Does Balance Dynamics Well Capture the Secondary Circulation and Spinup of a Simulated Hurricane?

MICHAEL T. MONTGOMERY\textsuperscript{a} and JOHN PERSING\textsuperscript{a}

\textsuperscript{a} Naval Postgraduate School, Monterey, California

(Manuscript received 22 September 2019, in final form 25 August 2020)

ABSTRACT: This study investigates a claim made by Heng et al. in an article published in 2017 and intimated soon after in their article published in 2018 that axisymmetric “balanced dynamics can well capture the secondary circulation in the full-physics model” during hurricane spinup. Using output from a new, convection-permitting, three-dimensional numerical simulation of an intensifying hurricane, azimuthally averaged forcings of tangential momentum and heat are diagnosed to force an axisymmetric Eliassen balance model under strict balance conditions. The balance solutions are found, inter alia, to poorly represent the peak inflow velocity in the boundary layer and present a layer of relatively deep inflow extending well above the boundary layer in the high-wind-speed region of the vortex. Such a deep inflow layer, a hallmark of the classical spinup mechanism for tropical cyclones comprising the radial convergence of absolute angular momentum above the boundary layer, is not found in the numerical simulation during the period of peak intensification. These deficiencies are traced to the inability of the balance model to represent the nonlinear boundary layer spinup mechanism. These results are contrasted with a pseudobalance Eliassen formulation that improves the solution in some respects while sacrificing strict thermal wind balance. Overall, the quantitative results refute the Heng et al. claim and implicate the general necessity of the nonlinear boundary layer spinup mechanism to explain the spinup of a hurricane in realistic model configurations and in reality.

KEYWORDS: Dynamics; Forcing; Hurricanes; Meridional overturning circulation

1. Introduction

A new model for tropical cyclone genesis and intensification, called the rotating convection paradigm, has been developed in recent studies and reviewed by Smith and Montgomery (2016b) and Montgomery and Smith (2014, 2017b). Within the context of a favorable atmospheric environment provided by a moist recirculating region of the low- to middle tropical troposphere (Dunkerton et al. 2009), the paradigm includes, but extends, the classical axisymmetric paradigm\textsuperscript{1} for vortex intensification articulated by Ooyama (1969). While the main focus of the rotating-convection paradigm has been on the dynamical mechanisms involved in the spinup process, the paradigm recognizes also the need for a modest elevation of surface enthalpy fluxes to sustain the deep convection required for vortex spinup.

The new model constitutes an overarching framework for understanding the complex convective and vortical processes in simulated and observed tropical cyclones, taking into account both azimuthal mean and eddy contributions to the dynamics and thermodynamics of vortex spinup. Localized, rotating deep convection is recognized as an integral component of the system-scale vortex evolution. Convection locally amplifies the vorticity by up to one or two orders of magnitude by vortex-tube stretching and tilting processes in the cyclonic circulation of an incipient storm.

The paradigm includes an explanation for the observed occurrence of the maximum tangential winds in the boundary layer, invoking nonlinear boundary layer processes as an essential component of the explanation. It includes also an explanation for the increasingly strong control exerted by the boundary layer as a storm intensifies and matures (e.g., KilRoy et al. 2016) and undergoes secondary eyewall formation (e.g., Abarca et al. 2016; Huang et al. 2018). The importance of nonlinear boundary layer dynamics in the spinup of a hurricane vortex appears to have been suggested for some time. Zhang et al. (2001, 106–107), for example, explained the dynamical importance of the nonlinear boundary layer in hurricane spinup as follows:

As the storm deepens, the cross-isobaric radial inflow in the [marine boundary layer] transports more [absolute angular momentum] from the hurricane environment into the eyewall region than frictional dissipation. The major radial inflow decelerates as it approaches the [radius of maximum wind] where the centrifugal force exceeds [the radial pressure gradient force]. . . . Then, all the inflow air mass must ascend in the eyewall, transporting [absolute angular momentum] upward to spin up the tangential flow above. This upward transport of [absolute angular momentum] could increase significantly the local centrifugal force, thereby causing the pronounced supergradient

\textsuperscript{1} In the classical paradigm, inflow in the lower troposphere induced by deep convection within the vortex circulation is argued to draw absolute angular momentum surfaces inwards. Above the frictional boundary layer, absolute angular momentum $M$ is approximately materially conserved on the system-scale so that the inward movement of these surfaces implies a local spinup of the tangential velocity component.

Corresponding author: Michael T. Montgomery, mmontgo@nps.edu

DOI: 10.1175/JAS-D-19-0258.1
acceleration and the development of radial outflow in the eye-wall. . . . It is evident that (a) the intensity of the radial outflow depends critically on the upward transport of [absolute angular momentum], and (b) the spindown of the eyewall by radial outflow must be overcompensated by the upward transport of [absolute angular momentum] if the storm is to deepen.

The importance of nonlinear boundary layer dynamics in the spinup process, once the vortex attains approximate hurricane strength, was challenged in a recent paper by Heng et al. (2017) through the use of a modified Eliassen framework. As is well known, the Eliassen model (Eliassen 1951; Willoughby 1979; Shapiro and Willoughby 1982; Bui et al. 2009; Heng et al. 2017) rests upon the approximations of hydrostatic and gradient wind balance for the axisymmetric hurricane. Heng et al. (2017) evaluated the balance model through a pseudobalance2 elaboration of the Eliassen model. Montgomery and Smith (2018) rebutted the challenges raised by Heng et al. (2017), but the reply by Heng et al. (2018) raises new questions about the applicability of their purported “balance vortex model” to capture a numerically simulated or observed hurricane undergoing rapid spinup. These questions require careful consideration.

It is widely acknowledged that the axisymmetric balance vortex model is a useful zero-order description of tropical cyclone intensification when eddy processes are dynamically subdominant. The research of Zhang et al. (2001), Smith, Montgomery, and collaborators indicate that nonlinear boundary layer dynamics is an essential element of the spinup of the maximum tangential winds in the hurricane. However, the apparent implication of Heng et al. (2018) would seem to be that the axisymmetric balance vortex model is sufficient for explaining the spinup of real or simulated hurricanes using realistic values of the subgrid-scale parameters. In other words, in their view, nonlinear unbalanced and asymmetric eddy effects are secondary to the spinup dynamics. If this conclusion were true, the rotating convection paradigm would serve little purpose and Occam’s razor would suggest that it be discarded in favor of just the simple, axisymmetric balance model. Adjudicating the veracity of the results of Heng et al. (2018) is pivotal toward obtaining a more complete understanding of simulated or real intensifying hurricanes.

In a fair and balanced debate, it is important that the debaters keep an open mind and construct constructive suggestions from the opposing “team.” In particular, we take to heart the recommendation by Heng et al. (2018) “to revisit their (meaning our) algorithm and see whether the findings of Heng et al. (2017) could have any implications to their results.”

One concern raised by Heng et al. (2018) was that our use of the balance solution to deduce the radial wind at the lower surface was inaccurate. Specifically, it was argued that our use of one-sided finite differences to approximate the required partial vertical derivative of the meridional streamfunction at the lower boundary ($z = 0$) would underestimate the radial inflow in the balanced boundary layer and consequently would underestimate the tangential wind tendency there.3 Here we have updated our methodology for calculating the balanced radial wind at the lower (and upper) boundary so that the finite-difference approximation there is second-order accurate in the vertical grid spacing $\delta z$. The results are shown in appendix B to be improved, as expected, but the peak inflow remains much weaker than the full-physics simulation, thus the conclusions drawn are largely unchanged relative to our prior first-order accurate formulation used in Bui et al. (2009) and Abarca and Montgomery (2014, 2015) for the radial wind at the lower boundary.

Another potential concern with our previously published balance solutions is the disparate vertical resolution employed in the numerical and balance models (e.g., Bui et al. 2009; Abarca and Montgomery 2014, 2015; Heng et al. 2017). In the process of reevaluating our balance model solutions, we have found that interpolation of data between a stretched vertical grid from the simulation and a fixed vertical grid for the balanced solver introduces unnecessary complications that can be removed readily by using the same vertical grid spacing as the numerical model solution. To achieve the goal of maximal simplicity, we have regenerated three-dimensional numerical solutions of an intensifying tropical cyclone using a vertical grid spacing everywhere equal to that of the companion Eliassen model solver. We show in the next section that this simplified numerical simulation captures all of the principal unbalanced effects from prior solutions by Persing et al. (2013) using a stretched vertical grid. Results from this simulation will be used in an Eliassen diagnostic analysis to examine the role of frictional imbalance in hurricane intensification and illustrate the deficiencies of the pure balance formulation in capturing this phase of intensification.

Finally, we note that during the review process, an important realization emerged based on the review comments of a coauthor of Heng et al. (2018). The upshot is that in their purported rebuttal of the concerns articulated by Montgomery and Smith (2018), Heng et al. (2018, 2500–2501) purportedly “[removed] the imbalance in the basic state in the boundary layer” by eliminating the “vertical shear of tangential wind in the boundary layer.”4 In their Fig. 3, they calculated the purported balanced, advective tangential wind tendency (with friction term omitted) and then compared this calculation against a similar one that did not modify the vertical shear of

---

2 Pseudobalance will be defined precisely below.

3 Since the radial velocity is proportional to the vertical partial derivative of the streamfunction, the concern is that the one-sided derivative at the lower boundary underestimates the radial inflow from the balance model and potentially underestimates the “true” balance solution and its corresponding tangential wind tendency there.

4 In the standard Eliassen balance vortex model, it is assumed that the flow is in gradient and hydrostatic balance everywhere, including the frictional boundary layer. The purported removal of “imbalance” in the basic state is non sequitur.
the tangential wind, but instead used the azimuthally averaged tangential wind of the simulation in the layer. In both of these calculations, however, the basic vortex does not solve the thermal wind equation, as would be required for a valid derivation of the standard Eliassen balance model. Consequently, these solutions should not be regarded as strict balance solutions.

It was only recently that we realized some of the artificial implications of the methodology of Heng et al. (2018), and this explains in part how Heng et al. (2018) show positive tangential wind tendencies in the inner-core boundary layer of their purported “balance solution” while our inferred balanced tendencies (including explicit friction) yield persistently negative tendencies (i.e., spindown) in the inner-core boundary layer, a markedly different result than intimated by Heng et al. (2018). Further details are provided in the results section.

The remaining paper is organized as follows: The numerical simulations are described in section 2, and the simulated evolutions are summarized in section 3. A brief overview of the Eliassen model, along with some pertinent details of the solution method, is provided in section 4. Results from the Eliassen model are presented in section 5. A summary of the principal results and conclusions are provided in section 6.

2. The simulations

The prototype intensification problem consists of an idealized, three-dimensional simulation of an initially cloud-free, circular vortex in thermal wind balance in a quiescent moist tropical environment. The simulations here use the Cloud Model 1 (CM1, version 14) and employ a fixed horizontal grid spacing of 3 km over an interior square subdomain of length 405 km. This grid configuration is adequate to represent the convective-vorticity organization process and the emerging eyewall and eye of the storm. Beyond the fine-grid interior region, the horizontal grid is progressively stretched into the complete, 2880 km \( \times \) 2880 km simulation domain. The control simulation (herein referred to as EX-1) employs a fixed vertical grid spacing of \( \delta z = 500 \) m so as to minimize interpolation errors between a stretched vertical grid simulation and the fixed-grid, Eliassen balance model as discussed in the introduction. The control experiment is a variation of the 3D3k simulation of Persing et al. (2013) who examined the qualitative and quantitative differences of vortex spinup in strictly axisymmetric and three-dimensional configurations for the prototype intensification problem. The 3D3k simulation of Persing et al. (2013) will be referred to here as EX-2.

In brief, both the EX-1 and EX-2 simulations use the following settings: 1) a near-moist-neutral initial sounding of Rotunno and Emanuel (1987); 2) an \( f \) plane with \( f = 5 \times 10^{-5} \text{s}^{-1} \), a Coriolis parameter corresponding to 20\(^\circ\)N latitude; 3) a Newtonian cooling capped at 2 K day\(^{-1}\) as a surrogate for longwave radiation; 4) a fixed SST of 299.3 K (26.15\(^\circ\)C); 5) a simplified rainfall scheme with fixed fall speed of 7 m s\(^{-1}\); 6) a constant sea-to-air moist enthalpy transfer coefficient \( C_k = 1.29 \times 10^{-3} \) and constant surface drag coefficient \( C_D = 2.58 \times 10^{-3} \) in a bulk aerodynamic heat and momentum transfer scheme; 7) no basic-state environmental wind; and 8) a horizontal mixing length \( l_h = 700 \) m and vertical mixing length \( l_v = 50 \) m in the Smagorinsky (1963) subgrid-scale turbulence scheme with stability modifications according to Lilly (1962).

The values for mixing lengths are based on the recent observational findings of Zhang and Montgomery (2012) and Zhang et al. (2011), respectively, and the resulting vertical and horizontal eddy diffusivities output in prior model simulations of Persing et al. (2013). These values are close to the value recommended by Bryan (2012) in order to produce realistic hurricane structure. For simplicity, these mixing lengths are assumed constant in both space and time. Persing et al. (2013) and Montgomery et al. (2019) provide detailed justification for these selected parameter values. A complete list of parameters used in executing the EX-1 CM1 simulation follows that of the appendix of Montgomery et al. (2019), except that the number of vertical points is reduced (\( n_z = 50 \)) and the vertical grid spacing is increased (\( \delta z = 500 \) m).

A simulation like EX-1, except with a 250-m fixed vertical grid spacing, is shown here also as the simulation EX-3. This third simulation serves as an additional dataset to test the robustness of the principal results of the Eliassen balance model to a higher vertical resolution.

3. Simulation summary

We give here a brief overview of the new simulation (EX-1), with the aim of showing that the uniform vertical grid simulation is sufficient to test the assertions of Heng et al. (2017, 2018).

The spinup of the system-scale vortex is summarized in Fig. 1, which shows a time series of the maximum azimuthally averaged tangential wind \( \nu \) defined relative to the moving circulation center. The center is defined as the location of the minimum of the temporally smoothed (\( \pm 1 \) h from nominal time using a sliding, gated, uniform average) surface pressure field. The time series shows that the vortex intensifies after 16 h from a tangential wind maximum of 11 m s\(^{-1}\) to hurricane intensity of 33 m s\(^{-1}\) by 50 h. Subsequently, the vortex continues to intensify progressively, albeit with superposed fluctuations, to 45 m s\(^{-1}\) by 68 h, and reaches a maximum intensity of 60 m s\(^{-1}\) by 250 h (not shown).

Here, intensity at any particular time in the simulation is defined by the maximum azimuthally averaged tangential wind. The intensification rate is defined by the rate of change of the intensity so defined. At all times shown in Fig. 1, the maximum mean tangential velocity typically resides in the vortex boundary layer at a height of 750 m and the maximum spinup rate in the low-level inflow layer of the vortex is approximately 5 m s\(^{-1}\) h\(^{-1}\). We will return to these solution properties a little later.

The use of the older version 14 of the CM1 model is motivated by the opportunity to compare the results directly with that of Persing et al. (2013).
The process of intensification found in EX-1 is similar to other documented simulations of the prototype intensification problem (Montgomery et al. 2006; Nguyen et al. 2008; Fang and Zhang 2011; Persing et al. 2013; Kilroy et al. 2016). Initialized without vertical wind or cloudiness, EX-1 develops an approximately annular envelope of isolated deep convective cells (not shown) near the radius of maximum initial tangential winds after a period of small weakening and moistening of the boundary layer. Subsequent convection organizes the preexisting relative vorticity and convectively generated vorticity via vortex-tube stretching through a process of aggregation into a nascent tropical cyclone.

For the purposes of this study as summarized in the introduction, it is necessary to test whether the coarse vertical resolution of the EX-1 simulation is sufficient to represent the frictionally induced unbalanced flow of the simulated boundary layer. Here we compare (Fig. 2) the boundary layer flow pattern found in EX-1 with the simulation, EX-2, which is otherwise identical except for employing a stretched grid in the vertical direction. Near the surface in EX-2, the vertical grid levels of all state variables except $w$ are 25, 90, 184, 308, 461, 644, and 856 m, etc., while in EX-1 only 250 and 750 m grid levels lie below the 1 km height level. The selected times shown in Fig. 2 correspond to a maximum gradient wind $v_g$ of approximately 30.5 m s$^{-1}$, and thus shows a similar stage of development in both simulations.

The key indicators of unbalanced boundary layer flow are shown in both EX-1 and EX-2, albeit in higher detail in EX-2 (Figs. 2c,d), for example, a maximum in azimuthally averaged tangential winds $v$ near the 1 km height level displaced radially inward of the maximum of $v_g$ by at least 3 km, and a peak inflow jet of 12 m s$^{-1}$ near the surface that decelerates just inside the radius of maximum $v$. [The gradient wind is computed according to Eq. (12) defined below.] The maximum in gradient winds, defined by $v_a = \sqrt{v^2 + v_g^2}$, lies just inside the radius of maximum $v$ in both EX-1 and EX-2. Therefore, despite the fewer vertical grid points in the boundary layer, EX-1 nonetheless exhibits the important features of the more resolved boundary layer in the experiment EX-2. Despite the similar gradient wind intensity at the selected times in these different simulations, we do

![Figure 2](https://example.com/fig2.png)

**Fig. 2.** Contours from (a),(b) EX-1 at 53 h and (c),(d) EX-2 at 74 h; the maximum gradient wind is approximately the same (30.3 m s$^{-1}$, EX-1; 30.6 m s$^{-1}$, EX-2) for each simulation at these times. In (a) and (c), the azimuthally averaged tangential wind $v$ is shown by shading with an interval of 10 m s$^{-1}$ and with thin black contours with an interval of 5 m s$^{-1}$; the azimuthally averaged radial wind $u$ is shown with blue contours (dark negative, light positive) with an interval 2 m s$^{-1}$; and azimuthally averaged vertical wind $w$ is shown with red contours (dark positive, light negative) with an interval 0.5 m s$^{-1}$. In (b) and (d), the same $v$ field is shown in shading and black contours; the gradient wind $v_g$ is shown in blue contours with an interval 5 m s$^{-1}$; the agradient wind $v_a = v - v_g$ is shown in red contours (dark positive, light negative) with an interval 2 m s$^{-1}$; and the radii of maximum $v$ and $v_g$ as a function of height are shown by white and yellow dotted lines, respectively.
not suggest that these simulations are exactly comparable at these times since the vertical grid spacing unquestionably influences the evolution of the simulated storms. We suggest simply that the important structures associated with gradient wind imbalance in the boundary layer are to a degree present in the lower-resolution experiment EX-1.

In summary, based on the foregoing results employing a fixed vertical grid spacing of \( \delta z = 500 \) m, the EX-1 simulation constitutes a physically viable solution that can be used to revisit our previous balance diagnoses. In particular, the simulation, as well as the experiment EX-3, will be used to test the assertion by Heng et al. (2017, 2018) that axisymmetric “balanced dynamics can well capture the secondary circulation in the full-physics model simulation even in the inner-core region in the boundary layer.”

4. The Eliassen vortex model in brief

The traditional Eliassen model is a balance model for the slow evolution of an axisymmetric vortex forced by sources of tangential momentum and heat. For a persistent coherent vortex, the model is formulated in cylindrical-polar coordinates. The model consists in part of a simplified radial momentum equation expressing strict gradient wind balance, and a simplified vertical momentum equation expressing strict hydrostatic balance. While the heat and momentum forcings in the potential temperature and tangential momentum equations would, themselves, tend to drive the vortex away from thermal wind balance, the vortex is assumed to evolve in a state of thermal wind balance. As a result, the meridional secondary circulation acts to oppose these forcings and keep the vortex in thermal wind balance. To ensure that the tangential velocity and potential temperature fields remain in thermal wind balance under the prescribed forcings, a compatibility equation must be satisfied between the tangential velocity and potential temperature equations. The compatibility equation is the Eliassen equation for the transverse circulation (given below).

As is usual in a balance model, for the model to remain self-consistent, the time scales of the heat and momentum forcing functions must be long in comparison to the intrinsic oscillation periods of the vortex so as not to excite large-amplitude, high-frequency inertial (centrifugal) and gravity modes. Being a balance model, only one of the two equations for tangential velocity and potential temperature may be used to predict the time dependence of the flow in a prognostic setting. As discussed in Smith et al. (2009, see footnote on p. 1718) the balance evolution is easiest to advance forward in time by using the tangential momentum equation. This is because of a restriction on the magnitude of the radial pressure gradient force when solving the gradient wind equation for the tangential velocity.

a. The balance model equations

Following closely the presentation of Smith et al. (2018), the specific axisymmetric balance equations are given as follows. The tendency equation for the tangential wind component \( \nu \) in cylindrical \( r-z \) coordinates is

\[
\frac{\partial \nu}{\partial t} = -u \frac{\partial \nu}{\partial r} + w \frac{\partial \nu}{\partial z} \frac{\mu w}{r} - f u - V,
\]

where \( \mu \) and \( w \) are the radial and vertical velocity components, \( t \) is the time, \( f \) is the Coriolis parameter (assumed constant), and \( -V \) is the azimuthal momentum sink associated with the near-surface frictional and subgrid-scale stress divergences, as well as resolved eddy flux divergences. Here, the variables \((u, v, w)\) denote the azimuthal wavenumber-0 flow component and, unless otherwise stated, \( v \) is assumed to be the gradient wind. Following Smith et al. (2005), the axisymmetric thermal wind equation has the general form

\[
\frac{\partial}{\partial r} \left[ A \frac{\partial \psi}{\partial r} + B \frac{\partial \psi}{\partial z} \right] + \frac{\partial}{\partial z} \left[ C \frac{\partial \psi}{\partial z} + \frac{1}{2} B \frac{\partial \psi}{\partial r} \right] = \dot{\Theta},
\]

where

\[
\begin{align*}
A &= -g \frac{\partial^2 \chi}{\partial z \partial r} \left( \frac{\chi}{\rho r} \right) N^2, \\
B &= -\frac{2}{\rho r} \left( \xi \chi \frac{\partial \nu}{\partial z} + C \frac{\partial \chi}{\partial z} \right), \quad (5) \\
C &= \left[ (\xi + f) \chi + C \frac{\partial \chi}{\partial r} \right] \frac{\chi}{\rho r^2}, \quad (6)
\end{align*}
\]

and

\[
\dot{\Theta} = g \frac{\partial}{\partial r} \left( \chi \dot{\theta} \right) + \frac{\partial}{\partial z} \left( C \chi \dot{\theta} \right) + \frac{\partial}{\partial z} \left( \chi \xi \dot{V} \right),
\]

where \( \xi = (1/r) \varphi (\alpha \varphi) / \partial r \) is the vertical component of relative vorticity, \( N^2 \) is the square of the Brunt–Väisälä frequency, defined as \( (g/\varphi \theta) \partial \theta / \partial z \), and \( \dot{P} = (\xi + f) + (C \chi) \partial \chi / \partial r \) is the square of the generalized inertial frequency. The quantities \( \bar{A} \) and \( \bar{C} \) are proportional to the static stability and inertial stability, respectively. The quantity \( \bar{B} \) characterizes, in part, the strength of the vertical shear of the gradient wind. The forcing term \( \dot{\Theta} \) represents a combination of, respectively, the diabatic heating and momentum forcing \( \theta \) and \( V \).

In the anelastic approximation, the transverse velocity components \( u \) and \( w \) are given in terms of \( \psi \) by
\[ u = -\frac{1}{\rho p} \frac{\partial \phi}{\partial z}, \quad w = \frac{1}{\rho p} \frac{\partial \phi}{\partial r} \] (8)

b. Specific forcing terms

The generalized tangential velocity forcing for the azimuthally averaged dynamics is computed by residual from the corresponding tendency and advection terms of the tangential velocity tendency equation:

\[ -V(r,z) = \frac{\partial v}{\partial t} + u(\xi + f) + w \frac{\partial v}{\partial z}. \] (9)

The resolved eddy advection and subgrid-scale flux derivative terms (associated with the vortex boundary layer and diffusion) are implicitly included in the forcing \(-V(r,z)\).\(^6\)

The generalized heat forcing for the azimuthally averaged balance dynamics is computed similarly from the right-hand sides of Eqs. (9) and (10) denote azimuthally averaged variables on constant height surfaces. These generalized tangential momentum and diabatic heat forcings are equivalent, respectively, to the forcings \(F_t\) and \(Q\) defined by Eqs. (16) and (17) of Bui et al. (2009) in which the eddy covariance and subgrid-scale terms are calculated explicitly.

c. Ellipticity and regularization

The discriminant \(\Delta\) of the Eliassen balance equation is given by\(^8\)

\[ \Delta = 4\gamma^2 \left[ -\frac{g}{\chi} \frac{\partial \chi}{\partial z} \left( \xi^a + \frac{C \partial \chi}{\chi} \right) + \left( \frac{1}{\chi} \frac{\partial \chi}{\partial z} \right)^2 \right], \] (11)

where \(\gamma = \chi/(\rho r)\) and \(\xi^a = \xi + f\) is the absolute vertical vorticity. The balance equation is elliptic when \(\Delta\) is everywhere positive.

When \(\Delta\) is everywhere positive, a unique solution is guaranteed for well-posed boundary conditions. However, \(\Delta < 0\) at isolated points, or extended regions of the mean vortex, the balance equation is locally hyperbolic and the flow there satisfies the conditions for symmetric instability. Technically speaking, the balance equation loses solvability as an elliptic (balance) problem. To achieve a plausible solution in such instances, a regularization method is required. The regularization scheme used here is similar in spirit to that proposed by Møller and Shapiro (2002) and used by Bui et al. (2009) and others (e.g., Abarca and Montgomery 2014, 2015). The regularization scheme, as well as the solution method for solving Eq. (3) is detailed in appendix A.

d. Interpolation

Interpolation is required to convert CM1 simulation data from the vertically staggered grid to the unstaggered grid used in the Eliassen solver. The simple extrapolations used to supply values at the surface are as follows: linear extrapolation is used for potential temperature, density, and pressure; a zero value is used for the heating rate at the surface; and because version 14 of CM1 uses no log-layer assumptions for the surface fluxes of momentum, the values of tangential wind and momentum forcing \(V\) found at \(z = 250\) m are replicated at the surface.

e. Options to obtain balance

When attempting to solve the Eliassen model at any particular time, or suitable averaging interval, it is necessary to recall that the dynamical and thermodynamical fields must be in axisymmetric, gradient wind, hydrostatic, and thermal wind balance. As discussed above, thermal wind balance is tacitly assumed in the derivation of the Eliassen equation. As noted in Montgomery and Smith (2018), this requirement was ignored in the solutions presented by Heng et al. (2017). In response to our comment paper, Heng et al. (2018) proposed that some of their new solutions preserve the basic characteristics of their original unbalanced solutions. However, as we will show below, our results confirm the discrepancy between the simulation and balance spinup diagnosis for realistic subgrid-scale parameters and surface exchange coefficients for heat and momentum.

Because of the intrinsic nonlinearity of gradient wind balance, there is some inherent fuzziness in defining the basic state, balanced vortex from the full-physics simulation. Two methods are employed here to assure thermal wind balance before producing a solution from the Eliassen equation. These resulting solutions help explore the range of variation of balance solutions. The first method, called “Smith balance,” based on Smith (2006), takes the mean tangential wind field \(v\) from the simulation and solves Eq. (2) to render density \(\rho\), pressure \(p\), and potential temperature \(\theta\) in thermal wind balance with \(v\). The second method, called “Holton balance,” based on Holton (2004), takes the mean thermodynamic fields from the model output and then deduces \(v_x\) from the gradient wind equation

\[ \frac{v_x^2}{r} + f v_x = \frac{1}{\rho} \frac{\partial p}{\partial r}. \] (12)

In general, \(v_x\) becomes imaginary, notably in the outflow layer. In such locations, a simple regularization is chosen by

\[^6\]In this equation, the tangential wind \(v\) and vorticity \(\xi\) come from the numerical solution and generally are not in gradient wind balance, especially in the boundary layer—as shown in section 3.

\[^7\]As a reminder, the individual quantities appearing on the right-hand sides of Eqs. (9) and (10) denote azimuthally averaged variables on constant height surfaces. These generalized tangential momentum and diabatic heat forcings are equivalent, respectively, to the forcings \(F_t\) and \(Q\) defined by Eqs. (16) and (17) of Bui et al. (2009) in which the eddy covariance and subgrid-scale terms are calculated explicitly.

\[^8\]Note Smith et al. (2018) inadvertently omitted the multiplicative factor of 4 in the typesetting of their Eq. (5).
setting $v_g = 0$ at these points. Given the $v_g$ field so defined, Smith balance is then performed on the regularized-$v_g$ field. Holton balance is summarized as three steps: 1) solve Eq. (12) for $v_g$, 2) regularize $v_g$, and 3) apply Smith balance.

Heng et al. (2017) present cases where no effort is made to assure gradient and thermal wind balance before solving the Eliassen equation. Since the input fields are not balanced, this methodology cannot be considered a solution to the traditional Eliassen balance model and all subsequent assertions about the sufficiency of balance dynamics for capturing the simulated spinup of the hurricane, including in the boundary layer, must be viewed with extreme caution. As a way of replicating the “unbalanced” methodology adopted in Heng et al. (2017) and reproducing their results, a third solution method is employed here that uses the azimuthally averaged tangential velocity ($\bar{v}$) and potential temperature ($\bar{\theta}$) from the numerical simulation to define the coefficients of the Eliassen equation. This method will be called the “pseudobalance” solution because the mean vortex is not strictly in thermal wind balance, but the solution solves the Eliassen balance equation [see Eq. (3)] with unbalanced coefficients.

The pseudobalance formulation can be given a rational justification by considering an extended Eliassen formulation in which the mean vortex is no longer required to be in thermal wind balance, but is assumed to vary sufficiently slowly in time. When such a generalization is formulated, one obtains extra time-dependent forcing terms on the right-hand side of the Eliassen Eq. (3). This result was demonstrated in Smith [2017a, p. 12, their Eq. (18)]. In this study, we neglect the extra forcing terms under the assumption that in a first approximation the time derivative forcing terms are sufficiently small in comparison to the traditional forcing terms involving spatial derivatives of heating and tangential momentum. Although the pseudobalance formulation emerges in this limiting approximation, it would be incorrect to refer to this formulation (in which the coefficients no longer satisfy thermal wind balance) as a traditional Eliassen balance formulation.\[9\]

\[ f. \text{Evaluation of intensification rate} \]

For the purposes of comparing the numerical and balance model predictions, the tendency of tangential wind inferred from the solution to the Eliassen equation is computed as the effect of the Eliassen solution upon the total vortex, via

\[
\frac{\partial \bar{v}}{\partial t} = -u_E(\xi + f) - \bar{w}_E \frac{\partial \bar{v}}{\partial z} - V, \tag{13}
\]

where $(u_E, \bar{w}_E)$ denote the solution to the Eliassen Eq. (3) and $\xi, f, \bar{v}$ are azimuthally averaged values of the tangential velocity, relative vorticity and tangential momentum forcing (including eddy covariance terms) from the model simulation.\[10\]

5. Results

a. S1—Smith-balance solution

The Eliassen solution using Smith-balance methodology is denoted as S1. The first example uses data from EX-1 with an output interval of 2 min averaged over the intensifying period between 53.3 and 53.7 h.

The forcing terms, defined by time averages of the simulation output and interpolated to the unstaggered grid, are shown in Fig. 3. The figure shows the separate diagnosed contributions from the generalized diabatic heating rate and tangential momentum forcing as defined by Eqs. (10) and (9), respectively, calculated over the averaging period. The principal features of the generalized heat forcing are an outward leaning annulus of heating inside a radius of about 60 km, with secondary and tertiary heating annuli centered around 70 and 105 km, respectively. There are also narrow regions of cooling along the inner developing eyewall and its exterior in the lower troposphere. In addition, two isolated, small-scale, patches of cooling are evident outside the eyewall in the midtroposphere and one patch in the upper troposphere above the main eyewall. The main regions of momentum forcing are in the boundary layer (negative associated with surface friction and flux derivatives of subgrid-scale momentum fluxes) and in the developing eyewall [positive in the lower troposphere above the boundary layer and variable in the upper troposphere, both features associated with eddy forcing as discussed in Persing et al. (2013)].

The regions regularized to assure a convergent solution to the Eliassen equation are shown in Fig. 4a. Regularization for $\mathcal{F} < 0$ (dry static instability) is shown only in the eye at 2.5 km height (red); this static instability is a result of relatively large adjustments to the mass field when using the simulated tangential wind field. The regularization of $\mathcal{B}$ occurs in the boundary layer between 21 and 125 km radius and in large

---

\[9\] A pseudobalance formulation was used also by Hendricks et al. (2004), Montgomery et al. (2006), and Persing and Montgomery (2003). In the first two studies, tropical cyclogenesis was being analyzed in part from an axisymmetric balance vortex perspective. During such an early stage of development, the maximum tangential wind speed is well below hurricane strength and consequently the nonlinear boundary layer dynamics are subdominant, with negligible difference between balance and pseudobalance models (see also Schecter and Menelaou 2020, p. 108). In the latter case, however, the gradient wind approximation breaks down in the inner-core boundary layer and in the upper outflow layer; the approximate agreement found between the simulation and balance model in the boundary layer of the superintense hurricane is recognized now to be a result of using the pseudobalance methodology, which, as noted above, should be regarded as a zero-order approximation for retaining unbalanced boundary layer effects in slowly evolving storms. N.B. This approximation will not be formally valid in rapidly intensifying storms.

\[10\] As noted in the introduction, the foregoing method for inferring the intensification rate with the Eliassen solutions contrasts with the methodology adopted by Heng et al. (2018, p. 2501), who found various solutions to the Eliassen Eq. (3) using an unbalanced vortex, but showed only the advective tangential wind tendency [with friction term $-\bar{V}$ omitted in Eq. (13)].
FIG. 3. Time-averaged (from 53.3 to 53.7 h) forcing of (a) heating rate, as defined by Eq. (10) (K h$^{-1}$), and (b) tangential momentum, as defined by Eq. (9) (m s$^{-1}$ h$^{-1}$), from the EX-1 simulation after interpolation to the unstaggered grid. These forcings are used as inputs into Eq. (7) to obtain the S1, H1, and P1 Eliassen solutions described in this section. Contours are ±{1, 2, 5, 10, 30} with blue contours denoting negative values and red contours denoting positive values.

FIG. 4. Depictions of the regions regularized in the solution (a) S1, (b) H1, and (c) P1. Red, green, and blue regions show regularizations of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively (see text for details). The tangential wind field $\mathbf{v}$ is contoured with interval 10 m s$^{-1}$, except the gradient wind $\mathbf{v}_g$ is shown in (b).
regions of the upper-tropospheric outflow layer (green). The regularization of $C$ (blue) is largely, but not precisely, coincident with the outflow regularization regions of $B$.

The radial and vertical winds from the balance solution $S1$ are shown in Figs. 5c and 5d, respectively. These balance solutions are to be compared with the corresponding winds from the simulation EX-1 in Figs. 5a,b. The broad features of the simulated secondary circulation are represented in the $S1$ solution, with inflow in the boundary layer, up-flow in the simulated eyewall, and outflow above 10 km height. There is also an inflow jet at approximately 8 km height just beneath the outflow jet. However, important differences between the simulated and balanced circulations become evident upon closer examination. To facilitate this comparison, the maximum and minimum values of the radial flow, as well as the location of the extrema, are shown in the figure and summarized in Table 1. The peak radial inflow interpolated to the 250-m height level is shown for direct comparison to the simulation on the staggered grid. Relative to the simulation EX-1, the peak inflow of the $S1$ solution at 250 m height is noticeably weaker by 34% and is displaced radially outward, away from the center by 12 km. The inefficacy of the balance flow to reach the interior radii is to be expected by the inability of balance theory to capture the nonlinear boundary layer spinup mechanism articulated by Smith and Vogl (2008), Smith and Montgomery (2016a), and Montgomery and Smith (2017b).

Another discrepancy between the balance and simulated circulation is evident by the significantly deeper low-level inflow (over 4 km height) in the $S1$ solution compared to the simulated inflow (confined below 1.5 km height). The $S1$ solution thus implicates a strong spinup tendency by the radial import of absolute angular momentum $M$ above the boundary layer where there is none in the simulation.

In the upper levels of the vortex, the radial outflow in $S1$ is roughly similar to that simulated in EX-1, except that a double outflow jet is suggested there with peak outflow at 11 and 13 km height. The simulation shows a single maximum at 13 km height. Also, the outflow jet in $S1$ extends to a lower height (9 km) than found in EX-1 (10.5 km) outside of 70 km radius. Wang and Smith (2019) show that regularization in the inertially unstable region of the outflow tends to flatten out the streamfunction of the secondary circulation—in $S1$, such a flattening would prevent the $S1$ solution from representing the slanting interface between the upper-level inflow and outflow jets shown in the simulation. As a result, both features would emerge from the eyewall at the lower altitudes. The vertical wind field shows strong similarity between $S1$ and EX-1.

The time rate of change of tangential wind $\partial \psi / \partial t$ in EX-1 across the time interval 53.3 to 53.7 h is shown in Fig. 6a, where positive (red) values are found at the radius of maximum tangential winds (RMW) ($r = 24$ km) at 1 km height. Large vertical stretches of the eyewall are also spinning up across the time interval. By comparison, the tangential wind tendency derived from the balance solution $S1$ shows (Fig. 6b) overall larger positive and negative magnitude of $\partial \psi / \partial t$. At the noted location of maximum tangential winds
(r = 24 km, z = 1 km), a “saddle point” in the $\partial v/\partial t$ field is evident. Of note, the simulated spinup tendency is found to reach the surface at the RMW, but the S1 solution shows a significant spindown tendency in the boundary layer at the RMW. In addition, the S1 solution shows a significant spinup tendency centered at (r = 27 km, z = 2 km) greater than 20 m s$^{-1}$ h$^{-1}$, associated with inflow above the boundary layer as noted above. Needless to say, these discrepancies between the simulation and the balance calculation are alarming.

Table 1. Metrics of the secondary circulation.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Balance</th>
<th>Min[$u(z = 0)]$</th>
<th>Min[$u(z = 250 m)]$</th>
<th>Max($u$)</th>
<th>R[Min($u$)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>EX-1</td>
<td>—</td>
<td>—</td>
<td>—16.07</td>
<td>14.01</td>
<td>27</td>
</tr>
<tr>
<td>S1</td>
<td>Smith</td>
<td>—14.55</td>
<td>—10.60</td>
<td>13.78</td>
<td>39</td>
</tr>
<tr>
<td>H1</td>
<td>Holton</td>
<td>—9.98</td>
<td>—8.85</td>
<td>9.88</td>
<td>36</td>
</tr>
<tr>
<td>P1</td>
<td>Pseudobalance</td>
<td>—20.10</td>
<td>—15.39</td>
<td>13.21</td>
<td>27</td>
</tr>
</tbody>
</table>

Fig. 6. Time rate of change of tangential velocity $\partial v/\partial t$ as defined by Eq. (13) from (a) the EX-1 simulation (time averaged between 53.3 and 53.7 h) and from the (b) S1, (c) H1, and (d) P1 solutions, with contours of ±[20, 10, 5, 2, 1] m s$^{-1}$ h$^{-1}$, blue contours negative, and red contours positive. The union of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ regularizations from the corresponding solutions are shown by gray hatching in (b)–(d).
As a final remark in the context of the S1 solution, in the boundary layer, at locations near regularization (hatching), we note a general spindown tendency in S1. One might inquire whether the solution methodology employed in these problem areas invalidates these comparisons between the simulations and balance model. Careful study of the regularization methodology by Wang and Smith (2019) reveals that the regularization in the boundary layer in regions of forcing does not invalidate the balance solution. On the other hand, regularization in the region of outflow and symmetric instability can produce spurious circulation structures locally. These points are discussed in more detail by Wang and Smith (2019), but the take home message is that the balance solutions in the boundary layer region are robust to the regularization procedure and the significant discrepancies between the simulation and balance model are not artifacts of the regularization methodology employed.

b. H1—Holton-balance solution

The solution to the Eliassen balance equation using the Holton-balance methodology over the same period as S1 is denoted here as H1. The regularized regions in H1 (Fig. 4b) are quite different than in S1 (Fig. 4a), with a gravity wave–like pattern through the model troposphere and stratosphere (all of the model stratosphere is not shown here).\(^{11}\) The H1 solution shows no \(\overline{\mathbf{A}}\) regularization, nor any coherent regularized region in the boundary layer or outflow layer. The gradient wind field computed from the EX-1 simulation data (averaged over the same time period and regularized for imaginary values as noted above) is shown by contours.

The H1 solution has many features in common with the S1 solution (Fig. 5), especially in comparison with the simulated secondary circulation of EX-1 such as: the discrepancy of the peak inflow and peak outflow in comparison with EX-1; a deeper inflow layer (5 km); and larger radius of peak inflow. The upper-level inflow jet in H1, however, is much weaker than in S1 and EX-1. The regularized regions aloft in H1 extend to lower altitudes in localized regions through the mid- and lower troposphere, while in S1 regularized regions are not found between 1 and 6 km height outside the eye. The diagnosed tendency of \(v\) derived from the solution H1 (Fig. 6c) shows, like S1, large magnitudes overall in the eyewall, with the location of peak tangential winds \((r = 24 \text{ km}, \ z = 1 \text{ km})\) lying on a sharp gradient. Like S1, the near-surface layer shows no spinup tendency.

Since Fig. 4b reveals that there is no coherent regularization of the Eliassen coefficients in the inner-core boundary layer, the concern raised by Heng et al. (2017) that one must rebalance the vortex in the boundary layer after regularization to verify the veracity of the findings is moot in this case.

To summarize: The foregoing results using two different balance vortex definitions clearly establish a significant discrepancy between the Eliassen balance vortex model and the numerical model. For reasons discussed above, the results as they pertain to the low-level radial inflow are not an artifact of the regularization procedure. Appendix B further verifies the results of this section using the simulation data of experiment EX-3, which employs a finer uniform vertical grid spacing of \(\delta z = 250 \text{ m}\).

c. P1—A pseudobalance solution

The Eliassen solver is applied also to the azimuthally averaged tangential winds of the EX-1 simulation. Since this vortex is unbalanced, this Eliassen solution, named P1, is a departure from the balance assumptions used to formulate that model, and is thus referred to as the “pseudobalance” solution. The regularized regions of P1 are very much like S1 (Figs. 4a,c), except that there is no \(\overline{\mathbf{A}}\) regularization. The solution P1, in contrast to S1 and H1 (Fig. 5), has inflow and outflow maxima that are close in value to the simulation at 250 m height (see Table 1); the location of the inflow maximum matches that of EX-1, but the outflow maximum in P1 is at a smaller radius (inside the outflow regularization region). The inflow depth of P1, like EX-1, is also confined below 1.5 km height. Because the P1 solution more closely matches the simulated secondary circulation (Fig. 5), the computed rate of change of \(v\) is nearly equal to that of EX-1.

On the whole, the closeness of the P1 solution to EX-1 provides confidence in the numerical methods employed herein and an assurance that the computed weaknesses of the secondary circulation shown in S1 and H1 are due to the limitations of axisymmetric balance dynamics.

d. Time variation of solutions

Now we consider the applicability of the prior tests, conducted over a short time window, to a more practical setting covering a several hour time interval, which includes a time of rapid intensification of the system-scale vortex. The time interval chosen, 53 to 56 h, is indicated by the vertical dashed red lines in Fig. 1.

We present three sets of multiple Eliassen solutions: MS1, corresponding to the Smith-balance solution S1; MH1, corresponding to the Holton-balance solution H1; and MP1, corresponding to the pseudobalance solution P1 as defined in the prior section. Each member of the sets consists of an Eliassen solution using 24 min averaged inputs from the outputs of the EX-1 simulation with an interval of 2 min, and the central time of the interval is noted here as the nominal time. The nominal times of all three sets are the same, spanning an interval from 53 to 56 h with members spaced by 2 min. Note that the Eliassen model is not being executed in an evolutionary sense, rather it is being solved diagnostically for 90 distinct subintervals of time comprising the 3-h analysis period. During the time frame from 53 to 54.7 h the simulation EX-1 shows sharp intensification, while from

\(^{11}\) Within the limited domain shown in Fig. 4b, the gradient wind is regularized at only two grid points, shown by a small zero-value contour (black) at \((r = 90 \text{ km}, \ z = 13 \text{ km})\). More extensive regions of \(v_y\) regularization in the H1 solution are found outside 140 km radius and above 16 km height.
54.7 to 56 h a quasi-steady intensity is seen (Fig. 7a). Note that the previously presented balanced and pseudobalanced solutions S1, H1, and P1 are members of the sets MS1, MH1, and MP1, respectively, corresponding to the nominal time 53.5 h.

From Fig. 7b, it can be seen that the underestimation of the peak inflow by the balanced S1 solution is a general feature, which is not sensitive to the exact analysis period. The peak upper-level outflow from MS1 (Fig. 7c) shows a larger range of variation than simulated, ranging from 12 to 19.5 m s\(^{-1}\) versus 13.5 to 18 m s\(^{-1}\) in the simulation. However, the variations in maximum outflow evident in MS1 bear little or no relation to the changes shown in the simulation. Recent work by Wang and Smith (2019) has shown a strong dependence of the solution characteristics in the upper-level outflow region on the regularization in the outflow. That work suggests caution be applied before attempting to infer any correlation between a balanced outflow feature with the simulated spinup. The rate of change of maximum tangential wind inferred from the MS1 solutions consistently predict weakening (Fig. 7d), while roughly tracking the temporal variations of the simulation. Arguably, a component of the temporal intensity variation of intensification is captured by the solver of the Eliassen equation, but the solver consistently produces a weakening bias.\(^{12}\)

Like MS1, the peak inflow in MH1 underestimates the simulated low-level inflow (Fig. 8), but MH1 shows less temporal variability than MS1 and the simulation. The peak upper-level outflow in MH1 is systematically weaker than in the simulation and does not increase with time after 53.6 h in the simulation. Also like MS1, the intensification rate is generally weaker and of opposite sign in comparison to the simulation.

The peak inflow from the pseudobalance (MP1) solutions to the Eliassen equation (Fig. 9b) provides an interesting illustration of the limitations of the Eliassen solutions. In this set of solutions the magnitude of the peak inflow is roughly captured by the MP1 solution, ranging from \(-13.3\) to \(-16.7\) m s\(^{-1}\) versus \(-14.3\) to \(-16.6\) m s\(^{-1}\) in EX-1. However, stepwise jumps in the peak inflow of the MP1 solutions are seen, most-prominently at 54.6 h, and at other times like 53.4, 55.0, and 55.2 h, among others. Since the inputs for each of the solutions of MP1 are time averaged, and since these time averages overlap, one would not anticipate a 2-min nominal time interval to lead to such a large change. With regularized regions being largely similar among members of MP1 (not shown), we have found that a change of a single grid point experiencing regularization in the boundary layer near the RMW can cause steplike behavior in solutions. Shown in green in Fig. 9b is a sample metric of regularization, namely, the number of \(\bar{F}\) regularized grid points at 500 m height between radii 0 and 30 km; so defined, this metric can only produce an integer value between 0 and 11, but never exceeds 3 in this period of analysis. Of course, this is not the exclusive metric of changes in regularization in this region of the simulation. Nevertheless, it is evident that this metric ranges from 0 to 3 in this time interval and that each of the stepwise changes in value of the metric is associated with stepwise changes in the peak inflow of the MP1 solutions. The size of the stepwise change in value of peak inflow provides an estimate of the uncertainty of the extended Eliassen solver. Specifically, this uncertainty can be as large as 2.5 m s\(^{-1}\) in this quantity. This finding confirms that regularization in the boundary layer using the unbalanced vortex limits the accuracy of the Eliassen solver in contrast to the balanced sets of solutions MS1 and MH1 (cf. Wang and Smith 2019). As a point of comparison, we note that the solutions MS1 and MH1 do not show any such jumps in peak inflow as found in MP1.

The peak upper-level outflow from the P1 solution shown above (Fig. 5g; corresponding to a nominal time of 53.5 h in Fig. 9) is quite close to the simulated peak outflow, but the MP1 solutions are relatively close to EX-1 only through the early period from 52 to 54.5 h (Fig. 9e). The MP1 solutions indicate a weakening of the peak outflow during the quasi-steady later period from 54.7 to 56 h, but this behavior is opposite the tendency found in the simulation. The intensification rate from the MP1 solutions (Fig. 9d) has an intensification bias less than 5 m s\(^{-1}\) h\(^{-1}\), but this bias varies with time, being less than 1 m s\(^{-1}\) h\(^{-1}\) at 53 h, greater than 4 m s\(^{-1}\) h\(^{-1}\) at 54 h, and near 55 h strengthening is shown by MP1 when the EX-1 simulation shows weakening.

6. Summary and conclusions

This study has reexamined a claim by Heng et al. (2017) that “balanced dynamics can well capture the secondary circulation in the full-physics model simulation even in the inner-core region in the boundary layer.” While this claim was rebutted by Montgomery and Smith (2018), the fundamental nature of the claim merits further examination.

To minimize interpolation errors between the numerical model and the balance model, a uniform vertical grid spacing of \(\delta z = 500\) m for both models was chosen and demonstrated to be adequate for detailed comparisons. Azimuthally averaged forcings of tangential momentum and heat are diagnosed from the numerical simulation and used to force an axisymmetric Eliassen balance model under strict balance conditions using two different definitions for the balanced vortex. The balance solutions are found to be deficient in several respects. To wit, the balance solutions

- greatly underestimate the peak inflow velocity in the boundary layer;
- systematically over predict the radial location of peak inflow;

\(^{12}\) Intensification here is evaluated over a region near the time-averaged radius of maximum tangential winds across the boundary layer. With the grid mesh used here for the numerical solver of the Eliassen balance equation, the average is conducted over nine grid points in a \(3 \times 3\) box; the radial positions are \(R - 3\) km, \(R\), and \(R + 3\) km, where \(R\) is the radius of maximum winds; and vertical positions are 0, 500, and 1000 m. In the vicinity of the RMW there are very large gradients evident in the calculated intensification rate from the S1, H1, and P1 solutions (Figs. 6b,c,d) that the simulation (Fig. 6a) does not exhibit. Very small location changes lead to very large variations of the solution-derived intensification rates, prompting the use of a more robust metric using a spatial average.
systematically exaggerate the vertical structure of inflow in the high-wind-speed region of the vortex; in accurately represent the radial and vertical structure of the upper-level outflow layer of the vortex.

The first three deficiencies are consequences of the intrinsically unbalanced and nonlinear nature of the vortex boundary layer. These deficiencies highlight the inability of the balance model to represent the nonlinear boundary layer spinup mechanism. The fourth deficiency can be attributed to significant degrees of inertial instability in the upper-tropospheric outflow layer. In both situations, a regularization method is needed to obtain weak solutions to the balance model. The relatively exaggerated layer of inflow predicted in the balance solutions results in an alarmingly exaggerated spinup tendency above the boundary layer in association with the exaggerated radial influx of absolute vorticity. Such a deep inflow layer, a hallmark of the classical spinup mechanism for tropical cyclones comprising the radial advection of absolute angular momentum above the boundary layer, is not found in the full-physics simulation during the chosen period of most rapid intensification.

When the Eliassen solver is run with vortex parameters that are the azimuthal average of the model output, the vortex is generally not in thermal wind balance, but the diagnosed overturning circulation and implied tangential wind tendency

![Fig. 7](image-url)

Time series of quantities from the MS1 set of solutions (red), plotted as a function of a nominal time at the center of the 24 min time bracket corresponding to each of the 90 solutions in the set, and similar quantities from the EX-1 simulation (black), time averaged to the same nominal time. (a) Maximum tangential winds \( v \) from the EX-1 simulation (dashed) and average \( v \) in an RMW box (solid, defined in the text). (b) Peak inflow \( u \) at the \( z = 500 \) m height level from the EX-1 simulation and MS1 solutions. (c) Peak outflow \( u \) aloft from the EX-1 simulation and MS1 solutions. (d) Rate of intensification from the EX-1 simulation and MS1 solution both averaged over the RMW box.

- systematically exaggerate the vertical structure of inflow in the high-wind-speed region of the vortex;
- inaccurately represent the radial and vertical structure of the upper-level outflow layer of the vortex.

The first three deficiencies are consequences of the intrinsically unbalanced and nonlinear nature of the vortex boundary layer. These deficiencies highlight the inability of the balance model to represent the nonlinear boundary layer spinup mechanism. The fourth deficiency can be attributed to significant degrees of inertial instability in the upper-tropospheric outflow layer. In both situations, a regularization method is needed to obtain weak solutions to the balance model. The relatively exaggerated layer of inflow predicted in the balance solutions results in an alarmingly exaggerated spinup tendency above the boundary layer in association with the exaggerated radial influx of absolute vorticity. Such a deep inflow layer, a hallmark of the classical spinup mechanism for tropical cyclones comprising the radial advection of absolute angular momentum above the boundary layer, is not found in the full-physics simulation during the chosen period of most rapid intensification.

When the Eliassen solver is run with vortex parameters that are the azimuthal average of the model output, the vortex is generally not in thermal wind balance, but the diagnosed overturning circulation and implied tangential wind tendency
reproduces many of the salient features of the azimuthally averaged numerical simulation. However, the improved agreement is the result of a pseudobalance solution, a crude, zero-order approximation of the nonlinear boundary layer dynamics during spinup, and incorrectly identified as a balance solution in prior works. The difference between the balance and pseudobalance formulations becomes significant at hurricane intensity and beyond when the nonlinear unbalanced boundary layer dynamics become dominant in the inner-core region. The results concerning the pseudobalance solution confirm the findings of Bui et al. (2009) examining the spinup of a hurricane vortex. All of the foregoing results have been essentially verified using doubled vertical resolution ($\delta z = 250$ m) sensitivity experiments (see appendix B). The only caveat is that the Smith balance vortex fails to produce a convergent solution to the Eliassen equation, at least using the relaxation algorithm. The Holton and pseudobalance formulations yield similar results to their coarse-resolution counterparts. We conclude that vertical resolution of the balance model is not a major factor limiting the veracity of our findings.

The comparison between the simulation and balance model was extended to 90 solutions during a 3-h spinup period spanning the time of maximum intensification rate. Again, the
balance model consistently predicts spindown in the inner-core vortex boundary layer where the simulated vortex is spinning up. In time, this discrepancy would yield a decoupling of the tangential winds in the boundary layer and interior and, ultimately, to a complete collapse of the spinup process. The pseudobalance solutions, while reducing the discrepancy at some individual times, are shown to overpredict the intensification rate at other times. The results underscore the unreliability of pseudobalanced solutions in consistently capturing the physics of rapid hurricane spinup in a realistic simulation.

The results support prior findings of Bui et al. (2009) and Abarca and Montgomery (2014, 2015) and refute the implication that the Eliassen balance model captures the primary characteristic of hurricane intensification in both the interior and boundary layer regions of an intensifying hurricane vortex. The results highlight the necessity of the nonlinear boundary layer spinup mechanism to complete the description of spinup in the boundary layer and eyewall of an intensifying hurricane (Montgomery and Smith 2018).

In closing, it should be recalled that if too large vertical diffusivities are used in the numerical simulation, then the
nonlinear boundary layer spinup mechanism will not be manifest (Smith and Thomsen 2010). The focus in this paper has been on realistic configurations using the latest observational guidance.

Acknowledgments. This paper arose out of stimulating exchanges we had with Dr. Yuqing Wang during our participation in the 33rd American Meteorological Society Conference on Hurricanes and Tropical Meteorology in Ponte Vedra, Florida, in 2018. MTM and JP acknowledge the support of NSF Grants AGS-1313948 and IAA-1656075, ONR Grant N0001417WX00336, and the U. S. Naval Postgraduate School. The Eliassen FORTRAN code and plotting routines are available by writing to the first author. The views expressed herein are those of the authors and do not represent sponsoring agencies or institutions.

APPENDIX A
Solving the Eliassen Equation

This appendix provides some details on the solution of the Eliassen equation, Eq. (3), given in section 3.

Since the vortex is assumed to be always in a state of thermal wind balance, the coefficients (\( A, B, C \)) at any time are assumed to be associated with an axisymmetric vortex \((u, p, \theta)\) in strict axisymmetric thermal wind balance.

For a vortex that is both statically stable and inertially stable, \( N^2 \) and \( \tilde{F} \) are both positive, and provided that the vertical shear is not too large, the criterion for ellipticity is everywhere satisfied. Typically, this is the case at early times in the CM1 simulations before rapid intensification ensues. However, as the vortex starts to intensify, small regions develop in the upper tropospheric outflow region where the flow is inertially unstable, \( \tilde{F} < 0 \). Regions of negative \( \Delta \) are found also in the vortex boundary layer under intense tangential winds (where \( \delta w/\delta z \) may become large in \( B \)), at the top of the boundary layer in the eye (where as a result of maintaining balance, \( \delta V/\delta z \) may become negative in \( A \)) and outside of the eyewall in the region of upper-level compensating subsidence.

a. Regularization details

In instances where \( \Delta < 0 \), the vortex is symmetrically unstable. To guarantee a unique solution balance, it is necessary to regularize the equation for \( \psi \) by removing the regions of symmetric instability. To accomplish the regularization we follow the spirit of the strategy presented by Möller and Shapiro (2002), whose overarching aim was to make a minimal alteration of the coefficients of the Eliassen equation so as to render the equation elliptic and hence solvable as an inversion problem. This procedure is quite unlike a more drastic redefinition of the basic-state vortex in the boundary layer so that it has zero vertical shear there (cf. Stern et al. 2015; Heng et al. 2018). The latter represents a global alteration of the vortex.

Our regularization procedure differs in only two small respects from Smith et al. (2018). First, having calculated the coefficients \( A, B, \) and \( C \) at a particular time or averaging interval, we search for points within the \((r, z)\) grid mesh at which \( \Delta < 0 \). At such points, either the product \( AC < 0 \) or \( |B| \) is large in relation to \( A, C \). Generally, \( A > 0 \) results from dry static stability, and \( C > 0 \) results from inertial stability. Where \( A < 0 \), \( A \) is redefined to a small positive value, that is, \( A = \pm 0.01 \). Generally, \( C > 0 \) results from dry static stability, and \( \Delta > 0 \) results from inertial stability. Where \( \Delta < 0 \), \( \Delta \) is redefined to \( \Delta = 0 \). At any remaining points where \( \Delta < 0 \), the vertical shear, and therefore \( B \), must be relatively large since \( C \) and \( A \) are both positive. At such points we redefine the vertical shear (and hence \( B \)) to zero. This latter step is slightly more stringent than the procedure adopted by Smith et al. (2018), which set \( B \) to an intermediate value \( \pm \sqrt{AC} \) and \( \Delta = \pm AC \). This more stringent regularization procedure yields robust solutions at any analysis time in the numerical experiments. Sensitivity experiments setting \( B = \pm 3.99AC \), giving \( \Delta = 4AC - 3.99AC \) strictly positive, provides essentially identical results (not shown) except in the boundary layer inflow (around a regularization point) where the peak inflow is a bit weaker (\( \sim 9.87 \text{ m s}^{-1} \)) than that of the S1 solution presented in section 5a.

b. Some numerical details

On the discrete \((r, z)\) mesh of points, the solver iterates for \( \psi \) by minimizing the quantity \( R \) (the residual):

\[
R = \frac{A}{(\Delta r)^2} + \frac{C}{(\Delta z)^2}
\]

via successive overrelaxation

\[
\psi_{k+1} = \psi_k + \frac{\omega R_k}{2} \left[ \frac{A}{(\Delta r)^2} + \frac{C}{(\Delta z)^2} \right]^{-1},
\]

where \( A, B, C, \) and \( \Theta \) are defined in section 4a, and where

\[
D = \frac{\partial A}{\partial r} + \frac{\omega B}{2 \partial z},
\]

\[
E = \frac{\omega B}{2 \partial r} + \frac{\partial C}{\partial z}.
\]

\( \omega = 1.8 \) is the relaxation parameter, and \( k \) is the iteration index.

Here, the operator \( \delta \) represents the discrete derivative in the direction of the subscript, with second-order derivatives with second-order subscripts, etc. The iteration is determined to have converged if the magnitude of \( R_k \) is less than \( 10^{-24} \text{ K}^{-1} \text{ s}^{-3} \) at all interior \((r, z)\) grid points in the solution domain; this is order \( 10^{-11} \) relative to the magnitude of the first term of (7).

As per a reviewer request, Fig. A1 shows the results of using (i) the second-order accurate finite-difference formulation in the interior and first-order accurate formulation of \( \psi \delta \delta \delta \) on \( z = 0 \) (Fig. A1a) employed in our prior studies (e.g., Bui et al. 2009, Abarca and Montgomery 2014, Abarca and Montgomery 2015), together with (ii) a second-order accurate formulation for \( \psi \delta \delta \delta \delta \) at both interior and boundary points (Fig. A1c). These calculations utilize the Holton balance vortex during the same time period examined in the main text. The second-order accurate formulation (Fig. A1c) shows a slightly stronger surface inflow compared with the mixed accuracy calculation.
Despite the marginally enhanced near-surface inflow, the balanced inflow is still significantly weaker than that of the simulation (cf. Fig. 5). The corresponding tangential wind tendency, calculated according to Eq. (13), is increased marginally at the surface (cf. Figs. A1b and A1d). Both tendency calculations exhibit only a very narrow subset of points with a positive spinup near the maximum tangential wind at the 750 m height level. However, the bulk of the boundary layer tangential winds are spinning down in contrast to the numerical simulation (see Fig. 6a). In summary, while the second-order accurate formulation of the balanced inflow yields marginally stronger inflow near the surface compared to the mixed accuracy formulation, the bulk of the boundary layer is spinning down, in stark disagreement with the numerical simulation.

APPENDIX B

Solutions at 250 m Grid Spacing

The EX-3 simulation is provided as a test of the above techniques when applied at higher vertical resolution. EX-3 is like EX-1, except for the use of a 250 m fixed vertical grid
spacing. The Eliassen solutions from EX-3 are computed also using 250 m grid spacing. The evolution of EX-3 is broadly similar to EX-1, except with about a 10 h delay in the timing of maximum intensification (see Fig. B1).

The Holton-balance solution using EX-3 is noted here as H3. The period 61.3 to 61.7 h is used for the H3 solution, at which time the EX-3 simulation has approximately the same intensity as EX-1 during the 53.3 to 53.7 h period used for the H1 solution (cf. Figs. 1 and B1). Figure B2 shows the H3 solution, to compare with Fig. 5. The regularization regions (shown in gray) are broadly similar for H1 and H3, but H3 shows an additional region at the 12-km level at the top of the eyewall updraft (Figs. B2c,d), which is a focal point for a local extremum of \( \psi \). This local extremum generates a sharp dipole in \( w \) and \( u \), with the peak inflow found in this region (\(-21.66 \text{ m s}^{-1}\) at 54 km radius and 12 km height), not in the boundary layer. The peak inflow for this period in the EX-3 simulation is \(-13.24 \text{ m s}^{-1}\) at 24 km radius and 125 m height; the H3 solution interpolated to 125 m height has a peak inflow of \(-8.4 \text{ m s}^{-1}\) at 45 km radius. The inflow from the solution H3 is much weaker than that of the EX-3 simulation (Figs. B2a,b), as is the rate of intensification (Figs. B3a,b) (which is actually negative in the boundary layer under the eyewall cloud). This finding of weaker inflow in the boundary layer and negative intensification rate is very similar to how the H1 solution has a weaker boundary layer inflow and intensification rate than EX-1 (see Table 1). Moreover, solution H3, like H1, shows a deeper inflow that extends to 4 km height in the radial interval from 50 to 90 km.

The corresponding Smith-balance solution, S3, for EX-3 fails to converge during this time period. However, the corresponding pseudobalance solution, P3, for EX-3 converges without any anomalous streamfunction extremum in the eyewall and resembles the simulated secondary circulation and intensification rate (Figs. B2e, B2f, B3c) approximately to the same degree that P1 resembles EX-1.

Per a reviewer request, we also provide a solution H3-A, the Holton balance solution using EX-3 data but solved with a 500 m vertical grid spacing. The reduction of resolution in the Eliassen solver would have the following conditions: 1) the descriptions of the mass field (pressure, density, and potential

---

**Fig. B1.** Time series of maximum azimuthally averaged tangential winds from the EX-1 and EX-3 simulations. Overlaid are dashed lines indicating the 53.3 to 53.7 h interval used for EX-1 and the 61.3 to 61.7 h interval used for EX-3 for computing Eliassen solutions.

**Fig. B2.** Contours of (a),(c),(e),(g) radial velocity \( u \) and (b),(d),(f),(h) vertical velocity \( w \) from the EX-3 simulation in (a) and (b), averaged between 61.3 and 61.7 h, and the solutions H3 in (c) and (d), P3 in (e) and (f), and H3-A in (g) and (h). Blue contours are negative, red contours are positive. The union of \( A \), \( B \), and \( C \) regularizations from the corresponding solutions are shown by gray hatching in (c)–(f). Green diamonds denote the locations of minimum and maximum of each contoured field.
temperature) are degraded in resolution; 2) the description of the gradient tangential wind is degraded in resolution [note here that the gradient wind we compute for the H3-A calculation is computed at model grid spacing (250 m), then regularized for imaginary values at model grid spacing, then degraded for the solver]; 3) the descriptions of the forcing terms (heating rate and momentum forcing) are degraded in resolution; 4) the solver executes at a reduced resolution. The areas of interest where resolution may be expected to affect the solution are (i) the upper-level eyewall region, where some dry static instability is shown in the 250 m grid spacing simulation and is not shown in the 500 m simulation, and (ii) the bottom boundary, where the momentum forcing term has an extremum that would be reduced in value after degraded interpolation.

The secondary circulation of the H3-A solution is shown in Fig. B2. Two aspects of the inflow are worth noting: 1) the radial inflow of H3-A is elevated to 4 km height (shown by the blue $1 \text{ m s}^{-1}$ contour at 60 km radius) in the same manner as H3; and 2) the peak inflow of the boundary layer of the H3-A solution, like H3, is much weaker than the simulation, with a value of 9.1 m s$^{-1}$ at 45 km radius. The value of peak boundary layer inflow in H3-A is nonetheless of slightly greater magnitude than that found in H3 ($8.4 \text{ m s}^{-1}$ at 45 km radius). Other features of note, the inflow jet at 7 km height is weaker in H3-A than in H3; and the dry static instability shown in the H3.
solution at 12 km height vanishes in H3-A, permitting an untainted calculation of the peak outflow maximum of 13.8 m s⁻¹, which exceeds that of the simulation but is at a smaller radius (60 vs 82 km). The intensification rate (Fig. B3) resulting from H3-A is broadly similar to H3, maintaining a negative tendency in the boundary layer at the RMW.

On the whole, the findings obtained with the Eliassen solver using EX-1 with 500 m vertical grid spacing apply also using EX-3 at 250 m vertical grid spacing (and its degraded version as discussed above), when such solutions can converge. Convergent solutions using a strictly balanced vortex consistently underestimate the strength of the boundary layer inflow and associated intensification rate in comparison with the nonlinear simulation. A practical advantage of the numerical experiment EX-1 is that the solution provides an opportunity to evaluate the limitations of the Eliassen balance model using two different balance states (S1 and H1).

REFERENCES


