interior and exterior coalescence points, the latter terminology borrowed from DS’08. Four examples of mode-$n$ transitions are presented for a system with monotonic energy input but with angular momentum held constant. In these examples the number of jets decreases slowly with time, but migrating poleward. Such behavior is very different from simulated transitions in which adjacent jets merge on a very long time scale.

As shown by the author at the 2015 Atmospheric and Oceanic Fluid Dynamics (AOFD; Dunkerton 2015) meeting of the AMS, any perturbation to $\mu$, of a regular staircase increases the kinetic energy because the diminution of wind speed in one retrograde parabola is overcompensated by an increase of KE in the adjacent one. Similarly, it was shown that staircase formation from rest requires less KE input if allowance is made (via the Taylor identity) for upgradient transport of PV in addition to lateral stirring. From this we infer that regular of prograde jets constitutes a relative minimum energy state; the question is raised as to why numerical simulations do not maintain this state, given the propensity of physical systems to relax to their ground state when left undisturbed. Further numerical study is warranted with experiments constructed to test the viability of mode transition pathways identified here, and/or to demonstrate why merger of adjacent jets is a preferred pathway, if indeed that earlier result is robust.

Finally, it is suggested that the asymptotic method, which is not trivial numerically, may be useful nevertheless to explore this question and others that arise along the solution trajectory bounded below by the exterior coalescence point, and the interior branch, through the interior coalescence point to its end where...
the identified mode transitions occur. The utility of the method is not so much that one can derive results valid for arbitrarily high mode index \( n \), but that the asymptotic results are quantitatively useful for \( n \) of \( O(10–20) \) as observed in planetary atmospheres nearby.

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## APPENDIX

### Further Details of the Sphere-Filling Method

**a. Differences with respect to equatorially trapped asymptotics**

The asymptotic analysis of DS’08 requires that the latitudinal extent of the staircase shrink to zero; that is, \( \mu_X, \mu_p \to 0 \). Solution trajectories in that paper were depicted with \( y \) axis linear in \( U \), rather than logarithmic as in this paper. The above limit is tantamount to \( U \to 0 \), at which point the trajectories \( C = C(U) \) converge approximately to the same value, weakly dependent on \( n \) and increasingly independent of \( p \). For all values of \( p \) considered, the limiting value of \( C \) is less than its value at the interior coalescence point (but only slightly so when \( p = -0.25 \)). By contrast, all results shown here (Figs. 2, 3, and 5) imply asymptotic values of \( C \) independent of \( U \), that is, vertical lines when the \( y \) axis is logarithmic in \( U \).

The logarithmic display gives the illusion of a contradiction with DS’08, which is resolved easily by noting that for the sphere-filling results shown here, \( \mu_p = 1 \), and convergence is with respect to increasing mode index \( n \). Even though \( U \to 0 \) in this limit, as in DS’08, it does so without reduction of \( \mu_p \).

For the purpose of functional analysis, it is helpful to keep in mind that solution trajectories \( C = C(U) \) may be regarded also as level contours of a function \( \mu_p = \mu_p(U, C) \) such that the trajectories shown here correspond uniquely to \( \mu_p = 1 \). Retaining the flexibility of variable \( \mu_p \) aids the functional analysis, for example, by allowing differentiation with respect to \( \delta\mu_p \), as done in the equatorially trapped asymptotics of DS’08; see the discussion at the upper left of page 1124 leading to (2.49) of that paper. On the preceding page the point was made also that the equatorially trapped asymptotics become independent of spacing exponent \( p \) to second order in \( \mu \), thanks to a fortuitous cancellation of terms in the Taylor series expansion. The latter result is reasonable geometrically because the notion of jet spacing dependent on \( \mu \) is of little consequence when all jets are located near the equator.

**b. Absolute vorticity, relative zonal wind, and kinetic energy**

For the record, the piecewise constant absolute vorticity (\( \zeta_a \)) segments of the staircase, derived from the piecewise linear angular momentum (\( m \)) segments of the corresponding “necklace of bamboo,” are

\[
\hat{\xi}_{aj} = \xi_{aj/2} = -\frac{m_{j+1} - m_j}{\mu_{j+1} - \mu_j} = -\frac{\mu_j^2 - \mu_{j+1}^2 + U \left[ \sqrt{1 - \mu_{j+1}^2} - \sqrt{1 - \mu_j^2} \right]}{\mu_{j+1} - \mu_j}
\]

\[
= (\mu_j + \mu_{j+1}) + U \frac{\mu_j}{\sqrt{1 - \mu_j^2}} + O(\epsilon_j) = 2 \sin \hat{\theta}_j + U \tan \hat{\theta}_j + O(\epsilon_j)
\]

(A.1a)

or, in dimensional terms,

\[
\zeta_{aj} = 2\Omega \sin \hat{\theta}_j + \frac{\bar{\pi}_m}{a} \tan \hat{\theta}_j + O(\epsilon_j)
\]

(A.1b)

in the usual meteorological notation, and \( \bar{\pi}_m \) refers to the westerly jet maxima, assumed uniform across the sphere (DS’08). It can be noted that the \( O(\epsilon_j) \) correction arises from the relative velocity/vorticity and not the planetary component: since the planetary component of absolute vorticity varies linearly in \( \mu \), the center position \( \bar{\mu}_j \) determines the planetary component at this location exactly: that is, \( \zeta_{aj} = 2\bar{\mu}_j \). To \( O(\epsilon) \), the absolute (planetary plus relative) vorticity is obtained from the corresponding terms evaluated at \( \hat{\theta}_j \).

Relative mean zonal wind profiles on stair steps are obtained by dividing the deviation of absolute angular momentum from the resting parabola

\[
\Delta m = m_j + \frac{m_{j+1} - m_j}{\mu_{j+1} - \mu_j} (\mu - \mu_j) - (1 - \mu_j^2)
\]

(A.2a)

[see (2.57a) of DS’08] by the local moment arm \( \sqrt{1 - \mu_j^2} \) for \( \mu \in [\mu_j, \mu_{j+1}] \). Neglecting a small local variation of the moment arm, these wind-profile segments may be visualized as small parabolas with variable bottom from
pole to pole (local minima which may, or may not, be
easterly) spanning the gap between westerly maxima.
Whereas the large resting parabola of \( m \) is concave
downward, the small zonal wind parabolas are concave
upward, as shown in Fig. 22 of DS'08. The discussion after
(2.57a) in that paper invoked some equatorial asymptot-
tics (needlessly, as it turns out) and the desired sphere-
filling alternative [identical in essence to (A.1a)] is

\[
\Delta m = 1 - \mu_j^2 - \epsilon_j^2 + \frac{1}{2} U \sqrt{1 - \mu_j^2} \left( \sqrt{1 - x_+} + \sqrt{1 - x_-} \right) \left( -4 \mu_j \delta_j + U \sqrt{1 - \mu_j^2} \left( \sqrt{1 - x_+} - \sqrt{1 - x_-} \right) \right)^2 \left( \frac{\delta_j^2}{2 \epsilon_j} \right) - 1 \\
+ (\mu_j^2 + 2 \mu_j \delta_j + \epsilon_j^2) = \frac{1}{2} U \sqrt{1 - \mu_j^2} \left( \left( \frac{1}{2} \delta_j \right) \sqrt{1 - x_+} \right) + \left( \frac{1}{2} \delta_j \right) \sqrt{1 - x_-} + \delta_j^2 - \epsilon_j^2 \tag{A.2b}
\]

without approximation, where \( \delta_j = \mu - \mu_j \) and \( \epsilon_j \) are
given by (2.6), such that \( \delta_j \in [\epsilon_j, \epsilon_j] \). This equation differs from (A.1a) only
superficially insofar as (i) the \( m \) deviation is evaluated about \( \mu_j \), which (unlike in DS'08)
is now the exact midpoint between \( \mu \) and \( \mu_{j+1} \), and (ii) the
variation of \( m \) between westerly jets is exactly linear
in \( \mu \), so the value of \( m \) at \( \mu_j \) is (for the same reason)
approximately the average of the adjacent westerly jet values \( m_j \)
and \( m_{j+1} \). This average value [employed in the dis-
cretized angular momentum conservation law; see
(2.18)] is contained in the first batch of terms on the lhs
of (A.1b). The second batch, varying linearly in \( \delta_j \), des-
cribes the poleward decrease of \( m \) on the stair step.
[The linear deviation averages to zero in each segment,
explaining why the average stair step \( m \) suffices for the
global summation (2.2) and (2.18).] The last batch of
terms contains the resting profile of \( m \) spanning this in-
terval; parabolic in \( \delta_j \), as in \( \mu \) itself.

The rhs of (A.1b) describes a parabola in the local
independent variable \( \delta_j \) since the remaining quantities
are constants evaluated at or near \( \mu_j \). This minor re-
formation of (A.1a) into (A.1b) offers the same ad-
vantages to the sphere-filling asymptotics as already
noted for the angular momentum conservation law in
(2.18) and jet-spacing rule in (2.19). The \( m \) deviation
is second order in \( \epsilon_j \):

\[
\Delta m = U \sqrt{1 - \mu_j^2} + \delta_j^2 - \epsilon_j^2 + O(\epsilon_j^3), \tag{A.3}
\]

noting that \( \delta_j = O(\epsilon_j) \) and \( U = O(\epsilon_j^2) \). Equation (A.2)
describes a local parabola about \( \mu_j \), which, to lowest
order, generates identical values of \( \Delta m \) at its endpoints
\( \delta_j = \pm \epsilon_j \). This is not true when the higher-order cor-
rections are retained, owing to a small change in the moment
arm from one endpoint to the other. The corresponding
values of maximum zonal wind remain exactly identical,
however, as required by the model (constant amplitude of
westerly jets spanning the staircase). This is seen at once
from (2.7a) and (2.7b) or (2.9a) and (2.9b) and there is no
point in proving the point from (A.1b) or (A.2), which
would require a backward step or two.

Solution trajectories for a particular mode \( n \) con-
touring \( \mu_p = 1 \) in the \( (C, U) \) plane describe staircase
profiles of mean zonal wind and \( m \) deviation with com-
parable maximum (westerly and easterly) amplitudes
over the range of \( C \) considered by DS'08. But it was
shown in Fig. 5 that the kinetic energy (KE) associated
with these profiles varies greatly within the same mode,
affording guidance for how staircases may evolve in
realistic planetary atmospheres. In the sphere-filling
asymptotics, each quasi-parabolic segment of relative
mean zonal wind contributes a finite meridional
increment of kinetic energy as measured in the resting
planetary frame:

\[
\hat{K}_j = \int_{\mu_j}^{\mu_{j+1}} \frac{1}{2} \pi \mu^2 d\mu. \tag{A.4}
\]

As it turns out, a global minimum KE state exists be-
tween the interior and exterior coalescence points
(Fig. 5).

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