NOTES AND CORRESPONDENCE

Estimating Changing Extremes Using Empirical Ranking Methods

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ABSTRACT

It is often useful to make initial estimates of changing extremes without the use of a specific statistical model, though a statistical model is likely to be desirable as a second step. A proof is given of a formula used by A. F. Jenkinson in the 1970s that converts data that are ranked according to their magnitude into an estimate of the associated cumulative probability. This formula is compared to its exact equivalent, based on a beta distribution of the first kind. It is also compared to similar ranking formulas, which have been recommended, mostly in hydrology, based on similar ideas. Some results concerning the effect of serial correlation on Jenkinson’s formula are reported. For initial estimates of return periods or percentiles of cumulative probability from time series of data, Jenkinson’s method performs as well as many of the other methods. Empirical ranking methods are not so useful for estimating the rarest percentiles in climatology, those in the most extreme 100/N% tails of the distribution, say, where N is the data length. To estimate such extreme percentiles, distributional models are essential. However, for moderate extremes it is suggested that Jenkinson’s or one of the similar methods are useful for an initial assessment of changing percentiles for a wide range of underlying data distributions.

1. Introduction

Horton et al. (2001) use an empirical ranking method for climatological data probably introduced by Beard (1943), derived in a similar form by Chegodaev (1953), and extensively used by Jenkinson in Volume II of a Natural Environment Research Council report (NERC 1975a,b). The method converts ranked values of a physical variable (e.g., temperature) into corresponding cumulative probabilities defined over a specific climatological period. Horton et al. (2001) used this as one method of estimating the cumulative probability corresponding to a given physical value and compared this with a statistical model–based approach. The aim of their work was to investigate whether the magnitude of various extremes shows trends during the warming climate of the last century or more. The same ranking method has recently been used by Bonsal et al. (2001) for studies of changing climate extremes over Canada. The purpose of this note is to discuss the origins of this empirical ranking method, to compare it with other empirical ranking methods that have been used, and to report some results concerning the effect of serial correlation.

Let there be N physical values taken from different years; let these values be ranked in increasing order of size from the smallest, \( m = 1 \), to the largest, \( m = N \). Thus \( m \) represents the rank of each of the \( N \) values, with \( m \) increasing as the value increases. Jenkinson (1977) gives an estimate of the cumulative probability \( P \) corresponding to rank \( m \) as

\[
\hat{P} = \frac{100}{N + 0.38} \left( \frac{m - 0.31}{N} \right). \tag{1}
\]

Jenkinson states that Chegodaev rounded the numerical value in the numerator to 0.3 and that in the denominator to 0.4. Jenkinson discussed this formula in the context of ranked annual maximum values of a variable. However, as shown below, the formula applies to any ranked series of continuously distributed measurements, provided it can be regarded as a random sample. Horton et al. (2001) used it when estimating the 95th percentile value in a given year from all values measured in the year ranked in order of their size. The question arises concerning the extent to which this estimate is independent of, or appropriate for, the underlying dis-
tribution of the physical variable. Horton et al. (2001) assumed their variables were gamma distributed, with many examples close to a normal distribution.

It is made clear in Jenkinson (1977), and in NERC (1975a, chapter 1), that empirical “plotting positions” or ranking formulas such as (1) are not quite exact. Their simplicity is sufficiently appealing, however, that if they can be used for most kinds of data to give consistent estimates of cumulative probabilities, they may be suitable for initial, reasonably self-consistent estimates of percentiles or return periods that do not require an investigation of the underlying distribution. An attraction of Eq. (1) lies in the fact that an estimate of return period can be obtained from it for different values of \( N \), and as indicated below, for a wide variety of distributions, though random uncertainties may be quite large. Thus if we wish to base estimates of percentiles on the 30-yr period, 1961–90, but the data are incomplete in that period (several years may be missing), estimates of \( P \) can still be made.

2. The distribution of data ranked by size

The arguments in this section are well known and added for clarity. See, for example, David (1981) or Johnson et al. (1994). Consider a sample of \( N \) values of a variable \( X \) such as temperature. Let the sample values be ranked in order of magnitude from the smallest to the largest:

\[
x_1 \leq \cdots \leq x_m \leq \cdots \leq x_N.
\]

These ranked observations are referred to as the order statistics of the sample. We will use the general notation that \( F \) is the (cumulative) distribution function of \( X \), with corresponding probability density function \( f \). Thus \( F(x) \) is the probability that \( X \) takes a value less than or equal to \( x \), and \( f(x) \) is the derivative of \( F(x) \). For a small increment \( dx \) we interpret \( f(x)dx \) as the probability that \( X \) takes a value close to \( x \) (a value in an interval of width \( dx \) around \( x \)).

For each value of \( \alpha \) (0 < \( \alpha < 1 \)) we define the \( (100\alpha) \)th percentile of \( X \) to be the value \( \xi_\alpha \) whose cumulative probability \( F(\xi_\alpha) \) is equal to \( \alpha \). Thus there is probability \( \alpha \) that \( X \) will be no larger than \( \xi_\alpha \). If \( F \) were known, then percentiles \( \xi_\alpha \) could be found directly for any \( \alpha \). On the other hand if, as is usually the case, \( F \) is not known but a sample of observations with distribution \( F \) is available, we can ask the following question: What is the probability, \( F(x_m) \), associated with the \( m \)th largest observation? This is the problem considered in this paper; given \( m \) and the observed value \( x_m \) of the \( m \)th order statistic in a sample of \( N \) observations, we estimate the probability, \( P = F(x_m) \), that a measurement of the variable \( X \) will be no larger than \( x_m \). This is the same as estimating the cumulative probability \( F(x_m) \). Jenkinson’s formula (1) is one estimate of \( P \), expressing it as a percentage. Another even simpler, intuitive estimate of \( P \) is \( m/N \), the proportion of the sample no greater than \( x_m \). Interest in estimating \( P \) stems from a wish to reverse the relation between \( P \) and \( x_m \), and to estimate the order statistic \( x_m \) whose associated \( P \) is close to a specified percentage 100\( \alpha \)% (e.g., 90% or 95%) as an approximation to the \((100\alpha) \)th percentile \( \xi_\alpha \) of \( X \). This is in the sense that we speak below about estimating the percentile of \( X \).

To understand the derivation of Jenkinson’s formula and why it might be preferred to the simple estimate \( m/N \), consider the \( N \) probabilities \( y_1 = F(x_1), \ldots, y_N = F(x_N) \). Because \( F \) is nondecreasing,}

\[
y_1 \leq \cdots \leq y_m \leq \cdots \leq y_N.
\]

(2)

We can think of the \( y_i \) as the ordered values of a sample of \( N \) observations. The variable now observed, however, is \( Y = F(X) \) rather than \( X \) itself. This transformed variable \( Y \) has a distribution that does not depend on that of \( X \) at all, provided only that the distribution function \( F \) is continuous. In fact, under this condition, \( Y \) has a uniform distribution over the interval \([0, 1]\), and so has a constant probability density function (equal to 1) over the whole interval. To see this, observe that the distribution function of \( Y \) is

\[
\text{Prob}(Y \leq y) = \text{Prob}[F(X) \leq y] = \text{Prob}[X \leq F^{-1}(y)] = F[F^{-1}(y)] = y,
\]

(3)

where \( F^{-1} \) is the inverse function of \( F \) (see Fig. 1). Differentiation of (3) confirms that the probability density function of \( Y \) is constant and equal to that of the uniform distribution. The conclusion, therefore, is that we can think of the object of interest, \( y_m \), as the observed value of the \( m \)th order statistic from a sample of \( N \) observations uniformly distributed over \([0, 1]\).

In repeated samples, \( y_m \) will vary, but not by as much as a single observation from the uniform distribution since it is constrained by the inequalities in (2). This variation of \( y_m \) is described by its probability density function, which we will denote by \( f_m(y_m) \) (Fig. 2 in...
section 5 below helps to visualize this). We might expect that values of \( y_m \) around \( mN \) would be the most likely, so that the density \( f_m \) would be concentrated in this region, but that densities corresponding to neighboring order statistics, \( y_m \) and \( y_{m+1} \), for example, might overlap. To calculate \( f_m(y_m) \) we can argue as follows. For the \( m \)-th order statistic in a sample of size \( N \) to take a particular numerical value \( y_m \), it is necessary that one of the \( N \) observations should have been equal to \( y_m \), a further \( m - 1 \) of them should have been less than or equal to \( y_m \), and the remaining \( N - m \) should have been greater than \( y_m \). If the individual observations were independent then the number of ways in which they could be ordered for this to happen (the number of ways of putting \( N \) labeled objects into three boxes so that there are \( m - 1 \) in the first, \( 1 \) in the second, and \( N - m \) in the third) is \( N! / [(m - 1)! (N - m)!] \). The probability for any single such ordering is \((y_m)^{m-1}(1 - y_m)^{N-m}\), so the probability of observing the \( m \)-th order statistic equal to \( y_m \) is

\[
f_m(y_m) = \frac{N!}{(m - 1)! (N - m)!} (y_m)^{m-1} (1 - y_m)^{N-m},
\]

\((0 \leq y_m \leq 1)\). \( (4) \)

This is the expression that Jenkinson (1977) uses as his starting point for deriving the formula (1), but without explanation. Equation (4) shows that a beta distribution of the first kind provides the probability density function of any transformed \( m \)-th order statistic \( y_m = F(x_m) \). The distribution does not depend on the underlying distribution of the observation \( x \), provided it is continuous. As shown in the next section, the Jenkinson formula (1) is based on the median of this distribution for different \( m \) and \( N \). The way in which these median values change with \( m \) and \( N \) again is the same for all continuous underlying distributions.

The argument leading to (4) also yields the probability density function, \( g_m \), for example, of the \( m \)-th order statistic \( x_m \) from the original sample. The result is

\[
g_m(x_m) = \frac{N!}{(m - 1)! (N - m)!} f(x_m) F(x_m)^{m-1} \times [1 - F(x_m)]^{N-m}.
\]

\( (5) \)

In contrast to (4), (5) clearly does depend on the underlying distribution of the observations \( x \).

3. Derivation of the Jenkinson ranking formula

Jenkinson’s derivation of Eq. (1) is based on the idea that a natural estimate for the (unknown value of) \( y_m = F(x_m) \) is the median of its distribution, the value with equal probabilities of being exceeded or not exceeded. Let the median value be denoted by \( \lambda \). By definition it satisfies

\[
\frac{N!}{(m - 1)! (N - m)!} \int_{\lambda}^{1} y^{m-1} (1 - y)^{N-m} dy = 0.5.
\]

\( (6) \)

Consider \( m = 1 \), the lowest ranked value, for which Eq. (6) simplifies to

\[
N \int_{\lambda}^{1} (1 - y)^{N-1} dy = 0.5,
\]

whence, expanding in powers of \( 1/N \),

\[
\lambda = 1 - 2^{-1/N} = 1 - e^{-(1/N) \log 2} = (1/N) \log 2 + \cdots.
\]

\( (7a) \)

by a standard result based on Maclaurin’s theorem.

Similarly it can easily be shown that for the highest ranked value, for which \( m = N \),

\[
\lambda = 2^{-1/N} = 1 - (1/N) \log 2 + \cdots.
\]

\( (7b) \)

Now we wish to determine for general \( m \) if \( \lambda \) can be represented well by a modification to the simple intuitive form \( m/N \). The simplest modification to both \( m \) and \( N \) is

\[
\lambda = (m - A)/(N + B),
\]

where \( A \) and \( B \) are constants expected to be small compared to \( N \). For \( m = 1 \), expansion in powers of \( 1/N \) gives

\[
\lambda = \frac{1 - A}{N + B} = \frac{1}{N} (1 - A) + \cdots\quad (7c)
\]

and for \( m = N \),

\[
\lambda = \frac{N - A}{N + B} = 1 - \frac{1}{N} (A + B) + \cdots\quad (7d)
\]

On matching coefficients of \((1/N)\) in Eqs. (7a) and (7c) for the case \( m = 1 \), we find \( 1 - A = \log 2 \), and similarly from Eqs. (7b) and (7d) for \( m = N \), we find \( A + B = \log 2 \). Hence \( A = 0.31 \) and \( B = 0.38 \) very closely. This suggests that the median value of \( y_m = F(x_m) \) will be approximated by \( \lambda \approx (m - 0.31)/(N + 0.38) \), Jenkinson’s formula (1).

4. A test of the accuracy of Jenkinson’s formula

Table 1 compares, for alternate values of \( m \) and for \( N = 30 \), the median values of the beta distribution \( f \) of \( y_m \) with the values given by Jenkinson’s formula (1). The agreement is excellent for all values of \( m \), despite small approximations involved in numerically integrating the beta density. The poorest agreement is around \( m = 5 \) and \( m = 25 \), but the error is tiny. Furthermore, similar calculations for \( N \) between 10 and \( 10^4 \) show that the maximum relative error of the formula over this range is no larger than 0.74%. The exact median of a beta distribution may be easily computed nowadays given suitable software. In S-PLUS, for example, the function qbeta may be used. With access to the software, such direct calculation would be the method of choice, but Jenkinson’s formula evidently gives a simple and accurate alternative.
Table 1. Values of the median of the cumulative probability \( y_m \) given by the Jenkinson formula and as calculated exactly from the beta distribution, for alternate values of \( m \), and \( N = 30 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>Value from beta distribution</th>
<th>Value from Eq. (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0229</td>
<td>0.0227</td>
</tr>
<tr>
<td>3</td>
<td>0.0881</td>
<td>0.0885</td>
</tr>
<tr>
<td>5</td>
<td>0.1540</td>
<td>0.1544</td>
</tr>
<tr>
<td>7</td>
<td>0.2200</td>
<td>0.2202</td>
</tr>
<tr>
<td>9</td>
<td>0.2858</td>
<td>0.2860</td>
</tr>
<tr>
<td>11</td>
<td>0.3518</td>
<td>0.3519</td>
</tr>
<tr>
<td>13</td>
<td>0.4176</td>
<td>0.4177</td>
</tr>
<tr>
<td>15</td>
<td>0.4836</td>
<td>0.4835</td>
</tr>
<tr>
<td>17</td>
<td>0.5496</td>
<td>0.5494</td>
</tr>
<tr>
<td>19</td>
<td>0.6154</td>
<td>0.6152</td>
</tr>
<tr>
<td>21</td>
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<td>0.6810</td>
</tr>
<tr>
<td>23</td>
<td>0.7472</td>
<td>0.7469</td>
</tr>
<tr>
<td>25</td>
<td>0.8132</td>
<td>0.8127</td>
</tr>
<tr>
<td>27</td>
<td>0.8790</td>
<td>0.8785</td>
</tr>
<tr>
<td>29</td>
<td>0.9447</td>
<td>0.9443</td>
</tr>
</tbody>
</table>

5. Characteristics of the \( m \)th-order statistic \( y_m \)

a. Probability density function

We show plots of the probability density function (4) of \( y_m \) for \( N = 30 \) in Fig. 2. This illustrates the idea of “support,” the range of values of \( y_m \) with a strictly positive probability density. As (4) shows, the support of \( f_m \) extends over the full range 0–1, but \( f_m \) is effectively 0 over much of this range.

Figure 2 shows this clearly. [In the same way the support of \( g_m \) is seen from (5) to coincide with that of the underlying probability density function \( f \), but \( g_m \) is effectively zero over much of this range.] Only selected values of \( m \) are shown in Fig. 2. First, for \( m = 1 \) the probability density function peaks at \( y = 0 \) though the median is at \( y = 0.0229 \) [Eq. (1) gives 0.0227]. So the median is distinctly greater than the mode. As \( m \) increases the median tends to the mode, becoming about equal at \( m = 15 \). For larger values of \( m \), the median is smaller than the mode. For a fixed \( N \) the density of the \( m \)th rank is the reflection about 0.5 of the density of the \((N - m + 1)\)th rank. In addition, when \( N \) is odd, the density for the rank corresponding to \( m = (N + 1)/2 \) is itself a symmetric density. The variance of these distributions for \( N = 30 \) is quite large, indicating that the precision with which the rank or percentile is estimated from 30 data is low. It is important to restate that we have worked with \( y_m = F(x_m) \) and not \( x_m \), whose probability density function would depend on the underlying distribution of \( X \).

b. Variance of the \( m \)th-order statistic \( y_m \)

From (4) the mean and variance of \( y_m \) are found to be

\[
E(y_m) = \frac{m}{N + 1} \quad \text{and} \quad \text{(8a)}
\]

\[
\operatorname{var}[F(x_m)] = \operatorname{var}(y_m) = \frac{m(N - m + 1)}{(N + 1)^2(N + 2)}. \quad \text{(8b)}
\]

With \( p = m/(N + 1) = E(y_m) \) we can rewrite Eq. (8b) as

\[
\operatorname{var}(y_m) = \frac{p(1 - p)}{N + 2}. \quad \text{(8c)}
\]

Equation (8c) shows that the dispersion of the distribution of \( y_m \) (as measured by its standard deviation)
Jenkinson’s choice of the median value of dependence in the observations be 100 or more. We now discuss the problem of serial large or small. Although the distribution is least dispersed for very large or small, the 95th percentile value can be estimated by any fixed value such as Jenkinson’s estimate (1) is seen from Fig. 2 to be rather poor for N = 30 [since there is evidently still considerable variability of y_m about (1)]. More specifically, we can measure the precision of (1) as an estimate of y_m by its root-mean-square error \( \text{rmse} (\hat{P}) = \sqrt{\text{mse} (\hat{P})} \), where

\[
\text{mse} (\hat{P}) = E (\hat{P} - y_m)^2 = \text{var}(y_m) + [\hat{P} - E(y_m)]^2,
\]

and compare precision for different N. For example, for estimation of the 95th percentile, for N = 10, this percentile is approximated by y_m with m = 10; for N = 30 by y_m with m = 29; for N = 50 by y_m with m = 48; and for N = 100 by y_m for m = 96. The corresponding rmse values of \( \hat{P} \) are 0.087, 0.044, 0.033, and 0.022. So using a century’s worth of data to estimate the cumulative probability gives an appreciable improvement over 30 years, but of course may not correspond to a stationary climate.

### 6. Discussion

This note explains the Jenkinson empirical ranking formula and shows that it is distribution independent. The formula’s usefulness evidently depends on the value of N in relation to the percentile of interest. For the 95th percentile, for example, as illustrated above, a sample size N = 10 gives too large an mse for practical use. When ranking daily temperature or rainfall, N will often be 100 or more. We now discuss the problem of serial dependence in the observations x, and also ask whether Jenkinson’s choice of the median value of y_m can be fully justified.

#### a. Serial dependence of x

The derivation of (1) assumes independence of the observations of x. To recap, the data we are particularly considering are either ranked averages (e.g., annual) measured yearly at a specific place or ranked values within a specific year. If they were in fact serially dependent, would the use of Eq. (1) be misleading? A small simulation experiment was carried out to investigate the question. Here we had in mind values sampled yearly. Samples of size N = 30 were generated from a stationary Gaussian autoregressive process

\[
x_t = \rho x_{t-1} + e_t,
\]

where the values of e_t are independently normally distributed. Ten thousand samples were generated for each value -0.4 (0.2) 0.4, respectively, of the first serial correlation coefficient, \( \rho \). The mean, median, and standard deviation of y_m = F(x_m) were obtained for m = 1 and m = 15, and the rmse that would result from use of the Jenkinson estimate (1) in these conditions was calculated. By symmetry the results for m = 30 would be similar to those for m = 1. Table 2 shows the results.

The mean and the median of the extreme lowest-order statistic \( (m = 1) \) are affected by serial dependence in the underlying distribution of x, though not grossly, but the central-order statistic \( (m = 15) \) is hardly affected at all. These central values of x are most likely to be consecutively placed in the time series than the extremes, and therefore more subject to persistence. Compared to an independent series, the standard deviations rise as positive serial correlations increase for both \( m = 1 \) and \( m = 15 \), and fall more slowly as serial correlation becomes more negative. The case \( m = 30 \) will be similar to that for \( m = 1 \). The rmse of \( \hat{P} \) as an estimator of y_m changes in the same way with \( \rho \) [indeed for \( m = 15 \) it is indistinguishable from standard deviation since the mean of y_m remains so close to (1)]. A broad conclusion is that for the small values of \( \rho \) likely to be encountered among high or low extremes in annually or seasonally averaged data, the use of (1) for estimation of y_m is unlikely to be systematically misleading and will affect precision of estimation no more than serial correlation changes the dispersion of the distribution of y_m.

Finally, one might note that if observations are believed to be autocorrelated and a suitable time series model can be identified, then it could be used for further analysis. One could, for example, estimate the quantiles of a future observation using the model, and test whether one or more subsequent observations show deviations from the model.

#### b. Comparison of Jenkinson’s ranking formula with other empirical ranking formulas

Numerous ranking formulas other than Jenkinson’s have been suggested: see Blom (1958). Many have been

| \( \rho \) | Mean | Median | Std dev | \( \text{rmse} (\hat{P}) \) | Mean | Median | Std dev | \( \text{rmse} (\hat{P}) \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.4 | 0.039 | 0.027 | 0.041 | 0.044 | 0.484 | 0.483 | 0.117 | 0.117 |
| 0.2 | 0.035 | 0.024 | 0.035 | 0.037 | 0.484 | 0.483 | 0.100 | 0.100 |
| -0.2 | 0.031 | 0.022 | 0.030 | 0.031 | 0.483 | 0.483 | 0.088 | 0.088 |
| -0.4 | 0.032 | 0.022 | 0.030 | 0.031 | 0.482 | 0.482 | 0.072 | 0.072 |

**Table 2.** Influence of serial dependence on characteristics of \( F(x_m) \) and its estimation by \( \hat{P} \) for \( m = 1 \) and \( N = 30 \).
derived for specific distributions in order that empirically ranked data could be plotted on special types of paper to best estimate the parameters of an underlying distribution of \( x \), for example, the parameters of the extreme value or gamma distributions or the standard deviation of the normal distribution. This is not the intended application of the ranked data here. We compare four ranking formulas with Jenkinson’s formula, in terms of estimation of the cumulative probability \( P = y_m = F(\hat{x}_m) \):

1) We first show a ranking formula (the average ranking method) that is equivalent to Eq. (1), but based on the mean of \( y_m \) instead of its median. From the expression for the mean in (8a):

\[
\hat{P} = \frac{m}{N + 1},
\]

which agrees with the mean values in Table 2. This is an early ranking formula discussed further by Blom (1958), but little used now (see below). It is very close to the simple estimate \( \hat{P} = m/N \) mentioned in section 2.

2) Blom (1958) shows that for an underlying normal distribution, an optimal unbiased estimate of the standard deviation \( \sigma \) may be obtained by suitably plotting ranked data using

\[
\hat{P} = \frac{m - 0.375}{N + 0.25}.
\]

The resulting estimate of \( \sigma \) has an mse no worse than that of the best linear unbiased estimate. NERC (1975a) gives similar “near optimal” ranking rules for the gamma and the exponential distributions, respectively (the latter being possibly suitable for the analysis of annual extremes).

3) Gamma distribution:

\[
\hat{P} = \frac{m - 0.40}{N + 0.20}.
\]

4) Exponential distribution:

\[
\hat{P} = \frac{m - 0.44}{N + 0.12}.
\]

These formulas are compared in Fig. 3. In order to clearly show the differences between the methods, the cumulative percentages have been converted to the return period \( R \) corresponding to a value of \( x \) as great as, or greater than the given order statistic:

\[
R = \frac{1}{1 - \hat{P}}.
\]

A similar graph could be shown for the return period corresponding to the value of \( x \) that is as small as, or smaller than, that value at a given rank. This would give symmetrical results. For most values of \( m \), all formulas give similar estimates of return period or cumulative percentage. For the largest values of \( m \), the average ranking method gives estimates of the percentage or return period that are smaller than the other methods. Hence it is not recommended. Jenkinson’s method gives a fairly similar estimate of \( R \) to that for a normal or gamma distribution, though it is a little further from that recommended for an underlying exponential distribution, noticeably for \( m = 30 \).

7. Conclusions

Jenkinson’s ranking method is likely to be satisfactory for consistently ranking time series of most climatological data when calculating changes of moderate extremes in the form of percentiles, even though it makes no assumptions about underlying distributions. It should be seen as a method for surveying such data and providing initial, though credible, results before the more difficult problem of fitting a parametric extreme value model is tackled. The method gives an estimate of the cumulative probability or return period independently of the underlying distribution of \( x \). Naturally, if the form of the underlying distribution were known, then it could be fitted, for example, by maximum likelihood methods, and estimates of quantiles could be obtained from the fitted distribution. Caution, however, would be needed with this approach, particularly for extreme quantiles, since knowledge of distributional form can rarely be exact at the extremes.

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REFERENCES

