An Upper-Ocean Model for Short-Term Climate Variability

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ABSTRACT

The authors propose and assess principles for the design of an upper-ocean model (UOM) suitable for studies of large-scale oceanic variability over periods of a few months to many years. Its essential simplification when compared with a conventional full-depth model (FDM) is the specification of an abyssal climatology for material properties. Observational analyses of temperature and salinity fluctuations demonstrate their degree of confinement to the upper ocean. Two idealized models for diffusive penetration of tracer fluctuations and for wind-driven currents show that the UOM approximations are usually accurate for the phenomena of interest. A UOM for the oceanic general circulation is constructed, and its solutions are compared with those of an equilibrium FDM. From a stratified resting state, the UOM spins up to an equilibrium state over a period of about 30 yr. The UOM and FDM solutions agree well in both the mean state and short-term climate fluctuations, even for cases for which the model parameters and forcing are modestly inconsistent with the UOM’s abyssal climatology. A UOM can therefore be a useful, efficient tool for studies of coupled climate dynamics and sensitivity to forcing fields and model parameters, and for hypothesis testing about the roles of the abyssal ocean.

1. Introduction

Climate is generally understood as the quasi-equilibrium state of the atmosphere that changes with changes in its lower-surface conditions, air composition, and solar forcing. The oceans are thus an important part of climate, through exchanges with the atmosphere that are dependent upon the thermal and chemical state of the sea surface. Oceanic surface conditions are shaped by dynamical and chemical processes throughout the oceans (e.g., the global thermohaline circulation and its associated meridional heat and water fluxes). However, fluctuations in oceanic material properties induced by anomalous air–sea fluxes of short duration tend to remain within the upper ocean until they are later reexchanged with the atmosphere. In contrast, long-term changes in climate—as over the course of an ice age or of anthropogenic global warming—have associated material changes in both the upper and abyssal oceans.

This suggests that the ocean’s roles in the dynamics of climate may be simplified by restricting the active domain to the upper ocean, if the focus is on large-scale, short-term variability over periods of a few months to many years. This range of scales is the one most relevant to current efforts at climate prediction (e.g., El Niño). However, in trying to make such a simplification, we must ask what comprises a dynamically consistent model for the short-term variability of material properties (i.e., “tracers”) in the upper ocean. Our dynamical hypotheses are the following.

• The variability of upper-ocean tracers depends importantly upon the time mean tracer distributions and circulation (which themselves cannot be consistently determined entirely within the upper ocean) because of advective nonlinearity.
• The variability of pressure and currents in the upper ocean is tightly coupled with that of tracers.
• Barotropic fluctuations in pressure and currents (i.e., averaged over the whole ocean depth) also have significant short-term climate variability coupled with that of tracers.
• Oceanic mesoscale eddies (which also cannot be consistently determined entirely within the upper ocean) and even finer-scale fluctuations are so loosely coupled to short-term climate variability that their transport effects can be parameterized rather than resolved explicitly.

In this paper, we present a simplified oceanic general circulation model, an upper-ocean model (UOM), based upon these hypotheses.
There are two reasons why such a model can be a useful alternative to a conventional, full-depth general circulation model (FDM). One reason is its value as a conceptual tool for testing the hypotheses enumerated above: if a UOM is skillful in its solution behavior, then we may conclude that the processes it excludes are unimportant for the phenomena of interest. The other reason is the relative economy of a UOM compared to an FDM. This is partly because of the smaller volume of upper-ocean domain, with a typically surface-intensified vertical grid, a UOM may have about half the number of levels as an FDM and thus require about half the computational work per time step. (Of course, because short-term tracer fluctuations have fine vertical scales near the surface, it will be tempting to reinvest the savings in a denser UOM grid.) The greater part of the economic advantage, though, comes from the much faster adjustment to equilibrium in a UOM. We show below (section 4c) that this occurs in a few decades instead of the millennia required for an FDM. In our view, having a well-defined equilibrium state greatly clarifies the interpretation of forced and intrinsic, short-term climate variability, by avoiding confusion with the undetermined trends present in a nonequilibrium FDM solution. Thus, even if one were to reinvest this savings in, say, a tenfold increase in the number of solutions—to better explore the parameter sensitivities, solution behaviors, and statistical reliability of climate variability—there would still remain a tenfold savings compared to using an equilibrium FDM for fewer solutions. A common practice in FDMs is to use accelerated convergence methods to obtain equilibrium, reducing computational expense (Bryan 1984). Even compared to such a model, a UOM can still achieve a tenfold savings. However, such acceleration methods may not always be acceptable, especially with high-frequency forcing, thus making the relevant comparison standard questionable.

There have been many upper-ocean models for particular phenomena (e.g., O’Brien and Hurlburt 1972; McCreary and Lu 1994; Gent and Cane 1989; Hurlburt and Thompson 1980; Schopf and Loughe 1995). Typically their focus has been on the wind-driven response, with quite limited roles for diabatic processes, and on regional rather than global circulation. Universally these models have been based on the assumption of a stagnant abyss, which distorts both the mean flow (by prohibiting vertical motions across the lower boundary) and the barotropic currents (by compressing them within too small a vertical extent, if not excluding them altogether). Such an assumption may be valid for tropical or coastal wave events, but it is inadequate for global climate variability and the general circulation.

Based on the dynamical assumptions listed above, the design principles for our UOM are discussed in section 2. We then make some a priori assessments of the plausibility of a UOM in section 3, firstly by analyzing measurements of the vertical structure of tracer fluctuations and then by solving idealized problems for diffusive penetration of tracers and for adiabatic, wind-driven currents. In section 4, a UOM for the global general circulation is specified, and its solutions are examined for their adjustment to equilibrium, mean state, and fluctuation behaviors in comparison with analogous FDM solutions. Our conclusions are presented in section 5. Some justification, equation, discretization formulas, and implementation details are given in the appendixes.

2. Model design principles

We specify the UOM domain by \(0 \leq z \leq -D'(x, y)\), subject to the constraint that \(D' \leq D(x, y)\), the full ocean depth (see Fig. 1). Here \(x\) and \(y\) are eastward and northward geographical coordinates, and \(z\) is the upward vertical coordinate. Within the UOM domain, the full hydrostatic primitive equations are solved for velocity, tracers, and pressure. Where \(D' < D\), the lower boundary conditions for horizontal velocity and tracers are such that their vertical advective and diffusive fluxes draw from abyssal reservoirs with specified climatological values. The reference climatology is a function of \(x, y, \) and \(t\), usually in the form of the mean seasonal cycle. The source of the climatology could be either from an observational analysis or from an FDM solution; we use the latter in our present implementations, because it provides spatially and temporally complete information that is dynamically consistent to the degree that the UOM and FDM equations are. Where the UOM extends to the bottom in shallow water, with \(D' = D\), its lower boundary conditions are the same as in an FDM.

In almost all UOM uses we can imagine, \(z = -D'\) will lie below the pycnocline, down to which short-term, large-scale variability is prominent in the ocean. We allow spatial variation in \(D'\) to take advantage of the geographical variability of the pycnocline depth, which can be as shallow as \(~100\) m in the Tropics and as deep as \(~1000\) m in subpolar regions. This led us to use a stretched \(\sigma(x, y, z)\) vertical coordinate, with both \(z = 0\) and \(z = -D'\) as iso-\(\sigma\) surfaces and a surface-intensified grid resolution. Although there is a well-known com-

![Fig. 1. Schema of the UOM.](image-url)
computational delicacy in $\sigma$-coordinate FDMs where the depth range and steepness of the bottom topography are especially large (Mellor and Blumberg 1985; Haidvogel et al. 1991; Haney 1991; Barnier et al. 1998), we can avoid this problem in our UOM by choosing $D'(x, y)$ to be both much less than $\max(D)$ and geographically smooth. Where $D' = D$, of course, our usage is subject to the same constraints as in FDMs.

In addition, we carry the depth-averaged (i.e., barotropic) horizontal velocity and pressure as dependent variables in the UOM. Their evolution is governed by depth-averaged momentum and continuity equations. The UOM simplification here is that all contributions to these equations from deviations in abyssal variables (i.e., with $z < -D'$) from their barotropic components are replaced by their climatological values. This UOM rule precludes, for example, fluctuations in the baroclinic topographic torque, $\alpha^*D'$ and $\beta^*S'$ and the impact of baroclinic and relief (JEBAR; Sarkesyan and Ivanov 1971).

This formulation permits a UOM solution to be equivalent to an FDM solution, including its mean and barotropic currents, in the special case where their model specifications and numerical grids are exactly the same within the UOM subdomain. Our model dynamics and discretization formulas match those of a current version of the National Center for Atmospheric Research (NCAR) Climate System Model Ocean Model (NCOM; Gent et al. 1998), except for the UOM modifications described above. NCOM has been used with some success in both uncoupled simulations of the general circulation (Danabasoglu et al. 1994; Large et al. 1997) and in coupled climate simulations (Boville and Gent 1998). Our UOM algorithm could be adapted to match any other FDM.

We envision a usage strategy in which many UOM solutions are calculated with only occasional changes in the abyssal climatology. This is practical, since only rarely is a new “benchmark” equilibrium FDM solution calculated, because of its computational expense, and a new observational climatology is likely to be even rarer. The variety of UOM solutions can include different choices in the forcing fields or atmospheric coupling, initial conditions, parameter values, parameterization forms, discretization formulas, and grid resolutions (interpolating the climatological fields as needed). Each of these differences, of course, implies some degree of dynamical inconsistency between the resulting UOM equilibrium climatology and the climatology used in its lower boundary conditions and abyssal barotropic dynamical influences. A UOM user must therefore be alert to whether this incompatibility is more damaging than the disequilibrium behavior of an FDM alternative. In section 4 we demonstrate that these types of inconsistencies can sometimes be unimportantly small.

3. Observational and theoretical assessments

a. Observational evidence

To assess the depth of penetration of potential temperature, $\theta$, and salinity, $S$, anomalies, we first analyze the Levitus and Boyer (1994) and Levitus et al. (1994) climatological datasets. Our analysis is based on the mean monthly data, and harmonic temporal interpolation (Epstein 1991) is used below 1000-m depth to construct monthly means from the seasonal Levitus data. The globally averaged variance profiles of $\alpha^*\theta'$ and $\beta^*S'$ are presented in Fig. 2, where primes denote monthly anomalies and $\alpha^*$ and $\beta^*$ are the local, time mean thermal expansion and saline contraction coefficients, respectively. In a linearization of the equation of state, $\alpha^*\theta'$ and $\beta^*S'$ express the respective contributions to the relative anomaly in density. The figure shows that the largest seasonal variability is confined to the upper 200 m, with a quasi-exponential decay by several orders of magnitude in the deep ocean.

We next analyze the $\theta$ and $S$ time series data at Hydrostation S (HS) located at about (32°N, 64°W) near Bermuda. This is a nearly continuous time series starting in 1954 (Michaels and Knap 1996). Joyce and Robbins (1996) show that the decadal variability at this site is representative of an extensive region in the North Atlantic Ocean. In the HS data we can distinguish the frequency content of the variability. By calculating mean monthly anomalies in $\alpha^*\theta'$ and $\beta^*S'$, we estimate the mean seasonal variance profiles in Fig. 3 both for HS data and for Levitus data evaluated at this location. There are two peaks in the variability, the largest at the surface and a secondary one in the main pycnocline around 750-m depth; the former may be mostly due to surface exchanges with the atmosphere, while the latter is probably mostly a result of adiabatic movements of the pycnocline caused by transient currents. The decay of var-
In summary, the short-term climate variability of tracers appears as white noise with the temporally sparse sampling at HS (Doney 1996); it may also indicate the effects of other tracer transport mechanisms than vertical advection, which may often be comparable in magnitude to the vertical (or, more fundamentally, diapycnal) diffusion in the low-frequency variability of the oceanic general circulation. The model represents horizontal variability only implicitly insofar as it might appear as energetic mesoscale eddy activity, whose nearly adiabatic tracer fluctuations may be due to energetic mesoscale eddy activity, whose nearly adiabatic tracer fluctuations may appear as white noise with the temporally sparse sampling at HS (Doney 1996); it may also indicate the effects of other tracer transport mechanisms than vertical diffusion. Also, there are problems at HS with measurement errors and sampling gaps. To diminish these biases, we use only annual mean data for further analysis of the low-frequency variability. First we apply a 5-yr running mean to obtain low-pass data. The subtraction of this from the annual-means yields the residual, bandpass data. The corresponding variance profiles are also included in Fig. 3. The low- and bandpass variances are similar in both $\theta$ and $S$ at all depths. Beneath the upper 200 m, they both are stronger than the seasonal variability. The amplitudes of the low-frequency variability are at least two orders of magnitude smaller in the abyssal ocean than near the surface in the pycnocline.

In summary, the short-term climate variability of tracers decays rapidly with depth, with relatively small amplitudes beneath the pycnocline. This suggests that a UOM may be able to capture the most important part of this behavior.

**b. Penetration by material diffusion**

A particularly simple conception of the ocean’s role in climate variability is as a passive receptor of surface material exchanges caused by atmospheric fluctuations. Hasselmann (1976) represents this behavior with an oceanic surface mixed layer, but this representation does not allow for variations in the boundary layer mixing rate and thickness nor for penetration into the interior pycnocline. A generalization of this conception that overcomes these limitations is to a vertically diffusive ocean. Here we use this simple model to assess the accuracy of the UOM approximations with respect to this process.

We consider a material tracer with concentration, $c(z, t)$, that evolves by linear diffusion:

$$\partial_t c = \partial_z [\kappa(z, t)\partial_z c],$$

with a vertical diffusivity, $\kappa$, whose $(z, t)$ variability represents the variability of turbulent mixing in the ocean. The independent variables in the subscripts denote partial derivatives with respect to those variables. Because the model is linear, we can consider fluctuations in $c$ separately from its mean distribution, $\overline{\mathcal{C}}(z)$ (if we ignore any correlated variability between $\kappa$ and $c$). The boundary conditions for (3.1) are an imposed transient flux at the upper surface, $z = 0$, and no flux at the ocean bottom, $z = -D$:

$$\kappa \partial_z c(0, t) = e(t), \quad \kappa \partial_z c(-D, t) = 0,$$

where we assume $\overline{\mathcal{C}} = 0$. This model ignores the effect of vertical advection, which may often be comparable in magnitude to the vertical (or, more fundamentally, diapycnal) diffusion in the low-frequency variability of the oceanic general circulation. The model represents horizontal variability only implicitly insofar as it might be applied independently at different $(x, y)$ points with different $\kappa$ and surface forcing. Horizontal advection is also absent here, but this process does not contribute directly to the vertical penetration of $c$.

The special case of constant $\kappa$ is particularly easy to analyze (see also Saravanan et al. 2000). We take the Fourier transform in time of (3.1)–(3.2), and denote transformed variables by

$$\tilde{c} = c(z, \omega), \quad -\kappa \partial_z \tilde{c}(0) = \tilde{e}(\omega), \quad \partial_z \tilde{c}(-D) = 0,$$

where $\omega$ is the angular frequency. The ordinary differential equation solutions of (3.3) have exponential dependences on $\gamma \omega$, where

$$\gamma = \left( \frac{\omega}{\kappa} \right)^{1/2},$$

where $I = \sqrt{-1}$. Without loss of generality we choose the $\gamma$ root with positive real part. Since $\gamma$ is complex, the vertical structure is thus a combination of oscilla-
tions and growth or decay. The explicit solution to (3.3) is
\[
\hat{c} = \frac{\hat{e} \cosh[\gamma(z + D)]}{\gamma \sinh[\gamma D]}.
\] (3.5)

For assessing the UOM, we are mostly concerned with \(|z| \ll D\), in which case we can approximate (3.5) by
\[
\hat{c} \approx \frac{\hat{e}}{\gamma} e^{\gamma z}.
\] (3.6)

The spectrum of the tracer fluctuations is
\[
C(\omega, z) = |\hat{e}|^2 = d_a^2 |\hat{e}|^2 e^{-2|z|/d_a},
\] (3.7)
where
\[
d_a = (\kappa/\omega)^{1/2}
\] (3.8)
is the penetration depth of the tracer fluctuations.

At the surface, the tracer spectrum is simply a reden
dened form of the surface flux spectrum,
\[
C(\omega, 0) = \frac{K}{\omega} |\hat{e}|^2.
\] (3.9)

The spectrum of tracer fluctuations at any interior level, \(z < 0\), is the product of the surface flux spectrum times a frequency-dependent vertical decay factor that is stronger for higher frequencies. The vertical decay length, \(d_a\), increases with the diffusivity, \(\kappa\), and decreases with the frequency, \(\omega\). Thus, at any given frequency the vertical structure of the tracer spectrum is exponentially decaying, and at any given depth the spectrum shape is exponentially decaying \(\propto e^{-\omega z/a}\) for some \(a > 0\). Together these behaviors imply that short-term climatic fluctuations of tracers penetrate only a limited depth into the upper ocean, \(O(d_a)\). This indicates that a UOM is likely to calculate such behavior accurately.

We now estimate the magnitude of \(d_a\). A canonical value for \(\kappa\) in the stable oceanic pycnocline is \(10^{-5}\) m\(^2\) s\(^{-1}\). For a frequency range of short-term climatic variability,
\[
5 \times 10^{-6} \geq |\omega| \geq 2 \times 10^{-5}\text{ s}^{-1},
\]
the corresponding range for the penetration depth is 25 \(\geq d_a \geq 1\) m. These depths are modest compared to the upper-ocean thickness in a typical UOM configuration (e.g., \(D' = 500\) m in the Tropics and 1250 m in high latitudes; see section 4). In the surface planetary boundary layer (PBL), \(\kappa\) values are larger by several orders of magnitude, but the PBL thickness is usually only \(O(100)\) m. Hence, even if tracers penetrate quickly through the PBL and the regime of small \(\kappa\) only begins below, there is still plenty of room in \(0 \geq z \geq -D'\) for the UOM to contain most of the short-term tracer variability. Only under the rare circumstances of deep convection, where \(\kappa\) has large values from the surface to beneath \(z = -D'\), will the tracer fluctuations not be well confined to the upper ocean. There, the UOM approximations imply a rapid mixing down to its bottom, with subsequent diffusive fluxes through the lower boundary based on a locally large \(\kappa\) value and a climatological abyssal-ocean tracer reservoir. The error in this representation is the inability to alter the abyssal tracer values for calculating the flux exchanged with the upper ocean. This error may often not be large, since abyssal tracer values do not vary much.

c. Wind-driven response

Here we assess the accuracy of the UOM approximations in the context of a wind-forced, steady-state, response with midlatitude, quasigeostrophic dynamics in Cartesian coordinates, \((x, y, z)\), linearized about a stably stratified state of rest. The standard of accuracy is a full-depth, quasigeostrophic model.

The quasigeostrophic equations are vertically discretized into \(N\) levels with thicknesses \(\tilde{h}_n\), \(n = 1, \ldots, N\). The density stratification is represented by the reduced gravitational accelerations across the interior interfaces between the levels, \(g'_j = (\tilde{\rho}_j - \tilde{\rho}_{j-1}) / \tilde{\rho}_j\), for \(j = 1, \ldots, N - 1\), or by the associated buoyancy frequency, \(B' V_j = (g'_j / \tilde{h}_j)^{1/2}\), where \(\tilde{h}_j = (\tilde{h}_{j+1} + \tilde{h}_j) / 2\). Here \(g\) is the earth’s gravitational acceleration, \(\tilde{\rho}_n\) is mean level density, and \(\rho_0\) is a reference value \((=10^3\text{ kg m}^{-3})\). The total mean depth of the ocean is \(D\), hence \(\sum \tilde{h}_n = D\).

We assume a rigid upper surface at \(z = 0\); a linear bottom slope at \(z = -D + s \cdot x\) (where bold face denotes a horizontal vector); a tangent-plane approximation to the Coriolis frequency, \(f + \beta y\); a surface wind stress, \(\tau\), a bottom stress, \(\mathbf{e}_n\), where \(e\) can be identified with the drag coefficient, \(C_P U\), in a linearized bulk formula, \(\mathbf{u}_n\) is the level horizontal velocity, and \(U\) is a mean value for \(\mathbf{u}_n\); a vertical eddy viscosity, \(\nu_z\); and a horizontal eddy viscosity, \(\nu_H\). We seek steady-state solutions to wind forcing that is periodic in \((x, t)\) with a vector curl of the form:
\[
\mathbf{e}_n \cdot \nabla \times \tau(x, t) = K \tau_0 e^{i(k \cdot \mathbf{x} - \omega t)},
\] (3.10)
where \(K\) is the horizontal wavenumber (with \(K = |k|\)), \(\mathbf{e}_n\) is the unit vertical vector, and \(\tau_0 = |	au|\). There is an analogous functional dependence in \((x, t)\) for all dependent variables.

Under these circumstances the FDM linear matrix system for the \(N\) geostrophic streamfunction amplitudes, \(\psi_n\), is
\[
\omega \left[ K^2 \psi_n + \nabla^2 \psi_n + \Gamma^2 \left( \psi_n - \psi_{n-1} \right) + \Gamma^2 \left( \psi_n - \psi_{n+1} \right) \right] + \frac{\beta k_{\perp}}{h_N} \delta_n \psi_n
\]
\[
= K \frac{\tau_0}{h_1} \delta_n - IK^2 \left[ \nu_z K^2 + \frac{e}{h_N} \delta_n \right] \psi_n
\]
\[
+ \mu_n (\psi_n - \psi_{n-1})
\]
\[
+ \mu_n^* (\psi_n - \psi_{n+1})
\] (3.11)
where \( \delta_{\alpha\beta} \) is the Kronecker delta function and \( k^{(\alpha)} \) is the \( x \) component of \( k \). The parameters in (3.11) are defined by

\[
K_2 = k \cdot k \\
\Gamma_n^- = f^2 h_n g_n' \quad \text{if} \ n > 1 \\
\Gamma_n^+ = f^2 h_n (g_n - z) \quad \text{if} \ n < N \\
\mu_n^- = \nu_n h_n \cos \theta_n \quad \text{if} \ n > 1 \\
\mu_n^+ = \nu_n h_n \cos \theta_n \quad \text{if} \ n < N, \quad (3.12)
\]

where \( s_0 = |s| \) and \( \theta_s \) is the horizontal angle of \( a \).

The system (3.10)–(3.12) and its UOM counterpart contain many parameters and thus exhibit a broad range of solution behaviors. In appendix A we survey a portion of these behaviors most relevant to the oceanic general circulation, based on both analytic and computational solutions. Our conclusions from this survey are that adiabatic, wind-driven responses are usually calculated quite accurately in a UOM for short-term climate variations in the parameter regime typically used in oceanic simulation models. The greatest errors with the UOM approximations occur either (a) near barotropic and baroclinic resonances [i.e., where the forcing \( k \) and \( \omega \) in (3.10) approximately coincide with the free-wave dispersion relations] or (b) with the combination of a baroclinic or barotropic near-resonance for large-scale or low-frequency wind forcing and a large topographic slope. Since both of these relatively erroneous circumstances depend strongly on the \( (x,t) \)-homogeneity assumptions of the test problem we have posed here, we can at least hope that they are not indicative of particularly important deficiencies in an oceanic general circulation model with a finite domain, inhomogeneous topography, and \( (x,t) \)-aperiodic wind forcing. This hope is bolstered by the results in section 4e(2) showing that UOM GCM solutions are accurate with large-scale topography and transient wind forcing.

4. Implementation and testing as a general circulation model

a. Model formulation

1) STRETCHED \( \sigma \) TRANSFORMATION AND MODEL EQUATIONS

We transform thin-spherical-shell coordinates, \((\lambda', \phi', z, t')\), onto a stretched \( \sigma \) coordinate set by

\[
(\lambda, \phi, t) = (\lambda', \phi', t') \\
\sigma = S(\xi) \quad \text{with} \quad \xi = 1 + \frac{z}{D'(\lambda, \phi)}, \quad (4.1)
\]

Here \( \lambda \) and \( \lambda' \) are longitude, \( \phi \) and \( \phi' \) are latitude, \( t \) and \( t' \) are time, \( z \) is height relative to the mean radius of the earth, \( a \). The stretching function \( S \) monotonically maps \( -D'(\lambda, \phi) \leq z \leq 0 \) onto \( 0 \leq \sigma \leq 1 \). Under most circumstances, a finer resolution in \( z \) near the upper surface in association with a uniform resolution in \( \sigma \) is desirable because oceanic vertical scales are smaller there. The Jacobian of the transformation is

\[
J = \frac{\partial(\lambda, \phi, \sigma, t)}{\partial(\lambda', \phi', z, t')} = \frac{\partial \sigma}{\partial z} = \frac{1}{h}. \quad (4.2)
\]

The dynamics of the model is governed by the primitive equations in stretched \( \sigma \) coordinates with hydrostatic, Boussinesq, and rigid-lid approximations. Here we present them in continuous form, and the discrete forms are in appendix C. The horizontal momentum equations are

\[
u_t + \nabla(u) - \left( f + \frac{u \tan \phi}{a} \right) v = \left( \frac{1}{a \cos \phi} \right) \left( D'_z - z \right) + F_{\sigma} + F_{\psi}, \quad (4.3)
\]

\[
\nu_t + \nabla(v) + \left( f + \frac{u \tan \phi}{a} \right) u = \left( \frac{1}{a} \right) \left( D'_z - z \right) + F_{\sigma} + F_{\psi}, \quad (4.4)
\]

Here \( u \) and \( v \) [\( u = (u, v) \)] are the longitudinal and latitudinal velocity components, respectively, and \( f = 2\Omega \sin \phi \) is the Coriolis frequency, where \( \Omega \) is the angular velocity of the earth. The terms \( F_{\sigma}, F_{\psi}, F_{\psi}, F_{\psi} \) are horizontal, \( H, \) and vertical, \( V, \) nonconservative terms. \( \nabla \) is the advection operator,

\[
\nabla(G) = \frac{1}{a \cos \phi} \left[ huG + (huG)_{\sigma} + (huG)_{\phi} \right] + \frac{1}{h} (WG), \quad (4.5)
\]

where \( G \) is a generic variable. In (4.5), \( W \) is the generalized vertical velocity,

\[
W = h \frac{D\sigma}{Dt} = w - z \left( \frac{u}{a \cos \phi} + v \frac{D'}{D'} \right), \quad (4.6)
\]

where \( w \) is the vertical velocity in the untransformed coordinate system.

In (4.3) and (4.4), \( \Phi \) represents the generalized pressure,

\[
\Phi = \frac{p}{\rho_0} + gz, \quad (4.7)
\]

and \( b \) is the buoyancy field,

\[
b = g \left( 1 - \frac{p}{\rho_0} \right), \quad (4.8)
\]

where \( p \) is pressure and \( \rho \) is the density of sea water.
The hydrostatic relation is
\[ \Phi_t = bh. \]  
(4.9)
The continuity equation is equivalent to \( h\zeta(1) = 0 \); that is,
\[ \frac{1}{a \cos \phi} \left[ (hu)_\lambda + (hv \cos \phi)_{\phi} \right] + (W)_\sigma = 0. \]  
(4.10)
With the mesoscale eddy, isopycnal transport parameterization of Gent and McWilliams (1990), the equation for \( \theta_t \) is
\[ \theta_t + \mathcal{L}^+(\theta) = \mathcal{S}(\theta_t, \theta) + F^e_t, \]  
(4.11)
with the effective-transport advection operator,
\[ \mathcal{L}^+(G) = \frac{1}{a \cos \phi} \frac{1}{h} \left[ (hu + u^*)_\lambda \right] 
+ [hv + v^*)_\phi \cos \phi \right]_{\phi} 
+ \frac{1}{h} [(W + W^*)_\sigma], \]  
(4.12)
and a diffusion operator along isopycnals, \( \mathcal{S} \) defined in (B.6) (Redi 1982; Cox 1987; Gent and McWilliams 1990; Danabasoglu and McWilliams 1995). Here \( A_{\lambda} \) is the isopycnal diffusion coefficient, \( F^e_t \) is a vertical mixing term, and \( u^*, v^*, \) and \( W^* \) are the longitudinal, latitudinal, and generalized vertical components of the nondivergent, eddy-induced transport velocity, respectively. This transport velocity is defined by
\[ u^* = -\frac{1}{h} [A_{\lambda T D} S]_\sigma, \quad w^* = \frac{1}{h} \nabla \cdot [A_{\lambda T D} S]. \]  
(4.13)
where the slope of the isopycnals with respect to the height coordinate, \( S \), is
\[ S = \nabla z - h \left( \frac{\nabla \rho}{\rho_\sigma} \right). \]  
(4.14)
Here \( \nabla \) is the horizontal gradient operator in the stretched \( \sigma \) coordinate, \( \rho \) is local potential density, \( A_{\lambda T D} \) is the isopycnal thickness diffusivity, and \( W^* = hW^* \). An identical conservation equation as (4.11) is used for \( S \). The set of governing equations is completed with the equation of state for seawater, which we denote symbolically by
\[ \rho = \rho(\theta, S, \sigma). \]  
(4.15)
The polynomial approximation from Bryan and Cox (1972) is used for (4.15). The continuous forms of all the nonconservative terms are given in appendix B.

2) BAROTROPIC AND BAROCLINIC VELOCITIES

In the UOM the depth-integrated velocity (or, equivalently, the barotropic streamfunction, \( \psi \)) is solved for prognostically, using the full (actual) ocean depth, \( D \), as in an FDM. We follow the same procedure as in Bryan (1969) and write (4.3) and (4.4) as
\[ u_t = \tilde{u}_t - \frac{1}{a \cos \phi} \left( \frac{\Phi^e}{\rho_\sigma} \right)_\lambda, \]  
(4.16)
\[ v_t = \tilde{v}_t - \frac{1}{a \cos \phi} \left( \frac{\Phi^e}{\rho_\sigma} \right)_\phi, \]  
(4.17)
where \( \Phi^e = p^e \) is the surface pressure. In (4.16) and (4.17), \( \tilde{u}_t \) and \( \tilde{v}_t \) are the velocity tendencies minus the surface pressure contribution.

The total velocity is decomposed into the barotropic component, \( (\cdot) \), and the baroclinic deviation, \( (\cdot)' \):
\[ u = \tilde{u} + u^*, \quad v = \tilde{v} + v^*. \]  
(4.19)
where
\[ (\cdot) = \frac{1}{D} \int_{-D}^{0} (\cdot) dz. \]  
(4.20)
We define the barotropic streamfunction, \( \psi \), by
\[ \psi = -\frac{1}{a \rho_\sigma D} \psi^e, \quad \psi^e = \frac{1}{a \rho_\sigma D} \psi^e. \]  
(4.21)
Vertically integrating (4.16) and (4.17) yields
\[ \pi_t = \tilde{u}_t - \frac{1}{a \cos \phi} \left( \frac{\Phi^e}{\rho_\sigma} \right)_\lambda, \]  
(4.22)
or, equivalently,
\[ \pi_t = \tilde{u}_t + u^*_t, \quad \tilde{v}_t = \tilde{v}_t + v^*_t, \]  
(4.23)
where \( u^*_t \) and \( v^*_t \) are the tendency contributions due to the surface pressure gradient.

A prognostic equation for \( \psi \) comes from taking the curl of (4.22) or (4.23),
\[ \left[ \frac{\psi_{\phi}}{D \cos \phi} \right]_{\phi} + \left[ \frac{\psi_{\theta}}{D \cos \phi} \right]_{\theta} = a \rho_\sigma \left[ \tilde{v}_t - \tilde{u}_t \cos \phi \right]. \]  
(4.24)
In our present implementation, we obtain \( \tilde{u}_t \) and \( \tilde{v}_t \) from (4.3) and (4.4), respectively, by neglecting the surface pressure contribution. Next, (4.21) and (4.24) are used to compute \( \pi_t \) and \( \tilde{v}_t \). Then, (4.23) is applied to evaluate \( u^*_t \) and \( v^*_t \).

Using \( u^*_t \) and \( v^*_t \), (4.16) and (4.17) can be rewritten as
\[ u_t = \tilde{u}_t + u^*_t; \quad v_t = \tilde{v}_t + v^*_t. \]  
(4.25)
Combining the tendency form of (4.19) with (4.25), we get the equations for the baroclinic velocity components

\[ u_i^* = \hat{u}_i + u_i^* - \sigma_i, \quad v_i^* = \hat{v}_i + v_i^* - \tau_i. \]  

(4.26)

Because the barotropic streamfunction is solved prognostically and because it uses \( D \) rather than \( D' \), the terms contributing to vorticity in (4.24) must be evaluated carefully (see appendix D). In our experience, any inconsistencies in the computations of the barotropic and baroclinic velocities and vorticity result in either various erroneous circulation patterns or model instability, affecting both the mean and variability of the solutions.

3) Pressure gradient

A well-known problem associated with \( \sigma \)-coordinate models is the discretization error in the calculation of the pressure-gradient force (PGF). In the transformed coordinates the PGF appears as the smaller difference of two larger terms, which makes it vulnerable to errors. Gary (1973) shows for an atmospheric model that substracting a horizontally uniform reference density state, \( \rho_s(z) \), from the model density prior to the PGF calculation diminishes the error by reducing the magnitude of the individual terms. Haney (1991) also shows a reduction of the PGF error in an oceanic application of this technique. This technique may be less useful in large-basin or global models, where the deviations of density from \( \rho_s(z) \) are often large (Barnier et al. 1998).

In \( \sigma \)-coordinate models that incorporate high-order, spectral discretization in the vertical direction (Haid-vogel et al. 1991), increasing the order of the horizontal discretization for the PGF computation may reduce the truncation error (McCalpin 1994; Chu and Fan 1997). In contrast, because the present model formally uses second-order finite differences in all directions, we find that increasing the order of the horizontal discretization for PGF does not reduce the error, suggesting that the error is mostly controlled by the vertical discretization. Therefore, following Leslie and Purser (1991) and Song and Haidvogel (1994), we use Lagrange Cardinal functions with five-point Gauss–Legendre quadrature in the PGF calculation.

A second problem associated with the PGF computation is “hydrostatic inconsistency” (Rousseau and Pham 1971; Janjić 1977; Mesinger 1982; Haney 1991). This condition arises if the local depth change over a horizontal grid box exceeds the thickness between two consecutive \( \sigma \) levels at that location. Otherwise, the two terms of the PGF use density information from different grid cells. For our model this consistency constraint is

\[ \frac{z_D}{h_D} \Delta \lambda \leq \Delta \sigma. \]  

(4.27)

In (4.27), \( \lambda \) can be replaced by \( \phi \). Note that this equation combines horizontal and vertical grid information. Therefore, increasing the steepness of bottom topography together with an increase in the vertical resolution may violate the constraint. In many applications of \( \sigma \)-coordinate models, this condition is satisfied by smoothing the bottom topography (Beckmann and Haid-vogel 1993; Barnier et al. 1998). In our experience, only extreme violations of (4.27) produce evidently spurious flow patterns (e.g., anomalous gyres in \( \psi \) locked to topographic features where the constraint is violated). In the present model configuration, we do not fully respect (4.27), partly because our upper-ocean \( \sigma \) coordinate is based on \( D' \) whenever it is smaller than \( D \) (Fig. 1). In addition to the above factors, this error strongly depends on the vertical stratification and the equation of state of the model. Unfortunately, the PGF errors are difficult to estimate in global models. Over the course of this study, we have tried many alternative continuous and discrete forms for the PGF. We believe that, with the discrete form given in (C.8) of appendix C, the errors are much reduced in the integration scheme above and when a \( \rho_s(z) \) subtraction is made, compared to the usual FDM integration forms and no \( \rho_s(z) \) subtraction. We calculate \( \rho_s(z) \) from the Levitus (1982) climatological \( \theta \) and \( S \) profiles, using the model’s equation of state.

4) Bottom boundary conditions

We specify \( \theta, S, u^\prime, \) and \( v^\prime \) fields as the climatological bottom boundary conditions in a UOM. In the present work, these fields, and others indicated below, are obtained from a companion FDM using linear interpolation to the upper-ocean bottom level, \( z = -D' \). The climatological density field is evaluated from these \( \theta \) and \( S \) distributions using the model’s equation of state with the tracer point depth of the bottom \( \sigma \) level as the reference pressure. Whenever the discrete equations require a bottom flux, it is computed using an extrapolative estimator [see (C.8) of appendix C]. The total horizontal velocity at \( z = -D' \) is constructed by adding the climatological baroclinic fields to the prognostic barotropic velocity (i.e., \( u = u_c + \bar{u}_c \)). Because our formulation of the barotropic component needs the climatological distributions of \( \bar{\pi} \) and \( \bar{\tau} \), [see (D.5) and (D.6) of appendix D], these fields are also provided. Finally, two more additional fields for the zonal and meridional momentum equations are computed, representing the summations of the integral quantities for the barotropic vorticity equation [see (D.1)–(D.6)], for example,

\[
\int_{-D'}^{-D} \left[ -\left(L(G)\right)_c - \nabla' \right] \rho_c \ dz' \\
+ \int_{-D'}^{-D} F_{\beta}(A_{MB}, G_c) + F_{\psi}(A_{MN}, G_c) \ dz. 
\]

(4.28)

We use a simple trapezoidal integration to evaluate (4.28) on the FDM \( \sigma \) grid. No vertical velocity distri-
b. Model configuration and parameters

1) Resolution, geometry, and topography

In this study, we use the $\times 3'$ resolution version of NCOM (NCAR Oceanography Section 1996; Gent et al. 1998). NCOM is derived from the Geophysical Fluid Dynamics Laboratory Modular Ocean Model (Bryan 1969; Cox 1984; Pacanowski et al. 1991, 1993). This version is very similar to the $\times 3$ resolution version used in Large et al. (1997) study, differing mostly in the meridional resolution. Here, the meridinal grid spacing is kept constant at 0.86° between 10°S and 10°N. It increases monotonically to 1.85° at 30°, using a cosine transition function. It remains constant at this value poleward of 30°. This meridional resolution is designed to resolve the equatorial currents better than in the $\times 3$ grid. The longitudinal resolution is constant at 3.6°. The maximum depth of the companion $z$-coordinate FDM is 5000 m, and there are 25 vertical levels, monotonically increasing in thickness from 12 m near the surface to about 450 m near the bottom. The model domain is global, extending from 79.9°S to 90°N.

In a UOM the bottom $\sigma$ level may coincide with the actual ocean bottom topography, particularly in shelf regions, and the FDM and UOM both use the same bottom topography in these areas. Because a primary purpose here is to calibrate the performance of the UOM by the FDM, we make both models as consistent as practically possible. Therefore, in order not to risk severe violation of the consistency constraint (4.27) in the UOM, we choose a smooth bottom topography in these shallow regions, and, to obtain consistent initial conditions datasets for the UOM, we use a similar topography for the FDM. Thus, we modify $D(\lambda, \phi)$ evaluated on the $\times 3'$ grid from the data from National Oceanic and Atmospheric Administration National Geophysical Data Center (Sloss 1988). First, we set the minimum depth to 200 m except in the Weddell and Ross Seas and the north-central Arctic Ocean region, where a minimum depth of 1000 m is used. Then, a local, five-point Gaussian filter is applied twenty times to smooth the bottom topography, eliminating completely any advectively isolated one gridpoint holes in the process. Finally, this is converted to the integer level-depth array used in the FDM.

We choose the UOM bottom depth to lie well below but approximately follow the climatological pycnocline depth. Thus, we choose $D' = 500$ m between 10°S and 10°N. In the interval 10°-35°, $D'$ monotonically increases to 1250 m, following a cosine function. Poleward of 35°, the depth is constant at 1250 m. In this first stage, the bottom depth is zonally uniform. Where this prescription for $D'$ exceeds the floating-point $D$, we then set $D' = D$.

In order to keep the horizontal boundary conditions simple in the UOM, we also adopt the restriction on our domain definition that

$$\frac{\partial D'}{\partial x_n} = \frac{\partial h}{\partial x_n} = 0,$$

where the horizontal coordinate $x_n$ is normal to a boundary direction. Thus, the coordinates locally revert to physical coordinates normal to the boundary, so that the usual boundary conditions suffice for the model.

The UOM cases have 15 vertical levels within the upper ocean. The normalized layer thicknesses [see (C.5) of appendix C] monotonically increase from 0.0144 near the surface to 0.119 at the bottom level. Consequently, the first level thicknesses are 2.9, 7.2, and 18 m for $D'$ values of 200, 500, and 1250 m, respectively. Similarly, the respective thicknesses of the bottom $\sigma$ level are 23.8, 59.5, and 148.8 m.

2) Model parameters

Having a model with somewhat refined resolution near the equator does not necessarily produce realistic equatorial currents. The horizontal eddy viscosity coefficients must also be reduced with the grid refinement. Large et al. (2000, hereinafter LDMGB) discusses in detail how these coefficients can be constructed and analyzes their effects on equatorial currents. Because this is not our primary focus, we initially follow a much simplified path here. We use only a single, equatorially symmetric coefficient with $B_{\text{MH}}(\lambda, \phi, \sigma) = A_{\text{MH}}(\lambda, \phi, \sigma) = A_{\text{MH}}(\lambda, \phi)$. With the present equatorial resolution, we set $A_{\text{MH}} = 8 \times 10^3 \text{ m}^2 \text{ s}^{-1}$ between 0° and 10°. In the interval 10°-30°, it gradually increases to $2 \times 10^3 \text{ m}^2 \text{ s}^{-1}$, following a cosine function. It is kept constant at this large value at higher latitudes. Next, considering a $5 \times 5$ stencil, $A_{\text{MH}}$ is set to $2 \times 10^5 \text{ m}^2 \text{ s}^{-1}$ at the center of the stencil if this point is within 2 grid points of a land point (regardless of direction) so that the lateral boundary layer may be resolved (Munk 1950). Finally, a local, five-point Gaussian filter is applied twice to smooth the distribution. North of 83.5°N, where the Courant–Friedrichs–Lewy diffusive stability criterion is violated because of polar convergence of meridians, $A_{\text{MH}}$ is reduced accordingly. This specification for $A_{\text{MH}}$ represents a useful improvement over a uniform viscosity, but it is too simple to be the optimal choice.

For the tracer equations, we set $A_t = A_{\text{TRP}} = 0.8 \times 10^3 \text{ m}^2 \text{ s}^{-1}$, following Danabasoglu and McWilliams (1995) and McWilliams et al. (1983). These coefficients are small enough to satisfy the diffusive numerical stability constraint near the North Pole. The vertical-mixing coefficients are determined by the K-profile parameterization vertical-mixing scheme (Large et al. 1994). In the ocean interior, below the boundary layer, the background values are $A_{\text{AV}} = 5.0 \times 10^{-2}$ and $A_{\text{IV}} = 0.5 \times$
$10^{-4} \text{ m}^2 \text{s}^{-1}$, but in the presence of convective and shear instability the values are much enhanced. Within the boundary layer, $\Delta u_0$ and $\Delta v_0$ usually are proportional to the wind speed and to the diagnosed boundary layer depth and may exceed $1000 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$.

To escape the severe restriction on the time step allowed for advective numerical stability with converging meridians near the North Pole, the model variables are Fourier filtered north of $75^\circ$N; no filtering is applied in the Southern Hemisphere. A quadratic bottom drag is used with a drag coefficient, $C_D = 1.0 \times 10^{-3}$ both in the FDM and in regions of the UOM where the last $\sigma$ level coincides with the bottom topography. We perform all the integrations with a model time step of 3600 s.

### 3) Surface boundary conditions

The surface forcing of the model is essentially identical to the bulk-forcing method described in Large et al. (1997), but with a few minor differences. In the present implementation, we take advantage of the full NCAR Climate System Model (CSM) design, and perform most of the data processing (i.e., reading, temporal and spatial interpolation, etc.) and flux calculations outside the ocean model, utilizing the NCAR CSM Flux Coupler (Bryan et al. 1996). Instead of monthly mean fluxes, we use the available 6-hourly National Centers for Environmental Prediction (NCEP)–NCAR reanalysis dataset (Kalnay et al. 1996) to calculate surface turbulent fluxes every 6 h. Cloud fraction, surface insolation, and precipitation datasets are still monthly mean, and a linear time interpolation is used to produce 6-h values. These fluxes are averaged by the coupler to form daily means, which are then passed to the ocean model once a day. In the ocean model the fluxes are kept constant over the next day of integration. The only exceptions to this are the modifications due to the restoring $\theta$ and $S$ fluxes that are diagnosed every ocean time step based on the monthly mean climatology. The restoring surface $S$ field is now based on the monthly mean Levitus et al. (1994) climatology and represents the mean over the upper two levels (10 m) of this dataset. Under diagnosed sea-ice (based on Shea et al. 1990 surface temperature climatology), strong restoring to these $\theta$ and $S$ fields is applied. In addition, the $S$ fields are used as a local constraint for prognostic model surface $S$, as represented by a weak restoring term in surface salt flux. We use atmospheric datasets for years 1985–88, with periodic repetition every 4 yr. Finally, because the model is integrated synchronously in all the experiments, we keep the global precipitation factor constant at 1.06. Further details of this bulk-forcing method are given in Large et al. (1997, section 3 and appendix A).

### 4) Boundary conditions and initial conditions

The momentum equations have a no-slip condition at the lateral boundaries. In the FDM, the normal velocity is zero at all boundaries. The same condition is also applied in the UOM, with the exception of the open bottom boundary. At the bottom of the FDM and in regions of the UOM where the last $\sigma$ level coincides with the bottom topography, a quadratic drag law is applied. The tracer equations are subject to the no-flux condition at all solid boundaries in both models, but not at the bottom of the UOM in open regions.

The FDM is initialized with an equilibrium solution obtained with identical physics and surface forcing on the original $\times 3'$ grid. This initial condition is modified only locally in several places where the present smoother and somewhat deeper topography created new ocean regions, by inserting reasonable climatological $\theta$ and $S$ from Levitus and Boyer (1994) and Levitus et al. (1994), respectively, and zero velocity values. The FDM is integrated for 48 yr, corresponding to 12 4-yr forcing cycles, to allow for adjustments (see Table 1). The model is then integrated for an additional 4 yr to obtain monthly mean bottom boundary conditions for the UOM. During this integration, the FDM produces 48 monthly mean bottom datasets. These sets are then averaged to get mean monthly fields, eliminating any interannual variability. Finally, for each bottom field, we construct 12 midmonthly datasets so that linear interpolation to each model time step preserves the monthly means of the original fields. These bottom boundary conditions are applied to all UOM integrations presented here.

The UOM case is initialized with the FDM solution at the end of year 48, interpolated to the $\sigma$ levels in the vertical by cubic splines. In this step, the baroclinic velocity components in regions where the bottom $\sigma$ level coincides with the actual ocean bottom are ensured to have zero vertical means by subtracting any residual due to interpolation. The UOM-C and FDM-C cases start from a state of rest with the January mean Levitus climatological $\theta$ and $S$ distributions interpolated to the model grids.

#### c. Approach to equilibrium

One of the primary goals of the UOM is to be able to reproduce an FDM equilibrium solution after a relatively short integration. In the present study, we first integrate both UOM and UOM-C cases for 48 yr each and calculate annual mean outputs. An additional 4-yr
integration is then performed for each case to obtain monthly mean datasets during one surface forcing cycle. The time series of the global and annual mean \( \theta \) and \( S \) are presented in Fig. 4 for both UOM and UOM-C cases, displaying their approach to equilibrium. For comparison purposes, the FDM-C time series are also included in the figure, showing significant trends and departures from the FDM equilibrium values even in the upper ocean, and thus, enforcing our earlier statement that the disequilibrium behavior in an FDM is problematic. For spatial mean (or spatial difference) computations such as presented in the figure, all UOM and FDM data are linearly interpolated in the vertical to the same level grid. In addition, the first \( \sigma \)-level \( \theta \) and \( S \) and the marginal sea data are excluded. The first one avoids any extrapolation and the latter eliminates advectively isolated regions. Also, we mask the FDM data in accordance with the UOM bottom depth distribution; for example, no data below \( z = -D' \) are included in analyzing the FDM for UOM comparisons.

Based on the last 4 yr of Fig. 4, we calculate the time and global mean \( \theta \) and \( S \) equilibrium values as \((9.182^\circ \mathrm{C}, 34.714 \text{ ppt})\) and \((9.175^\circ \mathrm{C}, 34.719 \text{ ppt})\) for UOM and UOM-C cases, respectively (Table 1). We also estimate the expected drifts in the global \( \theta \) and \( S \) during an additional 100 yr of integration as \((0.002^\circ \mathrm{C}, 0.005 \text{ ppt})\) and \((0.081^\circ \mathrm{C}, 0.005 \text{ ppt})\), respectively, based on the last 5 yr of these time series (Table 1). These linear estimates serve as a gross measure of the achieved equilibrium states. The actual changes in \( \theta \) and \( S \) would be much smaller than these estimates in such an additional integration, because the adjustment magnitudes decrease systematically with time.

We define an equilibrium time as the time it takes to go 90\% of the way from its initial to its equilibrium volume mean tracer values. Figure 4 shows that \( \theta \) equilibrium is achieved in about 25 yr for both cases. Equilibration in \( S \) seems to be a little slower, with about a 30-yr time scale.

Both Table 1 and Fig. 4 show that the models reach slightly different \( \theta \) and \( S \) equilibrium values. Besides the obvious discretization differences between the FDM and UOM cases, any differences in \( \theta \) and \( S \) values indicate possible differences and imbalances of surface and bottom net heat and freshwater fluxes. In particular, any model differences in the surface \( \theta \) will result in differences in surface buoyancy fluxes as it is directly used in these computations. The surface freshwater fluxes in all models are unconstrained. In addition, we specify only the abyssal reservoir distributions of \( \theta \) and \( S \) in the UOM, not the fluxes exchanged with the reservoir. Nevertheless, the equilibrium values in Table 1 are only slightly different among the models.

The value of a UOM is partly based on the value placed on having a well-defined equilibrium state. An FDM requires an adjustment period two orders of magnitude larger than a UOM to reach full equilibrium (Danabasoglu et al. 1996). This difference translates directly to two orders of magnitude more computational cost for synchronous FDM integrations. Even with the acceleration methods (Bryan 1984), an FDM requires about an order of magnitude more computational time than a UOM to achieve equilibrium. The upper ocean fields in an FDM will adjust on a timescale comparable to the UOM adjustment, albeit with larger and more persistent trends that reflect the continuing abyssal adjustments. In some locations and for particular analysis questions the residual trends will be unimportant, and in others they will be more detrimental. Beforehand, and even after any integration period of only a few decades, the residual trends will be difficult to assess in an FDM solution; only in a UOM can one have confidence that they are small everywhere.

\[ d. \text{ Mean comparisons} \]

1) Potential temperature and salinity

The root-mean-square (rms) differences in \( \theta \) and \( S \) between the UOM and FDM solutions and the UOM-C and FDM solutions can be used to test the success of the UOM in reproducing the FDM solutions. Here, we compare the upper-ocean and full-depth annual mean fields for years 49–52. This analysis produces \((0.24^\circ \mathrm{C}, 0.039 \text{ ppt})\) and \((0.26^\circ \mathrm{C}, 0.046 \text{ ppt})\) for rms (UOM – FDM) and rms (UOM-C – FDM), respectively, averaged over these 4 yr and the upper-ocean domain. A comparison of these rms values to those of Large et al. (1997, Table 3) for the FDM and Levitus climatology differences in the upper ocean shows that the present model to model differences in \( \theta \) and \( S \) are much less than the model to climatology differences and that the UOM reproduces the FDM solutions reasonably well.

The vertical profiles of the time and horizontal mean global \( \theta \) and \( S \) are presented in Fig. 5 in comparison to the climatology. Here the time average is for years 49–52 computed using the monthly mean datasets. All the
Fig. 5. Vertical profiles of the time and horizontal mean $\theta$ and $S$.

Fig. 6. Time and zonal mean $\theta$ and $S$ from the UOM and the corresponding UOM − FDM difference distributions. In the UOM $\theta$ and $S$ distributions, the contour intervals are 2°C and 0.2 ppt, respectively, and the regions where $\theta$ and $S$ exceed 10°C and 35 ppt, respectively, are shaded. In UOM − FDM, the contour intervals are 0.4°C and 0.04 ppt for $\theta$ and $S$, respectively, and the shaded regions represent negative contour values. The UOM maximum depth along a latitude is also shown.
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Fig. 7. Time mean SST from the UOM and the corresponding UOM - FDM difference. The contour intervals are (top) 2°C and (bottom) 0.5°C. The shaded regions represent negative contour values.

The time and zonal mean UOM θ profiles are virtually indistinguishable. Below about 100-m depth, the solutions display a warm bias compared to the climatology, and this bias is the largest (>1°C) between 200 and 400 m. The vertical gradients differ noticeably from the observations between 100- and 200-m depth. Deeper, the model gradients nearly match those of Levitus. Because these averages do not include any contributions below 500 m in the equatorial regions, the vertical profiles have a discontinuity at that depth.

The above results and other extensive analysis of the UOM and UOM-C cases (not reported here) show that both upper-ocean solutions are very similar in all aspects. Therefore, in the remainder of this study, we will not use the UOM-C case.

The time and zonal mean UOM θ and the corresponding UOM - FDM differences in Fig. 6 show that the UOM reproduces the FDM solution remarkably well. In general, the UOM distribution is slightly cooler than in the FDM. Between 60° and 70°N, the differences are somewhat larger, probably due to the delicacy of where deep convection occurs. The positions of the subarctic front are slightly displaced between the UOM and FDM solutions. In addition, the steplike representation of the Greenland–Iceland ridge topography in the z-coordinate model and the increased vertical resolution in the present σ-coordinate model in this region may influence these distributions. The UOM θ distribution is similar to that in Large et al. (1997), showing common features of coarse-resolution models. The warm bias of Fig. 5 is present at almost all latitudes, and in general, the model thermohaline is a little more diffuse than observed.

Similar to the θ profiles, the vertical profiles of the time and horizontal mean S for the model solutions are also very similar to each other (Fig. 5). Above 300 m, all model profiles are fresher than observations by as much as 0.07 ppt. All model solutions capture the sub-surface S maximum to a comparable degree with the maxima being less than the observed. Below 300 m, the model solutions are slightly more saline than observed, and the vertical gradients are in good agreement with the climatological gradients.

The time and zonal mean UOM S and the corresponding UOM - FDM difference distributions are also shown in Fig. 6. The UOM S distribution is similar to that of Large et al. (1997). The UOM - FDM distribution shows that the UOM solution is fresher than the FDM one, mostly south of 25°S and in the near-surface Arctic Ocean. In addition to the subarctic front region, the Arctic Ocean displays somewhat large difference patterns. We speculate that these Arctic Ocean differences may also be related to the Fourier filtering at high latitudes.

Figure 7 shows the time mean UOM sea surface temperature (SST) and the corresponding UOM - FDM difference distributions, as represented by the first model layer values. Because the model SST is strongly constrained by the imposed atmospheric temperature in our bulk-forcing formulation (Large et al. 1997), UOM and FDM differences are very small over much of the globe. For example, the mean UOM - FDM difference in the equatorial Pacific is about 0.02°C with the largest local difference of only about 0.11°C. In a few isolated convective regions in the Arctic Ocean and near the Labrador Sea and between Greenland and Iceland where sharp SST gradients exist, the local differences are a few degrees Celsius.

2) MASS TRANSPORTS AND EQUATORIAL CURRENTS

The vertically integrated mass-transport (barotropic) streamfunction from UOM in Fig. 8 shows typical features of coarse-resolution, large-viscosity models (see Large et al. 1997; Danabasoglu 1998). In particular, inertial boundary currents and recirculations are absent, and the gyres are approximately in Sverdrup balance with the wind stress curl. Detailed discussions on these
transports, including comparisons with some observational estimates, can be found in the above references.

The corresponding UOM – FDM difference distributions are also shown in Fig. 8. The differences are generally small, with a rms value of only 0.8 Sv (1 Sv = 10$^6$ m$^3$ s$^{-1}$). However, in some regions of the southern Indian and northern Pacific Oceans, the differences exceed 4 and 2 Sv, respectively (i.e., less than 10% of the total transport in these regions). The UOM solution procedure for the barotropic streamfunction is one of the primary design features of the present UOM [see section 4a(2) and appendix D], and we conclude that it is largely successful in reproducing both the mean (Fig. 8) and the variability (see section 4e) of $\psi$ in comparison with an FDM. The discrepancies are presumably a consequence of the differences in vertical discretization (resolution, coordinate transformation, PGF computation, etc.) between the UOM and FDM. Therefore, we do not feel that these differences are important to remove.

Figure 9 presents the UOM meridional overturning streamfunction and the corresponding UOM – FDM difference distributions. This transport indicates the strength of the thermohaline circulation except in the upper ocean where Ekman transport dominates. These distributions are obtained using the effective-transport velocity that represents the sum of the Eulerian mean and eddy-induced velocities. In the Northern Hemisphere, the overturning is dominated by downwelling in a relatively narrow region between about 55° and 70°N and slow upwelling over a broad region farther south. This thermohaline circulation mostly occurs in the North Atlantic, and it extends through the bottom of the UOM domain. Its maximum transports are 20.8 and 21.1 Sv in the UOM and FDM solutions, respectively. Two shallower cells on both sides of the equator are driven by the divergence of the Ekman transport due to easterly trade winds. The Deacon cell centered at about 50°S is partially cancelled by the eddy-induced transport, more so below about 200-m depth. The UOM – FDM differences are mostly much less than 1 Sv. Because of mass conservation, this overturning streamfunction also indicates the strength of the zonally integrated vertical velocity. Because the vertical velocity is not included as part of the bottom boundary conditions, the UOM is absolutely unrestrained to determine
its vertical velocity magnitudes in and out of this boundary subject to mass conservation.

Figure 10 shows the time mean, zonal velocity distributions for the equatorial Pacific Ocean. The eastward flow at depth with maximum core values of 38 and 39 cm s\(^{-1}\) in the UOM and FDM, respectively, is the equatorial undercurrent (EUC). These maxima are located at about 130-m depth in both. The westward surface flow penetrates below 50-m depth with maximum surface velocities of 64 and 60 cm s\(^{-1}\) in the UOM and FDM, respectively. The larger value for the UOM is due to the higher vertical resolution, resulting in a thinner surface layer. The structure and magnitude of the EUC, as well as of the westward surface flow, are strongly dependent on the meridional resolution and the associated horizontal eddy viscosity \(A_{\text{MH}}\), the vertical eddy viscosity \(A_{\text{MV}}\), and the zonal wind stress. Consequently, in comparison with Large et al. (1997), the present results are a significant improvement because of finer meridional resolution (0.86° vs 1.80°), lower equatorial \(A_{\text{MH}}\) (8 \(\times\) 10\(^3\) vs 300 \(\times\) 10\(^3\) m\(^2\) s\(^{-1}\)), and lower \(A_{\text{MV}}\) (5.0 \(\times\) 10\(^{-4}\) vs 16.7 \(\times\) 10\(^{-4}\) m\(^2\) s\(^{-1}\)) of the present models. Other known features of the equatorial current system are also evident in meridional sections (not shown), although with weaker magnitudes and broader spatial features than observed in both models.

Further improvements in these equatorial currents are possible. As in LDMGB we modify the UOM and FDM setups (designated as MUOM and MFDM): \(A_{\text{MH}} \neq B_{\text{MH}}\), the equatorial \(B_{\text{MH}}\) is reduced to 1.0 \(\times\) 10\(^{-3}\) m\(^2\) s\(^{-1}\); \(A_{\text{MV}}\) and \(A_{\text{T}}\) are decreased to 1.0 \(\times\) 10\(^{-4}\) and 0.3 \(\times\) 10\(^{-4}\) m\(^2\) s\(^{-1}\), respectively; and a blend of National Aeronautics and Space Administration scatterometer, European Space Agency (ERS-2) scatterometer, and NCEP analyses data is used for wind stresses at 6-h intervals for August 1996–July 1997 (Milliff et al. 1999). The remaining surface forcing fields also correspond to this time interval. Both cases start from a state of rest. However, for \(\theta\) and \(S\), MUOM uses climatology (as in UOM-C), whereas MFDM starts with a previous, equilibrium solution (see LDMGB for details).

The annual mean, equatorial Pacific zonal velocity

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**Figure 10.** Time mean zonal velocity for the Pacific Ocean at the equator for the UOM, FDM, MUOM, and MFDM cases. In all panels, the contour interval is 5 cm s\(^{-1}\), and the positive and negative (shaded) contours indicate eastward and westward flow, respectively. The near-surface westward flow contours in excess of 45 cm s\(^{-1}\) are not shown.
distributions for these two cases, obtained after 10 yr of integration with a repeating annual cycle, are also presented in Fig. 10. In both MUOM and MFDM solutions, the undercurrent maxima are larger than 90 cm s$^{-1}$ and located shallower (above 100 m) and farther east (at about 230°E) than in the UOM and FDM. The unrealistic deep penetration of the near-surface westward flow is now completely eliminated, and the maximum surface westward flow is about 35 cm s$^{-1}$ and located at about 260°E. All of these equatorial current features of the MUOM and MFDM are in good agreement with observations (as documented for the MFDM in LDMGB).

An important result here is that the above MUOM solution is obtained using the original bottom boundary conditions as in the UOM [section 4b(4)]. Therefore, the reproducibility of the MFDM solutions by the MUOM demonstrates the robustness of the upper-ocean model and verifies that it can be successfully used for parameter and surface-forcing sensitivity studies with a given set of bottom boundary conditions.

e. Variability comparisons

The lower boundary conditions of the UOM contain only the mean seasonal cycle information. How well does the UOM match the FDM for variability not present in these boundary conditions? To assess this we analyze the 48 monthly mean data for years 49–52. First, the 4-yr mean of each field is subtracted to form monthly anomaly data. The mean monthly anomaly fields (SEASONAL) are then computed by averaging over these four years. Finally, the residual anomaly fields (RESIDUAL) are evaluated by subtracting these mean monthly anomaly fields from the monthly anomaly data.

1) SST

The global SST variance distributions are shown for the UOM and FDM in Fig. 11. Because the model SST is strongly constrained by the imposed atmospheric temperatures and their variability, the UOM matches the FDM quite well in both SEASONAL and RESIDUAL.
The SEASONAL magnitudes are much larger than the RESIDUAL ones. The largest SEASONAL variability occurs in the Northern Hemisphere sub-tropics and off the east coasts of Asia and North America, mostly associated with the subtropical gyres and western boundary currents. Similarly, the Southern Hemisphere sub-tropics show somewhat high variability. Portions of the Arctic Ocean (i.e., south of Spitsberg and north of Iceland) and east of Drake Passage are also regions of high variability (>4°C). The strongest RESIDUAL variance exists in the equatorial Pacific with more than 2°C². In addition, the northeastern and southeastern Pacific and northern North Atlantic have regions where RESIDUAL variability exceeds 0.4°C². The largest discrepancies between the UOM and FDM distributions are in portions of the North Atlantic and Arctic Oceans where convection is common.

We anticipate that studies of equatorial Pacific variability in coupled or uncoupled oceanic models will be an important application for a UOM. Time series of the monthly anomaly equatorial Pacific SST from the UOM and FDM are presented in Fig. 12. (Because the model does not have a temperature point at the equator, the distributions represent the meridional average between 0.43°S and 0.43°N.) We see that both intra- and inter-annual variability of the FDM are fully captured by the UOM, even though the lower boundary conditions of the UOM have no interannual variability.

2) MASS TRANSPORTS AND EQUATORIAL CURRENTS

Variability of the barotropic streamfunction in Fig. 13 shows that the UOM matches the FDM SEASONAL and RESIDUAL variability remarkably well. Because the models are approximately in Sverdrup balance, this variability arises primarily from the wind stress curl, and it is mostly associated with meandering of currents. The largest SEASONAL variability occurs in the tropical and sub-tropical North Pacific with maxima of about 256 Sv² in each, in association with the tropical cyclonic circulation and the southern branch of the Kuroshio current. In the Southern Hemisphere, the SEASONAL variances of the Agulhas, East Australian, and Brazil Currents exceed 40 Sv². Along the path of the Antarctic Circumpolar Current (ACC) in the Southern Ocean, the SEASONAL variability is more than 100 Sv² in the Atlantic and Indian sectors and in the extreme eastern Pacific sector. The large SEASONAL variability in the western Indian Ocean centered at the equator is due to the monsoonal forcing. It is interesting to note that the North Atlantic and, in particular, the Gulf Stream do not show comparable variability.
The RESIDUAL variability magnitudes are in general comparable to and in some areas (e.g., northern North Atlantic, northern North Pacific, eastern Australian) larger than the SEASONAL one. In the Northern Hemisphere, the UOM and FDM RESIDUAL variability magnitudes are alike. In contrast, although the spatial extents of variability are identical in the UOM and FDM, the UOM peak variance values are about 10%–25% larger than in the FDM in some areas of the Southern Hemisphere. These variance differences translate into 1–2-Sv RESIDUAL anomaly differences between UOM and FDM solutions along the ACC. A likely source for these differences is the absence of abyssal baroclinic JEBAR variations in the UOM, although we cannot rule out the effect of discretization differences (as in Fig. 8).

The FDM meridional overturning streamfunction variability is also well reproduced in the UOM (Fig. 14). The SEASONAL variability is almost exclusively confined to the tropical regions with maximum variances of 699 and 638 Sv$^2$ in the UOM and FDM, respectively. This larger maximum in the UOM and some small differences between the UOM and FDM distributions are most likely due to enhanced vertical resolution of the UOM in these regions. This tropical variability is induced by the seasonal cycle of the zonal wind stress, producing Ekman transport anomalies. Obviously, these are compensated by interior return flows that are well captured in the UOM. This strong tropical variability indicates that the SEASONAL overturning distributions have quite different equatorial circulation patterns than in the mean. Elsewhere, the SEASONAL variance exceeds 25 Sv$^2$ only in the subtropical regions, indicating an anomaly of about 6 Sv. The RESIDUAL variability is much less than the SEASONAL one. The peak values are 60 and 63 Sv$^2$ in the UOM and FDM, respectively, occurring near the surface at the equator. In contrast with the Northern Hemisphere subtropics, the RESIDUAL variability in the Southern Hemisphere subtropics is in excess of 15 Sv$^2$. Here the UOM variability is larger than in the FDM. A similar increase in variability in the UOM is also observed at depth in the Southern Hemisphere Tropics.

The equatorial Pacific zonal velocity variability is shown in Fig. 15, verifying that both the SEASONAL

![Fig. 13. Mean seasonal and residual barotropic streamfunction variance distributions for the UOM and FDM. For the seasonal fields, the contour interval is 20 Sv$^2$, and the regions where variance exceeds 40 Sv$^2$ are shaded. For the residual fields, the contour interval is 10 Sv$^2$, and the shading indicates regions where variance exceeds 20 Sv$^2$.](image)
and RESIDUAL variability of the FDM are captured by the UOM. They are comparable in both magnitude and spatial extent, with the largest values confined to above 100-m depth. The maximum SEASONAL variability occurs in the eastern Pacific, with peak values of 515 and 528 cm$^2$ s$^{-2}$ in the UOM and FDM, respectively. In contrast, the RESIDUAL distributions indicate the western Pacific as the largest variability site where the peak values are about 710 cm$^2$ s$^{-2}$ in both. The RESIDUAL distributions display somewhat larger variability below about 100-m depth in the UOM than in the FDM.

These variability magnitudes and distributions strongly depend on the model parameters and, especially, the zonal wind stress. Consequently, the mean SEASONAL variability distributions from MUOM and MFDM cases (Fig. 16) differ considerably from those of the UOM and FDM. In particular, the variance magnitudes are now almost an order of magnitude larger. Near-surface MUOM variability is larger than in the MFDM in the eastern Pacific, resulting in sharper variance gradients between 50- and 80-m depth. Higher vertical resolution in the MUOM may partly be responsible for this. Farther west, both models have similar variability.

3) MIDDEPTH VARIABILITY

As a final assessment of the UOM variability, we examine time series of $\theta$, $S$, $u$, and $v$ at a middepth level of 417 m. We show time series only for the North Pacific (Fig. 17), but other locations show similar features. All frequencies resolvable by our monthly mean datasets and present in FDM solutions are well reproduced in the UOM. The largest discrepancy is evident in $\theta$ and $S$, reflecting slight vertical offsets of the mean thermocline and halocline (as in Figs. 4–6).

5. Summary and discussion

We have developed design principles for a UOM to be used for studies of short-term climate variability. The salient features are a full dynamics for the upper-ocean domain, an abyssal climatological reservoir for tracers and baroclinic velocity for calculating fluxes through...
Fig. 15. Mean seasonal and residual equatorial Pacific zonal velocity variance for the UOM and FDM. In all panels, the contour interval is 25 cm$^2$ s$^{-2}$, and the shading indicates regions where variance exceeds 100 cm$^2$ s$^{-2}$.

The principles are assessed with observations of the $(z, t)$ structure of $u$ and $S$ that show the variability to be strongly decaying beneath the pycnocline. They are also assessed with idealized models for transient tracer penetration by vertical diffusion and for adiabatic response to wind stress variations. Diffusion yields surface-trapped variability with a depth scale that varies inversely with frequency, and short-term climate variability should be confined to the pycnocline and shallower except for regions of deep convection. Wind-driven fluctuations of both barotropic and upper-ocean currents are generally well modeled with the UOM approximations, except for modest shifts in the resonances and in the presence of large-scale topographic slopes.

We have constructed a global upper-ocean general circulation model. The model is an extension of the NCAR CSM Ocean Model, utilizing stretched $\sigma$ coordinates in the vertical direction. Mean monthly fields from an equilibrated FDM solution are used as the lower boundary conditions in our UOM solutions. Starting from different initial conditions, but subject to the same surface forcing, the UOM produces nearly the same equilibrium solution as in the FDM. The equilibration time in the UOM is about 30 years. This represents a reduction in computational expense by about two orders of magnitude compared to achieving a full equilibrium state with an FDM. This reduction is about an order of magnitude, still substantial, if the FDM uses acceleration methods to achieve equilibrium, but such an integration is not always the relevant comparison standard. Although the upper-ocean fields in an FDM may adjust on shorter timescales, with small residual trends in the upper ocean; only in a UOM can one have confidence that the trends are small everywhere. Another benefit of the UOM comes from a reduction in computational expense due to having fewer vertical levels compared to an FDM; however, this reduction is not linear because of additional coordinate-transformation computations, and the present UOM achieves only about half of this potential savings. We have assessed the accuracy of the UOM by comparing its mean and variability solution behaviors to those of similar FDMs. The differences are generally quite small, and they seem to be more due to...
the differences in discretization procedures than to errors in the UOM design principles. This success occurs even for situations where the solutions are significantly influenced by parameter or forcing changes not included in the UOM lower boundary conditions.

An alternative approach to our UOM might be an FDM with relaxation in the deep ocean toward climatological $\theta$ and $S$. It too could have a rapid adjustment toward an equilibrium if the relaxation is strong, albeit not one in which any claim for dynamical consistency could be made. With strong relaxation, based on previous experience with diagnostic models, we expect strong baroclinic bottom torques (JEBAR) that give rise to erroneous barotropic currents. This is a manifestation of the dynamical inconsistency. If the relaxation is weak then JEBAR may not be large, but deep baroclinic adjustments will be slow and require a long equilibration time. Therefore, it is unclear to us how long such an FDM should be integrated and how our UOM would compare.

The present results confirm that the UOM is dynamically consistent, verifying our initial design hypotheses. Because the UOM is skillful in its solution behavior, we believe that the processes it excludes are unimportant for short-term climate variability investigations for which it can be a useful and robust tool. Given a suitable abyssal climatology, usually obtained from an equilibrium FDM solution, the UOM can be used for forcing, coupling, parameter-sensitivity, and hypotheses-testing studies without the confusion of slow model drifts inherent in disequilibrium FDM solutions.

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APPENDIX A

Wind-Driven Response in a UOM

a. Problem formulation

Here we pose the UOM counterpart to the FDM quasigeostrophic, wind-driven model defined by (3.10)–(3.12), and we compare UOM and FDM solutions to

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**APPENDIX A**

**Wind-Driven Response in a UOM**

**a. Problem formulation**

Here we pose the UOM counterpart to the FDM quasigeostrophic, wind-driven model defined by (3.10)–(3.12), and we compare UOM and FDM solutions to
assess the accuracy of a UOM model for this type of dynamics. We define the depth-averaged (barotropic) streamfunction by

$$\psi = \frac{1}{D} \sum_n \bar{\psi}_n$$

(A.1)

and its governing equation from (3.11) is

$$[\omega K^2 + \beta k^{(\omega)}] \psi = \frac{K \tau_0}{D} - IK^2 v_{rj} \psi$$

$$-\frac{1}{D} (K \sigma + IK^2 \epsilon) \psi_n.$$

(A.2)

In the FDM, (A.2) is redundant with the system (3.11) because of (A.1). The control parameters here are $N$, $\bar{\psi}_n$, $g$, $f$, $\beta$, $\epsilon$, $v_{rj}$, $\tau_0$, $s_r$, $\hat{\theta}_r$, $K$, $\hat{\theta}_k$, and $\omega$. We now specify a UOM that is analogous to the FDM above. Its depth range is $-D' \leq z \leq 0$, with $D' \leq D$; typically we choose $D'$ to extend somewhat beneath the pycnocline. The dependent variables of the UOM are $\psi_m^c$, $m = 1, \ldots, M < N$, and $\psi^c$, and, instead of (3.11) and (A.2), we solve a linear matrix system for the $M$ geostrophic streamfunction amplitudes and a depth-averaged equation for the barotropic streamfunction. These governing equations differ both by truncation in $n \leq M$ and by the replacement of any excluded dependent variables by their climatological values plus the departure of the barotropic component from its climatological value. Here the climatological state is one of rest, with $\psi = 0$. Thus, the governing equations for the UOM are

$$\omega [K^2 \psi_m^c + \Gamma_m (\psi_m^c - \psi_{m-1}^c)]$$

$$+ \Gamma_m^c (\psi_m^c - \psi_{m-1}^c) + \beta k^{(\omega)} \psi_m^c$$

$$= \frac{K \tau_0}{h_1} \delta_{m-1} - IK^2 [v_{rj} K \psi_m^c + \mu_m^c (\psi_m^c - \psi_{m-1}^c)]$$

$$+ \mu_m^c (\psi_m^c - \psi_{m-1}^c).$$

(A.3)

where

$$\psi_m^c = \psi_{m-1}^c [1 - \delta_{m,M}] + \psi^c \delta_{m,M}.$$  

(A.4)

and

$$[\omega K^2 + \beta k^{(\omega)}] \psi$$

$$= \frac{K \tau_0}{D} - IK^2 v_{rj} \psi - \frac{1}{D} (K \sigma + IK^2 \epsilon) \psi.$$  

(A.5)

In the UOM these two governing equations are not redundant since $\psi^c$ is not a linear combination of the $\psi_m^c$. The control parameters for the UOM are $M$, $D$, $\bar{\psi}_n$, $g$, $f$, $\beta$, $\epsilon$, $v_{rj}$, $\tau_0$, $s_r$, $\hat{\theta}_r$, $K$, $\hat{\theta}_k$, and $\omega$, which substantially coincide with those for the FDM.

Comparing the FDM and UOM equations (A.2) and (A.5), we see that the barotropic dynamics are equivalent if $\epsilon = s_r = 0$, so that any barotropic errors in the UOM are due to topography and bottom drag. Note that (A.5) fully determines $\psi^c$ independently from the other $\psi_m^c$, unlike in the FDM where all the $\psi_m^c$ are coupled. Comparing the FDM and UOM equations (3.11) and (A.3), we see that the dynamics for the departure in $\psi_m^c$ and $\psi^c$ from $\psi$ and $\psi^c$ (i.e., the baroclinic component) are never equivalent unless the departure is zero.

Because the control space is so large, we shall not attempt an exhaustive comparison of the FDM and UOM solutions to the problem posed above. So we narrow the scope of the assessment by concentrating on parameter values like those commonly used in global-ocean general circulation models for calculating climate variability. We shall now make representative, albeit somewhat arbitrary, parameter choices that we believe illustrate the important features of the differences in FDM and UOM responses to forcing.

Short-term climate variability involves spatial scales larger than the mesoscale up to the basin size and temporal scales ranging from synoptic to interannual. Thus, we are interested in assessing the UOM for the approximate wavenumber and frequency ranges,

$$3 \times 10^{-5} \leq K \leq 3 \times 10^{-2} \text{ m}^{-1},$$

$$5 \times 10^{-6} \leq |\omega| \leq 2 \times 10^{-8} \text{ s}^{-1}.$$  

(A.6)

We know that Rossby waves are among the homogeneous solutions of this model problem, and thus they permit the possibility of a resonant-forced response. The well-known dispersion relation for barotropic Rossby waves implies

$$|\omega| K \leq \beta \approx 2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}.$$  

(A.7a)

The range of values implied by (A.6) for $|\omega| K$ is $6 \times 10^{-5} - 15 \times 10^{-5} \text{ m}^{-1} \text{ s}^{-1}$, indicating that a barotropic resonance is possible for the faster and shorter forcing scales. An analogous relation for baroclinic Rossby waves is

$$\frac{|\omega|}{K} \leq \beta R_0^2.$$  

(A.7b)

when $KR < 1$, where $R$ is the deformation radius for baroclinic mode $\alpha$. For a typical midlatitude value for the first mode, $R_1 \approx 4.5 \times 10^4 \text{ m}$, the value of the right-hand side of (A.7b) is $4 \times 10^{-2} \text{ m s}^{-1}$, while the value of the left-hand side ranges from $1 \times 10^{-3} - 2 \times 10^0 \text{ m s}^{-1}$. This again indicates a possible resonance for the slower and shorter forcing scales. Because slower variability in the atmosphere tends to be associated with longer scales, however, the important part of the range for the left-hand side of (A.7b) probably begins well above $1 \times 10^{-3} \text{ m s}^{-1}$. Higher baroclinic modes reduce the right-hand side bound in (A.7b) by a factor $\sim \alpha^2$, which therefore makes them less likely to provide important resonances for the ranges in (A.6).

Oceanic simulation models typically have a large vertical grid with resolution concentrated in the upper ocean; hence $N, M \gg 1$. We shall focus on a represen-
accelerations are responding layer thicknesses and reduced gravitational achieves this is BV 0 is chosen so that R1 = 4.5 × 104 m, where R is the eigenvalue in the conservative vertical-structure equation associated with (3.11); namely,

\[
(\Gamma_n^+ + \Gamma_n^- - R^{-2}) F_n - \Gamma_n^+ F_{n+1} - \Gamma_n^- F_{n-1} = 0,
\]

where \( F_n \) is the eigenmode for \( \psi_n \). For the midlatitude value of \( f = 0.7 \times 10^{-3} \) s\(^{-1} \), the value of \( BV_0 \) that achieves this is \( BV_0 = 6.27 \times 10^{-3} \) s\(^{-1} \), and the corresponding layer thicknesses and reduced gravitational accelerations are \( h_i = (74.0, 77.8, 82.0, 86.8, 92.1, 98.1, 105.0, 112.9, 122.1, 132.9, 145.9, 161.6, 181.2, 206.1, 239.1, 284.7, 352.1, 462.1, 510.3, 675.3, 1308.1 \) m and \( g'_i = (2.62, 2.49, 2.36, 2.23, 2.10, 1.97, 1.84, 1.71, 1.58, 1.46, 1.33, 1.20, 1.07, 0.94, 0.81, 0.68, 0.55, 0.42, 0.29) \times 10^{-3} \) m s\(^{-2} \). We choose the lower boundary of the UOM at \( D' = 1291 \) m, corresponding to \( M = 12 \).

This model has several mechanisms for damping associated with the parameters \( \nu_v, \nu_H, \) and \( \epsilon \). We expect them to diminish the magnitude of the forced response compared to a conservative model, and in particular to limit any resonant responses. We estimate some typical damping rates as follows:

\[
\nu_v d^2 \sim (10^{-3} \text{ m}^2 \text{ s}^{-1})(750 \text{ m})^{-2} = 2 \times 10^{-9} \text{ s}^{-1},
\]

\[
\nu_H K^2 \sim (2 \times 10^4 \text{ m}^2 \text{ s}^{-1})(10^{-6} \text{ m}^{-1})^2 = 2 \times 10^{-8} \text{ s}^{-1},
\]

\[
\epsilon D^{-1} = C_0 UD^{-1} \sim (10^{-3})(10^{-4} \text{ m s}^{-1})(5000 \text{ m})^{-1} = 2 \times 10^{-8} \text{ s}^{-1}.
\]

The estimated magnitudes increase in order here, suggesting that the associated mechanisms are similarly ordered in their likely importance. All of these magnitudes are bounded from above by the \( |\omega| \) values in (A.6), although only marginally so with \( \nu_v \) and \( \epsilon \). This indicates that the damping mechanisms are often likely to make only modest changes in the forced response (except at a resonance). There is considerable scope, however, for changing the estimates in (A.10), particularly for horizontal diffusion because of a wide range of possible values both for \( \nu_v \) in GCMs (often varied as the cube of the horizontal grid size to resolve viscous boundary currents) and for \( K \) in (A.6). So damping effects will not always be quantitatively unimportant in oceanic simulation models.

Oceanic topography varies on all scales. In our quasigeostrophic model for a forced response with a Fourier shape in \( (x, t) \), only topographic scales large compared to \( K^{-1} \) can be included with dynamical consistency. We are considering here only large \( K^{-1} \) values in (A.6) that are associated with important atmospheric forcing, and these are often comparable to oceanic basin scales. Thus, there is not much room on earth for the very large topographic scales for which our model is relevant. Nevertheless, we can at least formally analyze its influence on the differences between the UOM and FDM. From (A.5) we see that a relevant standard by which to measure the influence of a given topographic slope, \( s_n \), is \( \beta D f \) (ignoring directional effects). Therefore, for the typical values given above, we estimate that \( s_n \) values of order \( 10^{-3} \) are likely to have an important influence on the forced response.

**b. Analytical assessments**

Before making quantitative assessments of the UOM accuracy for typical parameter values, we analytically derive some asymptotic approximations using a severe vertical truncation to \( N = 2 \) and \( M = 1 \). Although we do not expect such a truncation to be quantitatively accurate relative to \( N, M \to \infty \), the truncated model does expose the qualitative basis for and degree of UOM accuracy.

To facilitate the analysis we first define several new parameters:

\[
\xi = \omega K^2 + \beta k^{(o)} + \nu_\psi K^4, \quad \delta = \frac{1}{D}(Ks + IK^2\epsilon), \quad \Gamma = \Gamma^+_i + I \frac{K^2}{\omega} \mu_i^+ \quad \chi = \frac{h_i}{D},
\]

With these parameters we can write the two-layer FDM as

\[
(\xi + \omega \Gamma) \psi_1 - \omega \Gamma \psi_2 = \frac{K \tau}{h_1}
\]

\[
-\omega \Gamma \left( \frac{\chi}{1 - \chi} \right) \psi_1 + \left[ \xi + \omega \Gamma \left( \frac{\chi}{1 - \chi} \right) + \frac{\delta}{1 - \chi} \right] \psi_2 = 0
\]

\[
\xi \psi_1 + \delta \psi_2 = \frac{K \tau}{D} \psi_1
\]

\[
\chi \psi_1 + (1 - \chi) \psi_2 = \psi_1
\]

and the analogous UOM as

\[
(\xi + \omega \Gamma) \psi_1^r - \omega \Gamma \psi_2^r = \frac{K \tau}{h_1}
\]

\[
(\xi + \delta) \psi_2^r = \frac{K \tau}{D} \psi_1^r.
\]

We will compare the solutions of (A.12) and (A.13) with particular attention to the regime with \( \chi \ll 1 \), which
represents strong surface intensification of the density stratification [i.e., \( d \ll D \) in (A.8)].

Both of the FDM and UOM have the formal possibility of an infinite response to finite forcing in the absence of damping, (i.e., a resonance). This occurs when real-valued \((\omega, k)\) satisfy the dispersion relation for \( \nu_r = \nu_H = \epsilon = 0 \). In the UOM, the dispersion relation factors easily into barotropic and baroclinic modes with \( \omega = \omega^* \) and \( \omega^\tau \), respectively, where

\[
\omega^* = -\frac{\beta k^{(*)}}{K^2 + \Gamma},
\]

\[
\omega^\tau = -\frac{\beta k^{(*)}}{K^2 + \Gamma}.
\]  

(A.14)

In the FDM, the factorization is not as evident, but we can write its dispersion relation as a quadratic equation,

\[
\omega^2 + P\omega + Q = 0,
\]

(A.15a)

where

\[
P = -\frac{K^2 + \Gamma}{K^2\left(K^2 + \Gamma\frac{1}{1 - \chi}\right)}
\times \left[K^2 + \Gamma\frac{\chi}{1 - \chi}\right]\omega^* + K^2\omega^\tau + \frac{\delta\chi}{1 - \chi}\]

\[
Q = -\frac{K^2 + \Gamma}{K^2\left(K^2 + \Gamma\frac{1}{1 - \chi}\right)}\omega^\tau\left(K^2\omega^* + \frac{\delta\chi}{1 - \chi}\right).
\]  

(A.15b)

The corresponding coefficients in the UOM are, of course, \( P^* = -\left(\omega^* + \omega^\tau\right) \) and \( Q^* = -\omega^*\omega^\tau \). The FDM resonant frequencies are clearly different from those in the UOM in general. However, they are equivalent if \( \chi = 0 \). There is also an approximate equivalence if \( \chi \ll 1 \) and \( \omega^\tau \ll \omega^* \), as occurs when \( K^2 \ll \Gamma \), that is, large scale, as in (A.6) above, and \( \beta k^{(*)} + Ks/D \) is not small compared to \( \beta k^{(*)} \), that is, an absence of cancellation between the planetary and topographic potential vorticity gradients perpendicular to \( k \). Thus, the resonances
in the FDM and UOM usually occur close to each other in their control spaces.

Near a resonance the oceanic response is large. Between any two models with small differences in their conditions of resonance, the solution differences near a resonance will be large, both absolutely and relatively. Because this behavior is generic to all model differences—for example, the stratification strength, the forcing wavenumber, or a diffusivity—it should not be considered as a serious flaw of the UOM in particular.

In each of the systems (A.12) and (A.13), we can eliminate the other dependent variables and obtain an explicit formula for the upper-ocean streamfunction:

\[
\psi_1 = \frac{K\tau_0}{h_1} \frac{\xi + \delta \frac{1}{1 - \chi} + \omega\Gamma \frac{\chi}{1 - \chi}}{(\xi + \omega\Gamma)(\xi + \delta - \frac{1}{1 - \chi}) + \xi \omega\Gamma \frac{\chi}{1 - \chi}}
\]

Thus, \(\psi_1 = \psi_1\) if \(\chi << 1\), as long as none of the denominators in (A.16) are close to zero. The latter occurs only if either

\[\xi + \delta \approx 0 \text{ [barotropic near resonance]} \quad (A.17a)\]

or

\[\xi + \omega\Gamma \approx 0 \text{ [baroclinic near resonance]}, \quad (A.17b)\]

as can be confirmed by comparison with (A.14). Here \(\approx 0\) is to be understood in the sense of \(O(\chi)\) relative to individual left-hand terms. Thus, the UOM is accurate in its calculation of \(\psi_1\) if \(\chi << 1\), except near resonances where it should not be expected to be (see above).

We can similarly eliminate dependent variables in favor of the barotropic streamfunction:

\[
\psi^r_1 = \frac{K\tau_0}{h_1} \frac{\xi + \delta + \omega\Gamma \chi}{(\xi + \omega\Gamma)(\xi + \delta)}
\]

Thus, \(\psi^r_1 = \psi_1\) if \(\chi \ll 1\), as long as none of the denominators in (A.16) are close to zero. The latter occurs only if either

\[\xi + \delta \approx 0 \text{ [barotropic near resonance]} \quad (A.17a)\]

or

\[\xi + \omega\Gamma \approx 0 \text{ [baroclinic near resonance]}, \quad (A.17b)\]

as can be confirmed by comparison with (A.14). Here \(\approx 0\) is to be understood in the sense of \(O(\chi)\) relative to individual left-hand terms. Thus, the UOM is accurate in its calculation of \(\psi_1\) if \(\chi \ll 1\), except near resonances where it should not be expected to be (see above).

We can similarly eliminate dependent variables in favor of the barotropic streamfunction:

Fig. A3. UOM relative errors in the strongly damped case. The contour intervals are (top) 0.01 and (bottom) 0.1.

Fig. A4. Barotropic response for the case with \(s_0 = D\beta/2f\) and \(\bar{\theta}_1 = \pi/2\). (top and middle) The contour interval is 0.5. (bottom) The contour interval is 0.1.
Here there is not equivalence as $\chi \rightarrow 0$. To see more clearly the discrepancy, we make the approximation of $\chi \ll 1$ in the FDM:

$$\psi = \frac{K\tau_0}{D} \frac{\xi(1 - \chi) + \omega\Gamma + \delta}{\xi(1 - \chi) + \xi(\delta + \omega\Gamma) + \delta\omega\Gamma}. \quad \text{FDM}$$

$$\psi^o = \frac{K\tau_0}{D} \frac{1}{(\xi + \delta)}. \quad \text{UOM.} \quad \text{(A.18)}$$

Thus, comparing (A.19) with (A.13), we have $\psi^o = \psi$ only if $\chi \ll 1$ and

$$\delta \ll \xi + \omega\Gamma,$$

except near the resonance conditions (A.17). Therefore, we conclude that sufficiently large values for either topographic slope or bottom drag can make $\psi^o$ inaccurate, even though $\psi^o$ can remain accurate. The earlier discussion, following (A.10), indicates that the topographic influence is the one more likely to be large enough to sometimes matter. An estimate for when this will occur is when $s_0$ is not small compared to $(o(\ell))(\text{DIR})(1/\text{KR})$.

In summary, the solutions of the UOM are usually close to those of the FDM because of the surface intensification of oceanic stratification. Exceptions occur near barotropic and baroclinic resonances, but these circumstances are ones with sensitive dependence in general; the resonance conditions are also close in the UOM and FDM when the forcing scale is large compared to the baroclinic deformation radius. A more important exception can occur for the barotropic streamfunction, but not the upper-ocean streamfunction, where topographic slopes are substantial and the forcing is near the small-scale, low-frequency end of the ranges of interest in (A.6).

c. Computational assessments

We now return to the higher vertical resolution in section a of Appendix A to make a quantitatively relevant assessment of the UOM. We shall present only a few examples, since a qualitative survey of the parameter dependences was made in section b of appendix A.

We make the assessment in terms of the amplitude of the wind-driven response. For the barotropic response, the measures we use are

$$A = |\psi|, \quad A^o = |\psi^o|, \quad \Delta A = \frac{|\psi - \psi^o|}{|\psi|}, \quad \text{(A.20)}$$

and for the upper-ocean response, the measures we use are

$$B = \left( \frac{1}{D'} \sum_{m=1}^{M} h_m |\psi_m^o|^{1/2} \right),$$

$$B^o = \left( \frac{1}{D'} \sum_{m=1}^{M} h_m |\psi_m^o|^{1/2} \right),$$

$$\Delta B = \left( \frac{\sum_{m=1}^{M} h_m |\psi_m^o - \psi_m^o|^{1/2}}{\sum_{m=1}^{M} h_m |\psi_m^o|^{1/2}} \right). \quad \text{(A.21)}$$

As a primary case, we choose many of the same parameter values as in section a of appendix A: $[N = 20, M = 12, (\ell, g')$ as listed above, $f = 0.7 \times 10^{-4} \text{s}^{-1}, \beta = 2 \times 10^{-11} \text{m}^{-1} \text{s}^{-1}, \epsilon = 10^{-4} \text{m}^{-1}, \nu_f = 10^{-3} \text{m}^2 \text{s}^{-1}, \nu_b = 2 \times 10^{-4} \text{m}^2 \text{s}^{-1}, \tau_v = 10^{-4} \text{m}^2 \text{s}^{-2}, s_0 = 0.0, \delta_0 = 0.0].$ In addition we survey $(K, \omega)$ values within the ranges indicated in (A.6) for the particular orientation $\phi_0 = 0.0$.

Figure A1 shows the barotropic responses in the FDM and UOM for the primary case. They are quite similar.

The response is generally larger with smaller $K$, and it is especially large near the barotropic resonance. The relative difference between the barotropic responses is very small everywhere except near the baroclinic resonances. Analogous displays of the upper-ocean responses and their relative difference are shown in Fig.
A2. The responses are larger for smaller $|\omega|$ and $K$, and they are especially large near both barotropic and baroclinic resonances. However, the relative difference in response is large only near the baroclinic resonances. All of these features confirm the expectations from the analysis in section b of appendix A.

We next assess the sensitivity to the damping rates. If each of $\epsilon$, $\nu_\psi$, and $\nu_H$ are multiplied by an arbitrary factor of 100, the responses (not shown) have qualitatively similar patterns but comparably reduced amplitudes. The relative differences (Fig. A3) are again usually small, somewhat less so compared to the primary case for the barotropic response and more so for the upper-ocean response. The differences are largest near the baroclinic resonances, which have now been appreciably shifted from their inviscid locations.

Finally, we consider topographic influences in a case that is otherwise like the primary case. We choose a bottom slope magnitude, $s$, equal to half the equivalent planetary $\sin f$. This slope magnitude is sufficiently large to appreciably modify the response patterns from their flat-bottom shapes, in part by shifting the locations of the baroclinic resonances.

Thus, we conclude that the UOM usually yields an accurate approximation to FDM solutions for adiabatic, wind-driven response in the problem defined by (3.10)–(3.12) in the forcing regime defined by (A.6). Exceptions occur near resonances and in particular situations with a large-amplitude topographic slope.

APPENDIX B
Continuous Forms of the Nonconservative Terms

The continuous forms of the horizontal viscous terms in (4.3) and (4.4) with spatially inhomogeneous, anisotropic eddy viscosities, $A_{\text{MHz}}$ and $B_{\text{MHz}}$, are

\[
F_H^y = \frac{1}{a^2 \cos^2 \phi} (A_{\text{MHz}} u_x)_\psi + \frac{1}{a^2 \cos \phi} (B_{\text{MHz}} \cos \phi u_x)_\phi + \frac{(1 - \tan^2 \phi)}{a^2} B_{\text{MHz}} \tan \phi u_x - (A_{\text{MHz}} + B_{\text{MHz}}) \frac{\tan \phi}{a^2 \cos \phi} v_y
\]

\[
+ \left[ \frac{\tan \phi}{a^2} u_x + \frac{v_x}{a^2 \cos \phi} \right] (B_{\text{MHz}})_y - \frac{\tan \phi}{a^2 \cos \phi} v (A_{\text{MHz}})_\psi - \frac{u_{\psi}}{a^2 \cos \phi} (B_{\text{MHz}})_x.
\]

\[
F_H^z = \frac{1}{a^2 \cos^2 \phi} (B_{\text{MHz}} u_x)_\psi + \frac{1}{a^2 \cos \phi} (A_{\text{MHz}} \cos \phi u_x)_\phi + \frac{B_{\text{MHz}} - \tan^2 \phi A_{\text{MHz}}}{a^2} v + (A_{\text{MHz}} + B_{\text{MHz}}) \frac{\tan \phi}{a^2 \cos \phi} u_x
\]

\[
+ \left[ \frac{\tan \phi}{a^2} v + \frac{u_x}{a^2 \cos \phi} \right] (B_{\text{MHz}})_y + \left[ \frac{\tan \phi}{a^2} u_x + \frac{u_{\psi}}{a^2 \cos \phi} \right] (B_{\text{MHz}})_y.
\]

(Wajsowicz 1993; LDMGB). In (B.1) and (B.2), all the spatial derivatives are evaluated using the differential relations

\[
G_{\psi} = G_\psi - \frac{z}{h} \frac{D_y'}{D_x'} G_{\omega}, \quad G_{\phi} = G_\phi - \frac{z}{h} \frac{D_y'}{D_x'} G_{\omega}.
\]

Also, we evaluate the first two terms of (B.1) and (B.2) as the divergences of diffusive fluxes; for example, for (B.1),

\[
\mathbf{\nabla}' \cdot \mathbf{B}_\psi = \frac{1}{a \cos \phi} \frac{1}{h} \left[ (h B_{\psi})_x + \frac{D'_x}{D_x} (h B_{\psi})_y + (h \cos \phi B_{\psi})_\phi \right.
\]

\[
- \cos \phi \frac{D'_x}{D_x} (z B_{\psi})_\phi \left].
\]

with

\[
\mathbf{B}_\psi = [B^y_\psi, B^z_\psi]
\]

\[
= \left[ A_{\text{MHz}} \left( u_x - \frac{z}{h} \frac{D'_y}{D_x} u_y \right) \right. \quad \frac{B_{\text{MHz}}}{a} \left( u_\phi - \frac{z}{h} \frac{D'_y}{D_x} u_y \right).
\]

This flux-preserving form satisfies the integral conservation relation for the horizontal diffusion operator.

The isopycnal diffusion operator, $\mathcal{K}$, in (4.11), is defined by

\[
\mathcal{K}(A, G) = \mathbf{\nabla}' \cdot (A \mathcal{K} \cdot \mathbf{\nabla}' G),
\]

where $\mathbf{\nabla}'$ is the three-dimensional gradient operator in spherical coordinates, and $\mathcal{K}$ is the second-rank tensor, $\mathcal{K} = \mathbf{I} \otimes \mathbf{S} \otimes \mathbf{S}$. (B.7)
Here $I$ is the $2 \times 2$ identity matrix, and $T$ denotes a matrix transpose.

Because the isopycnal-diffusion operator has the same value in any coordinate system, we may alternatively use

$$
\mathcal{R}(A, G) = \frac{1}{h} \nabla_{3D} \cdot (A \frac{\partial}{\partial \sigma} \nabla_{3D} G),
$$

where

$$
\mathcal{R} = \begin{pmatrix} 1 & S^r \cdot S^r \\ S^r & S^r \cdot S^r \end{pmatrix}, \quad \nabla_{3D} = \begin{pmatrix} \nabla & \frac{\partial}{\partial \sigma} \end{pmatrix},
$$

and $S^r$ is the slope of the isopycnals with respect to a surface of constant $\sigma$,

$$
S^r = -\frac{\nabla p}{\rho^r}.
$$

In our present implementation, we use (4.14), (B.6), and (B.7), because components of $S$ can readily be used in the eddy-induced velocity computations to save computational work. Also, the flux-preserving form in (B.4) and (B.5) is used for $R$.

The general forms of the vertical mixing of horizontal momentum and of any tracer $\tau$ are

$$
F^\tau = \frac{1}{h} \left[ \frac{A_{MV}}{h} u^\tau \right]_{\sigma}, \quad \text{and}
$$

$$
F^\nu = \frac{1}{h} \left[ \frac{A_{SV}}{h} \left( \frac{\tau^\nu}{h} - \gamma^r \right) \right]_{\sigma},
$$

respectively. Here, the KPP vertical-mixing scheme (Large et al. 1994) is used to determine the profiles of the vertical viscosity, $A_{MV}$; vertical diffusivity, $A_{SV}$; and the nonlocal (countergradient) boundary layer transport, $\gamma^r$.

**APPENDIX C**

**Discretization of the Model Equations**

All of the continuous equations of the stretched $\sigma$ coordinate model are discretized on the Arakawa B grid, consistent with the NCAR CSM Ocean Model (NCOM) (NCAR Oceanography Section 1996; Gent et al. 1998). Here, we provide a brief summary of our discrete forms, especially focusing on the ones that differ substantially from the standard NCOM, and refer the reader to these references for further details.

Here we adopt generic shorthand notations for spatial differencing and averaging as

$$
\delta_i(G) = \frac{1}{\Delta_i} (G_{i+1/2} - G_{i-1/2}) \quad \text{and}
$$

$$
\overline{G} \Delta_l = \frac{1}{2} (G_{i+1/2} + G_{i-1/2}),
$$

respectively, where the subscript $i$ denotes the discrete spatial location at which the gradient or average is computed, and $\frac{1}{2}$ refers to one-half grid point offset with respect to the point $i$. Terms $\Delta \lambda$, $\Delta \phi$, and $\Delta \sigma$ are the grid spacings in the longitudinal, latitudinal, and vertical directions, respectively. In the following discrete relations, we omit the time index for clarity.

The model bottom depth is fundamentally defined at the tracer points, denoted by superscript $\tau$,

$$
D^\tau = D^\tau(\lambda, \phi).
$$

We also define depths at the model horizontal velocity points, denoted by superscript $v$, using

$$
D^v = \overline{D^\rho}^{\lambda, \phi}.
$$

The vertical grid is uniform in $\sigma$, with $km$-layer grid points and spacing $\Delta \sigma = 1/km$. The corresponding coordinate values are

$$
\sigma_k = 1 - \left( k - \frac{1}{2} \right) \Delta \sigma, \quad k = 1, \ldots, km.
$$

Thus the grid index increases downward, while the vertical coordinates increase upward. The fundamental (i.e., on the tracer grid) layer thicknesses are defined by

$$
\Delta z_t^\tau = D^\tau \cdot \Phi(\sigma_k), \quad k = 1, \ldots, km,
$$

where $\Phi$ is the normalized, stretched layer thickness. Although in our present implementation, we use a numerical $\Phi$, it can be analytically defined. Using (C.5), we can now write the discrete forms for $h$,

$$
h^\tau = \frac{\Delta z_t^\tau}{\Delta \sigma}; \quad h^v = h^\tau \overline{D^\rho}^{\lambda, \phi}
$$

The discrete form for the Eulerian mean advection operator at the tracer points is

$$
L^G = \frac{1}{ah^\tau \cos \phi^r} \left[ \delta_i \left( \frac{\Delta h^\tau \Delta \phi^r \Delta \lambda^\tau}{\Delta \phi^r} \right) G
\right]
$$

$$
+ \delta_i \left( \frac{ah^\tau \cos \phi^r \Delta \lambda^\tau}{\Delta \lambda^\tau} \right),
$$

$$
+ \frac{1}{h^\tau} \delta_i \left( \overline{WG} \right).
$$

The continuity equation is fundamentally evaluated for the tracer grid, using $h^\tau L^t(1) = 0$, with $W$ defined at the bottom of the tracer grid boxes. Following Danabasoglu and McWilliams (1995), this vertical velocity is then area averaged to obtain the vertical velocity for the momentum equations. The related advection operator for the momentum equations is computed from (C.7) as area-averaged fluxes to maintain the flux and kinetic energy conservation properties.

The zonal and meridional pressure gradient terms are discretized as
Thus, the discrete form for \( \nabla \cdot \mathbf{B}_e \) is discretized as follows

\[
\nabla \cdot \mathbf{B}_e = \frac{1}{a \cos \phi^v} \left[ \delta \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \right) - \frac{z^w}{h^v} \frac{\delta}{D^v} \frac{\delta}{z^w} \left( \frac{B_e^\lambda}{D^\lambda} \right) \right] + \delta \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \cos \phi^v \right) + \delta \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \cos \phi^v \right) - \cos \phi^v \delta \left( \frac{B_e^\lambda}{D^\lambda} \right)
\]

(C.10)

In (C.10), \( z^w \) represents the level depth at the vertical velocity locations associated with the horizontal velocity points.

In order to compute the isopycnal diffusion tensor components, the associated local potential density gradients must be obtained first. The evaluation of the (1–3) component of the tensor, \( K_{31} \), requires the computation of these gradients at the eastern faces of the tracer grid boxes. Consequently, we use

\[
\rho_{\lambda} = \frac{1}{a \cos \phi^v} \left[ \delta \left( \frac{\lambda^w}{D^\lambda} \right) - \frac{z^w}{h^v} \frac{\delta}{D^v} \frac{\delta}{z^w} \left( \frac{\lambda^w}{D^\lambda} \right) \right] \quad \rho_{\phi^v} = \frac{1}{a} \left[ \delta \left( \frac{\phi^v}{D^v} \right) - \frac{z^w}{h^v} \frac{\delta}{D^v} \frac{\delta}{z^w} \left( \frac{\phi^v}{D^v} \right) \right] \quad \rho_{\phi^\lambda} = \frac{1}{h^v} \delta \left( \frac{\phi^\lambda}{D^\lambda} \right)
\]

(C.11)–(C.13)

In (C.11)–(C.13) and in some of the following equations, the braces are used to denote that the expressions inside have the same discretization for a given set. The discrete forms of the density gradients, obtained at the northern faces of the tracer grid boxes, for \( K_{33} \), computation follow from the above equations, and are not repeated here. The \( K_{31}, K_{32}, \) and \( K_{33} \) components of the tensor are computed at the vertical faces of the tracer grid boxes. The discrete forms are

\[
\rho_{\lambda} = \frac{1}{a \cos \phi^v} \left[ \frac{\delta (\lambda^w \delta \lambda^w)}{D^\lambda} - \frac{z^w}{h^v} \frac{\delta (\lambda^w \delta \lambda^w)}{D^v} \right] \quad \rho_{\phi^v} = \frac{1}{a} \left[ \frac{\delta (\phi^v \delta \phi^v)}{D^v} - \frac{z^w}{h^v} \frac{\delta (\phi^v \delta \phi^v)}{D^v} \right] \quad \rho_{\phi^\lambda} = \frac{1}{h^v} \delta (\phi^\lambda \delta \phi^\lambda)
\]

(C.14)–(C.16)

where the superscript \( \omega \) represents the variables located at the vertical velocity locations associated with the tracers.

In stretched \( \sigma \) coordinates, to satisfy the integral conservation relations, the horizontal flux components of the isopycnal tracer diffusion are discretized at the eastern and northern faces of the tracer grid boxes, respectively.

\[
B_e^\lambda(G) = A_{\phi^v} \left[ \frac{1}{a \cos \phi^v} \left( \delta \left( \frac{\phi^v \delta \phi^v}{D^\lambda} \right) - \frac{z^w}{h^v} \frac{\delta}{D^v} \frac{\delta}{z^w} \left( \frac{\phi^v \delta \phi^v}{D^\lambda} \right) \right) + A_{\phi^\lambda} \frac{1}{h^v} \delta \left( \frac{\phi^\lambda \delta \phi^\lambda}{D^\lambda} \right) \right] \quad B_e^\phi(G) = A_{\phi^v} \left[ \frac{1}{a} \left( \delta \left( \frac{\phi^v \delta \phi^v}{D^\phi} \right) - \frac{z^w}{h^v} \frac{\delta}{D^v} \frac{\delta}{z^w} \left( \frac{\phi^v \delta \phi^v}{D^\phi} \right) \right) + A_{\phi^\lambda} \frac{1}{h^v} \delta \left( \frac{\phi^\lambda \delta \phi^\lambda}{D^\phi} \right) \right]
\]

(C.17)–(C.18)

Thus, the discrete form for \( \nabla' \cdot \mathbf{B}_e \) becomes

\[
\nabla' \cdot \mathbf{B}_e = \frac{1}{a \cos \phi^v} \left[ \delta \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \right) - \frac{z^w}{h^v} \frac{\delta}{D^v} \frac{\delta}{z^w} \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \right) + \delta \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \cos \phi^v \right) - \cos \phi^v \delta \left( \frac{h^\lambda}{D^\lambda} B_e^\lambda \cos \phi^v \right) \right]
\]

(C.19)

The vertical components of the isopycnal diffusion tensor are discretized as
The eddy-induced transport volume components are computed at the centers of the eastern, northern, and vertical faces of a tracer grid box, respectively. The discrete forms are

\[ u^* = -\frac{1}{h^r} \frac{1}{a \cos \phi^r} A_{173} \chi_{13}^{r} \delta_{\phi}(\frac{\Delta \phi^r}{\Delta y^r}) \cdot \left( \frac{\Delta \phi^r}{\Delta y^r} \delta_{z}(G) - \frac{\Delta \phi^r}{\Delta y^r} \frac{1}{h^r} \delta_{z}(G) \right) \]

\[ v^* = -\frac{1}{h^r} \frac{1}{a \cos \phi^r} A_{173} \chi_{23}^{r} \delta_{\phi}(\frac{\Delta \phi^r}{\Delta y^r}) \cdot \left( \frac{\Delta \phi^r}{\Delta y^r} \delta_{z}(G) - \frac{\Delta \phi^r}{\Delta y^r} \frac{1}{h^r} \delta_{z}(G) \right) + A_{233} \frac{1}{h^r} \delta_{z}(G) \cdot \left( \frac{\Delta \phi^r}{\Delta y^r} \delta_{z}(G) - \frac{\Delta \phi^r}{\Delta y^r} \frac{1}{h^r} \delta_{z}(G) \right). \]

Finally, the discrete counterparts of the vertical-mixing terms (B.11) and (B.12) are

\[ F_u^*(G) = \frac{1}{h^r} \delta_{z}(\frac{A_{MV}}{h^r}\delta_{\phi}(G)), \quad \text{and} \]

\[ F_v^*(G) = \frac{1}{h^r} \delta_{z}(\frac{A_{SV}}{h^r}(\frac{\delta_{\phi}(G)}{h^r} - \gamma)). \]

The discrete forms listed above necessitate the values of either a variable itself or its vertical gradient at both the ocean surface and the lower boundary. We assume that the surface values of these variables are the same as the ones at the first model layer. Also, because the gradients are multiplied by \( z \), they are set to zero at the surface. At the ocean bottom boundary, most of the required variables are prescribed as boundary conditions in the upper-ocean model. However, in some regions the bottom depth of the upper ocean represents the actual ocean bottom depth where no bottom values are available as boundary conditions. In these instances we use

\[ G_{-D'} = \frac{1}{\Delta z_{im-1}} [G_{im}(\Delta z_{im} + \Delta z_{im-1}) - G_{im-1} \Delta z_{im}], \]

where \( G_{-D'} \) is the extrapolated value of a variable at the ocean bottom. Also, \( \Delta z^* (= \Delta \rho h^*) \) is the vertical level thickness at the vertical velocity points located between the tracer points. Similarly, another extrapolative estimator, \( DZ \), is applied to evaluate the unknown vertical gradients at the bottom,

\[ DZ(G) = \frac{(2h_{im} + h_{im-1})^2(G_{im} - G_{-D}) - h_{im}(G_{im-1} - G_{-D})}{\Delta \rho h_{im}(h_{im} + h_{im-1}/2(h_{im} + h_{im-1}))}. \]

As stated in section 4a(1), a third-order polynomial approximation from Bryan and Cox (1972) is used for the equation of state. In this approach, the expansion coefficients as well as the reference \( \theta \) and \( S \) profiles are all functions of depth. In a \( \sigma \)-coordinate model, each grid point can potentially be located at a different depth than all the other grid points, necessitating an efficient method for density computations. Here, we first compute and store these expansion coefficients together with \( \theta \) and \( S \) at the standard Levitus data depths. They are then linearly interpolated to each model gridpoint depth. Finally, the polynomial equation is applied to compute density. To expedite the interpolation procedure, all the bracketing indices are precomputed and stored.

**APPENDIX D**

**Formulation of Vorticity Terms**

We formulate the right-hand side terms in the vorticity equation, (4.24). Here, we use the subscripts \( U, L \), and \( C \) to denote the upper-ocean, prognostic variables; lower (abyssal) ocean variables; and climatological and/or from a corresponding FDM variables, respectively. In the following, we decompose the depth integral of a variable into an upper- and lower-ocean part where the latter will be provided as a lower boundary condition into a UOM. Therefore, we write the equations in the \( z \)-coordinate system for simplicity, keeping in mind that any variable with an \( U \) subscript is evaluated in the transformed coordinate system in the model.

**Advection**

\[ \frac{1}{D} \int_{-D}^{0} \mathcal{L}(G) \, dz = \frac{1}{D} \left\{ \int_{-D}^{0} [\mathcal{L}(G)]_{U} \, dz + \int_{-D}^{0} [\mathcal{L}(G)]_{L} \, dz \right\}. \]

For these nonlinear terms, we assume
\[ [\mathcal{L}(G)]_c = [\mathcal{L}(\overline{G} + G')]_c \tag{D.2} \]

that is, we ignore the fluctuating part of advection. An alternative approach may be to linearize these terms in the abyssal ocean. The nonlinear metric terms are treated similarly.

**Coriolis force**

\[
\frac{1}{D} \int_{-D}^{0} f \overline{G} \, dz = \frac{1}{D} \int_{-D}^{0} f \overline{G} \, dz + \frac{1}{D} \int_{-D}^{0} f G' \, dz \\
= f \overline{G} + \frac{1}{D} f \int_{-D}^{0} G' \, dz \\
= f \overline{G}_v, \tag{D.3} \]

where we used the definition of the baroclinic velocity. Note that there is no lower boundary condition for these terms, and \( \overline{G}_v \) is the prognostic upper-ocean barotropic velocity based on \( D \).

**Pressure gradient**

\[
\frac{1}{D} \int_{-D}^{0} \frac{g}{\rho_0} \nabla \left( \int_{z}^{0} \rho \, dz^+ \right) \, dz \\
= \frac{g}{\rho_0} \frac{1}{D} \int_{-D}^{0} \nabla \left( \int_{z}^{0} \rho \, dz^+ \right) \, dz \\
= \frac{g}{\rho_0} \frac{1}{D} \int_{-D}^{0} \nabla \left( \int_{z}^{0} \rho_v \, dz^+ \right) \, dz \\
+ \int_{-D}^{0} \nabla \left( \int_{z}^{0} \rho_c \, dz^+ \right) \, dz \\
+ \int_{-D}^{0} \nabla \left( \int_{z}^{0} \rho \, dz^+ \right) \, dz. \tag{D.4} \]

Here, only the last term comes from climatology (i.e., it is provided as a lower boundary condition). Again, we ignore \( \rho_c \) fluctuations.

**Horizontal nonconservative terms**

Note that all of the nonconservative terms are linear and that the vertical average and the horizontal gradient operators commute with the barotropic–baroclinic decomposition. Therefore, we can write

\[
\frac{1}{D} \int_{-D}^{0} F_{uv}^{(y)}(A_{uv}, G) \, dz \\
= \frac{1}{D} \left( \int_{-D}^{0} F_{uv}^{(y)}(A_{uv}, G_v) \, dz \right) \\
+ \left( D - D' \right) F_{uv}^{(y)} \left( \overline{A}_{uv}, (\overline{G}_v - \overline{G}_c) \right). \tag{D.5} \]

The form of the last term on the right-hand side of (D.5) assumes that the horizontal eddy viscosity coefficient, \( A_{uv} \) (or \( B_{uv} \)), is either independent of the vertical coordinate or obtained by depth averaging (denoted by \( A_{uv} \)), and thus it is different than the baroclinic \( A_{uv} \).

**Vertical nonconservative terms**

\[
\frac{1}{D} \int_{-D}^{0} F_{v}^{(y)}(A_{v}, G) \, dz \\
= \frac{1}{D} \left( \int_{-D}^{0} F_{v}^{(y)}(A_{v}, G_v) \, dz \right) \\
+ \int_{-D}^{0} F_{v}^{(y)}(A_{v}, G) \, dz \\
= \frac{1}{D} \left( \int_{-D}^{0} F_{v}^{(y)}(A_{v}, G_v) \, dz \right) \\
+ \int_{-D}^{0} F_{v}^{(y)}(A_{v}, G_c) \, dz - \epsilon (\overline{G}_v - \overline{G}_c). \tag{D.6} \]

The surface wind stress is included in the first term. In this implementation, we assume a linear drag law. The drag due to the climatological \( u \) is included in the middle term of (D.6). The drag due to the prognostic barotropic component appears in the last term with \( \epsilon = C_D 0.1 \text{ m s}^{-1} \), where \( C_D \) is the ocean bottom drag coefficient.

**REFERENCES**


