ABSTRACT: In a recent paper Rousseau-Rizzi and Emanuel (2019) presented a derivation of an upper limit on maximum hurricane velocity at the ocean surface. This derivation was based on a consideration of an infinitely narrow (differential) Carnot cycle with the warmer isotherm at the point of the maximum wind velocity. Here we show that this derivation neglected a significant term describing the kinetic energy change in the outflow. Additionally, we highlight the importance of a proper accounting for the power needed to lift liquid water. Finally, we provide a revision to the formula for surface fluxes of heat and momentum showing that, if we accept the assumptions adopted by Rousseau-Rizzi and Emanuel (2019), the resulting velocity estimate does not depend on the flux of sensible heat.

KEYWORD: Hurricanes

1. Introduction

Rousseau-Rizzi and Emanuel (2019, hereafter RE) presented a new derivation of surface potential intensity (PI). They based this on consideration of an infinitely narrow Carnot cycle in the vicinity of maximum wind speed. Cyclonic storms, especially powerful hurricanes, represent a serious threat to human lives in many regions of the world but our ability to understand and anticipate these storms remains incomplete. Finding theoretical constraints on maximum hurricane velocities is an important goal, and we welcome the new work and its subsequent discussion by Montgomery and Smith (2020) and Rousseau-Rizzi and Emanuel (2020).

In their derivation, RE considered a configuration of closed airstreamlines as shown in Fig. 1a. Montgomery and Smith (2020) pointed out that, for their derivation to be valid, the air does not have to actually move as shown in Fig. 1b; in particular, it does not have to actually move along the warmer isotherm from point B’ to point B. Here we reconsider these arguments and show that the streamline configuration does matter for the derivation of RE and that accounting for realistic air motion (see Fig. 1 of Makarieva et al. 2019) leads to the appearance of a significant term characterizing the outflow region.

Furthermore, Rousseau-Rizzi and Emanuel (2020) clarified that RE’s derivation of surface PI assumed reversible thermodynamics whereby the total water content of air parcel \( q_r \) is not supposed to change. RE neglected the last term in their Eq. (13). This term is proportional to \( dq_r/dt \) and, in the general case, describes the power needed to lift liquid water. If \( q_r \) is constant, this term should be zero and could be discarded as RE did. However, Sabuwala et al. (2015) estimated this term in the real atmosphere to be significant leading to up to a 30% reduction of PI. This would make RE’s assumption of reversible thermodynamics of limited relevance for real PIs. We discuss the other available estimate of the power needed to lift liquid water by Makarieva et al. (2018), which revises the estimate of Sabuwala et al. (2015) and shows the corresponding term to be small thus restoring the practical relevance of RE’s derivation in this aspect.

Finally, Montgomery and Smith (2020) question how RE’s resulting formula for surface PI, RE’s Eq. (15), is obtained from the consideration of the differential Carnot cycle summarized in RE’s Eq. (14). Indeed the expressions in those equations have different units, W m\(^{-3}\) for the integrand in RE’s Eq. (14) versus W m\(^{-2}\) in RE’s Eq. (15). In their reply to Montgomery and Smith (2020), Rousseau-Rizzi and Emanuel (2020) did not provide an explicit derivation to show how RE’s Eq. (15) can be derived from RE’s Eq. (14). We clarify the physical assumption behind this transition and show that, once the definition of entropy is explicitly considered for the warmer isotherm of the differential Carnot cycle, the resulting revised formula relates surface PI to latent heat flux only and PI is independent of the flux of sensible heat.

2. Integrals over closed contours

RE noted that the material derivative of pressure, \( dp/dt \), enters both the definition of the material derivative of moist entropy \( ds/dt \) [Eq. (9) of RE] and the scalar product of the equation of motion with the three-dimensional velocity \( V \) [Eq. (10) of RE]:
To arrive at their Eq. (14), RE assumed that the integrals over closed contours of one another, RE concluded that, along the infinitesimal inner loop $B^*$, the contour integrals of $Tds/\partial t$ and of the friction power $\mathbf{F} \cdot \mathbf{V}$ ($\mathbf{F}$ is the “frictional source of momentum”) coincide, as summarized by RE’s Eq. (14). To arrive at their Eq. (14), RE assumed that the integrals over closed contours of any $dX/\partial t$ (in particular, of $X = |\mathbf{V}|^2$) are zero.

Montgomery and Smith (2020) pointed out that RE did not indicate the integration variables in their integrals. In their reply Rousseau-Rizzi and Emanuel (2020) did not clarify this omission. In a steady state, material derivative is defined as follows:

$$
\frac{dX}{\partial t} = \mathbf{V} \cdot \nabla X.
$$

The integral of the material derivative over a closed contour is equal to zero only if the contour is a streamline. Moving with the parcel over the contour and taking into account that $\mathbf{V} = d\mathbf{A}/\partial t$, where $d\mathbf{A}$ is directed along the streamline, one has for any scalar quantity $X$:

$$
\int \frac{dX}{\partial t} = \int \mathbf{V} \cdot \nabla X \, dt = \int d\mathbf{A} \cdot \nabla X = \int d\mathbf{A} \cdot \nabla \mathbf{A} = \iint dX = 0.
$$

If the closed contour is not a streamline, the integral is not zero. Since streamlines are parallel to velocity and since velocity is unambiguously defined at each point, there cannot be two different streamlines emanating from or entering a single point, like B’ and C’ in Fig. 1 of RE (Fig. 1a). The configuration shown in Fig. 1a is therefore impossible: the inner “differential” loop B’BC’B’ is not composed of streamlines or their parts. A realistic configuration of streamlines is shown in Fig. 1b.

The point about the unrealistic configuration of streamlines was put forward by Montgomery and Smith (2020). However, they did not specify the consequences of this assumption for RE’s derivations. As we discuss below, the main implication is that the integral of $d|\mathbf{V}|^2/\partial t$ over $d\mathbf{A}$ for the inner “differential” cycle is not zero.

There is an essential difference between Eqs. (1) and (2) [Eqs. (9) and (10) of RE]. Equation (1) represents a definition of entropy, to which the operator $d\mathbf{A}/\partial t$ has been applied. Instead of $d\mathbf{A}/\partial t$, one can apply to the definition of entropy an operator $d\mathbf{A} \cdot \nabla$, where $d\mathbf{A}$ is an arbitrary vector (not necessarily parallel to velocity vector). Formally it is equivalent to replacing $d\mathbf{A}/\partial t$ in Eq. (1) with $d\mathbf{A} \cdot \nabla$.

If one assumes, following RE, that the inner loop B’bcC’b is a Carnot cycle, with two adiabats bc and B’C’ and two isothersms B’b and cC’, one can write an analog of Eq. (1) for this loop as

$$
\int (1 + q_0) \alpha \, dp = \int T \frac{ds^*}{\partial t} = \int_{w}^b \delta Q.
$$

Here $q_0$ is the mixing ratio for total water, $\alpha = 1/\rho$, $\rho$ is air density, $s^*$ is saturated moist entropy corresponding to relative humidity $\mathcal{H} = 1$, $\delta Q = Tds^*$ is heat increment, and $e_c = (T_b - T_b)/T_b$ is Carnot efficiency. The temperatures $T_b$ and $T_c$ correspond to the points b and c, respectively (see Fig. 1b). The first integral in Eq. (5) represents work of the cycle per unit mass of dry air.

Montgomery and Smith (2020) criticized RE for not accounting for ice melting. However, Eq. (5) is valid for any Carnot cycle, whether or not it includes ice. The Carnot cycle efficiency does not depend on the latent heat of vaporization.
and (10)—yields is analogous to RE eliminating work associated with lifting liquid water and changing its entropy. See Eq. (1). So if the cycle is reversible, the result is uninfluenced by ice.

Importantly, Eq. (5) relates state variables and is thus valid independent of whether the air actually moves along the considered closed contour or not. Indeed, Rousseau-Rizzi and Emanuel (2020) pointed out that, for the thermodynamics, it is not essential whether the air actually moves from point B' to point B.

But Eq. (2), unlike Eq. (1), is not a definition; it is a law of motion. Replacing \( \frac{d\alpha}{dt} \) with \( \alpha \cdot \nabla \) and \( \nabla \) with \( \partial \) in Eq. (2) is only possible if \( \partial \) is part of a streamline. This replacement transforms Eq. (2) into the Bernoulli equation that is only valid on a streamline:

\[
\frac{1}{2} \partial \left| \nabla \right|^2 = -\alpha dp + \mathbf{F} \cdot \partial - g \partial z. \tag{6}
\]

Assuming, again following RE, that along the streamlines B'C and B'C there is no friction, \( \mathbf{F} = 0 \), and applying Eq. (6) to these two streamlines, one can express the integral of \( \alpha dp \) over the closed contour B'bcC'B' as kinetic energy. For a closed streamline with \( \mathbf{F} = 0 \), one has from the Bernoulli equation

\[
\frac{1}{2} \partial \left| \nabla \right|^2 = -\oint_\partial \left( \frac{1}{2} \left| \nabla \right|^2 + g \partial z \right) \tag{11}
\]

(For details, see Makarieva et al. 2019). This term corresponds to the second term in the right-hand side of Eq. (13) of RE who wrote, "The last term in Eq. (13) represents the irreversible entropy loss associated with lifting water mass against gravity and changing its kinetic energy. It is quantitatively small compared to the other terms in Eq. (13) and we henceforth neglect it."

Rousseau-Rizzi and Emanuel (2020) clarified that in their derivations RE assumed reversible thermodynamics under which \( q_\alpha \) should not change. In this case \( dq_\alpha = 0 \) and the corresponding term is zero. In the real atmosphere condensed moisture is removed from the rising air parcels by precipitation and \( q_\alpha \) is not constant. If the corresponding term in RE's Eq. (13) were large, neglecting it would limit the application of their results to the real atmosphere.

RE did not support their statement about the term being quantitatively small. Emanuel (2018) quoted two studies that evaluated how the estimate of the potential intensity of tropical cyclones can be changed by accounting for lifting water, those of Sabuwala et al. (2015) and Makarieva et al. (2018). Among the two, Sabuwala et al. (2015) indicated that lifting water can decrease potential intensity by as much as 30% This runs counter to the statement of RE that the corresponding term is "quantitatively small"; RE did not quote Sabuwala et al. (2015).

Makarieva et al. (2018), on the other hand, showed that the analysis of Sabuwala et al. (2015) was in error and reported the first ever, to our knowledge, estimate of the gravitational power of precipitation (lifting water) in tropical cyclones. In making use of the size of this term, an essential aspect of their analyses, RE did not cite where this result had been established.

3. The power to lift water

The second term in the right-hand side of Eq. (10) represents work associated with lifting liquid water and changing its (sublimation, melting) or heat capacity \( c_v \) of liquid water that all enter the definition of entropy; see Eq. (1). So if the cycle is reversible, the result is uninfluenced by ice.

In the last equality of Eq. (7) the hydrostatic equilibrium \( \alpha dp/\partial z = -g \) was applied along the vertical path C'C.

Eliminating \( \alpha dp \) between Eqs. (5) and (7)—this procedure is analogous to RE eliminating \( dp/\partial t \) between their Eqs. (9) and (10)—yields

\[
\int_T \partial s = -\int_B \alpha dp - \frac{V_x^2 - V_x^2}{2B} + \frac{V_y^2 - V_y^2}{2B} - \oint_\partial q_\alpha dp. \tag{8}
\]

Assuming hydrostatic equilibrium at point B, assuming that we are in the region of maximum wind velocity (\( \partial |\nabla|^2 \partial = 0 \)) and considering the streamline b'Bb in the limit \( B' \rightarrow B' \) of two infinitely close streamlines, with use of the Bernoulli Eq. (6) one obtains

\[
\int_B^{B'} \left( \alpha \frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial z} \partial z + g \partial z \right) = \int_B^{B'} \mathbf{F} \cdot \partial = \int_B^{B'} \alpha dp. \tag{9}
\]

Using Eq. (9) one can write Eq. (8) as

\[
\lim_{B' \rightarrow B} \int_T \partial s = -\int_B \mathbf{F} \cdot \partial + \oint_\partial q_\alpha dp + \frac{V_x^2 - V_x^2}{2}, \tag{10}
\]

The first integral in the right-hand side is taken along the streamline b'Bb. This equation is analogous to RE's Eq. (13) but it correctly takes into account the change of kinetic energy in the outflow region. Makarieva et al. (2019, see their appendix C) showed that the last term in Eq. (10), neglected by RE, is significant when the outflow radius \( r_c \) (defined in the above derivation as the radius where the ascending air reaches the tropopause, i.e., the vertical isotherm \( \partial T/\partial z = 0 \)) is close to the radius of maximum wind, \( r_c \sim r_0 \). The implication is that RE's derivation should be less relevant when the rising air reaches the tropopause close to the radius of maximum winds.

4. Relationship between surface and volume fluxes

Montgomery and Smith (2020) noted that RE did not explain how their final Eq. (15) relating surface fluxes of heat and
turbulent friction was obtained from their Eq. (14). Here we briefly clarify and revise such a derivation. Following RE, we neglect the last two terms in Eq. (10). Then, by using the last equality in Eq. (5), Eq. (10) can be written as

\[ e_c \lim_{b \to \infty} \int_0^b \delta Q = - \int_0^b F \cdot dA. \]  

(12)

Under the additional assumption that the adiabats below B’b are vertical, such that only horizontal air motion is associated with heat input into the air parcel, the above equation can be interpreted as follows: any time an air parcel moves from b’ to b, the work of the friction force \(-F \cdot dA\) equals \(e_c\) times the heat the air parcel receives. The units of these variables are joule per kilogram of dry air.

Assuming that the same ratio characterizes the surface fluxes of heat and momentum, one can conclude that the surface flux of momentum \(D = \rho C_{d} V^2\) equals \(e_c\) times the surface flux of heat \(J = \rho C_{k} V(k_s^e - k)\) (W m\(^{-2}\)):

\[ e_c = -\frac{F \cdot dA}{\delta Q} = \frac{D}{J}, \]  

(13)

where \(C_d\) and \(C_k\) are surface exchange coefficients, \(k_s^e\) is saturated enthalpy at surface temperature \(T\), and \(k\) is the actual enthalphy in the near-surface layer. This assumption is key for RE’s derivation: RE’s Eq. (15) could not have been obtained otherwise from their Eq. (14). A similar relationship between surface-specific and volume-specific energy fluxes was adopted by Makarieva et al. [2019, their Eq. (18)]. We consider it reasonable. Indeed, if air motion above the boundary layer is adiabatic, all the surface heat flux should be accommodated into the volume of air parcels moving above the surface within the boundary layer.

In their Eq. (15), RE formulated the surface heat flux as the flux determined by the difference in enthalpies plus the so-called dissipative heating equal to the flux of momentum \(D\). However, inspection of Eq. (12) reveals that the heat increment \(\delta Q\) already contains the work of the friction force, which cannot be accommodated into the heat input yet another time; see Eq. (9):

\[ \int_0^b \delta Q = \int_0^b (L_e dq - adp) - \int_0^b L_e \frac{\partial q}{\partial r} dr - \int_0^b \delta Q = \int_0^b F \cdot dA. \]  

(14)

Here \(L_e\) is the latent heat of vaporization and it is assumed that \(q_s \ll 1\). Heat input from the surface arrives to within an air parcel in two forms: latent heat \(L_e dq\) that is associated with change \(dq\) of the water-vapor mixing ratio and sensible heat \(-adp\) that the air parcel accommodates by expansion (to remain isothermal). Thus, if, as Eq. (9) prescribes, \(-adp\) is equal to the work of friction \(-F \cdot dA\), and if all this work of friction dissipates to heat within the air parcel, no more heat can be accommodated in sensible form from the ocean. As we previously argued, there is no enhancement of hurricane intensity due to dissipative heating (see discussions by Makarieva et al. 2010; Bister et al. 2011; Bejan 2019, 2020).

Using Eq. (14) the assumed relationship in Eq. (13) can be rewritten as

\[ 1 - e_c = \frac{-F \cdot dA}{L_e (aq/\partial r) dr} = \frac{-F \cdot V}{J_L}, \]  

(15)

where \(J_L = \rho C_k |V| (q_s^e - q)\) is the surface flux of latent heat, \(q_s^e\) is the saturated vapor mixing ratio at sea surface temperature, \(q\) is the vapor mixing ratio in the near-surface layer, and \(u = dr/dt\) is radial velocity. The third ratio in Eq. (15) is the ratio of the volume-specific fluxes of frictional dissipation and heat consumption (W m\(^{-2}\)).

One essential point is that RE assumed \(F = 0\) everywhere above B’B’ on their contour (Fig. 1a). This assumption can be valid if point B’ is chosen sufficiently close to the top of the boundary layer, above which friction is commonly assumed to be negligible. If point B’ is located at the top of the boundary layer, such that friction tends to zero, \(F \to 0\), in the considered limit \(b' \to B'\), the second and third ratios in Eq. (15) can nevertheless remain finite (not zero) if the air leaving the boundary layer has zero radial velocity \(u\), i.e., if \(|dr/dt| \to 0\) as \(b' \to B'\). In the view of Eq. (15) the revised version of RE’s Eq. (16) becomes

\[ |V|^2 = \frac{T_s - T_c}{T_c} \frac{C_k}{C_d} L_e (q_s^e - q), \]  

(16)

where \(T_s\) and \(T_c\) are temperatures of the warm and cold isotherms, respectively. The wind speed \(|V|\) at the radius of maximum winds, vapor mixing ratio \(q\), and the exchange coefficients \(C_k\) and \(C_d\) pertain to 10-m altitude. This expression relates latent heat input to losses due to friction and avoids consideration of the temperature difference between the sea surface and the adjacent air, as explained in detail by Makarieva et al. (2019).

REFERENCES


