Estimating Extremes in Transient Climate Change Simulations

VIATCHESLAV V. KHARIN AND FRANCIS W. ZWIERS

Canadian Centre for Climate Modelling and Analysis, Meteorological Service of Canada, Victoria, British Columbia, Canada

(Manuscript received 6 January 2004, in final form 13 September 2004)

ABSTRACT

Changes in temperature and precipitation extremes are examined in transient climate change simulations performed with the second-generation coupled global climate model of the Canadian Centre for Climate Modelling and Analysis. Three-member ensembles were produced for the time period 1990–2100 using the IS92a, A2, and B2 emission scenarios of the Intergovernmental Panel on Climate Change. The return values of annual extremes are estimated from a fitted generalized extreme value distribution with time-dependent location and scale parameters by the method of maximum likelihood. The L-moment return value estimates are revisited and found to be somewhat biased in the context of transient climate change simulations.

The climate response is of similar magnitude in the integrations with the IS92a and A2 emission scenarios but more modest for the B2 scenario. Changes in temperature extremes are largely associated with changes in the location of the distribution of annual extremes without substantial changes in its shape over most of the globe. Exceptions are regions where land and ocean surface properties change drastically, such as the regions that experience sea ice and snow cover retreat. Globally averaged changes in warm extremes are comparable to the corresponding changes in annual mean daily maximum temperature, while globally averaged cold extremes warm up faster than annual mean daily minimum temperature. There are considerable regional differences between the magnitudes of changes in temperature extremes and the corresponding annual means. Changes in precipitation extremes are due to changes in both the location and scale of the extreme value distribution and exceed substantially the corresponding changes in the annual mean precipitation. Generally speaking, the warmer model climate becomes wetter and hydrologically more variable. The probability of precipitation events that are considered extreme at the beginning of the simulations is increased by a factor of about 2 by the end of the twenty-first century.

1. Introduction

Weather and climatic extremes are significant departures from a normal state of the climate and may have significant societal and ecological impacts. An important question is whether climate changes caused by anthropogenic activities will change the intensity and frequency of extreme events. Possible changes in extreme value statistics have been a subject of several recent studies (e.g., Hennessy et al. 1997; Meehl et al. 2000; McGuffie et al. 1999; Durman et al. 2001). Several methods for detecting a trend in the magnitude of extreme values are compared using Monte Carlo simulations in Zhang et al. (2004). The primary tools to study possible future climate changes are global coupled models and the transient climate change simulations that are obtained when those models are run with projected anthropogenic radiative forcings (Houghton et al. 2001).

In a previous study, Kharin and Zwiers (2000, hereafter KZ2000) examined changes in extremes of several near-surface climate parameters as simulated by the first version of the Canadian Centre for Climate Modelling and Analysis (CCCma) coupled global climate model (CGCM1). Extreme values were described in terms of return values of annual extremes. These are values that are exceeded, on average, once every specified number of years. KZ2000 computed the return values from a generalized extreme value (GEV) distribution fitted to a sample of annual extremes by the method of L moments (Hosking 1990, 1992). This method assumes that annual extremes are identically distributed. This is a fair assumption in equilibrium climate simulations (e.g., such as in Zwiers and Kharin 1998) but it is true only approximately in transient simulations when the climate responds to gradual changes in radiative forcing. Return values in KZ2000 were estimated from 21-yr time slices using the assumption that climate statistics do not change substantially within a given 21-yr period. However, the possibility
that a climate trend within the 21-yr periods might have an affect on the estimated return values was not fully explored.

The purpose of the present study is twofold. First, integrations with the second version of the CCCma coupled global climate model (CGCM2) for several scenarios of projected changes in the radiative forcing are now available. The updated model includes a change in the ocean mixing parameterization from horizontal/vertical diffusion scheme to the isopycnal/eddy stirring parameterization of Gent and McWilliams (1990). It also includes Flato and Hibler (1992) sea ice dynamics. The differences between CGCM1 and CGCM2 in the mean climate response simulated under global warming are discussed in Flato and Boer (2001).

The present study examines changes in the extremes of near-surface air temperature and precipitation simulated in these new integrations. The CGCM2 monthly and daily output is available online from the CCCma Web site (http://www.cccma.ec.gc.ca) and is disseminated widely in the international climate change and climate impacts research communities, providing an additional motivation for documenting an important aspect of the behavior of an extensively used model.

Second, we revisit the method of L moments in the context of slowly changing climate and examine its sensitivity to deviations from the stationarity assumption. This reevaluation prompted us to consider the method of maximum likelihood for fitting a GEV distribution with time-dependent parameters. The latter method possesses a number of attractive properties and appears to be more suitable for situations when the rate of climate change within a sample cannot be ignored.

The outline of the paper is as follows. First, we give a brief description of the coupled model and the climate change integrations in section 2. The maximum likelihood methodology for estimating extreme values is introduced and discussed in section 3. The performance of the L-moment and maximum likelihood methods is evaluated in section 4. Simulated changes in the distribution parameters and in the corresponding return values and periods are presented in sections 5 and 6, respectively. The results are briefly summarized in section 7.

2. Transient climate change simulations

CGCM2 is based on the earlier CGCM1 (Flato et al. 2000; Boer et al. 2000a, b) but with some improvements aimed at addressing shortcomings identified in the first version. In particular, the ocean mixing parameterization has been changed from a horizontal/vertical diffusion scheme to the isopycnal/eddy stirring parameterization of Gent and McWilliams (1990). Sea ice dynamics have also been included following Flato and Hibler (1992). The atmosphere component (McFarlane et al. 1992) is essentially the same as in the previous model.

A description of CGCM2 and a comparison, relative to CGCM1, of its response to increasing greenhouse gas (GHG) forcing can be found in Flato and Boer (2001). CGCM2 was also used in a number of simulations of the last glacial maximum (Kim et al. 2002a, b).

The net radiative effect of all greenhouse gases is represented in CGCM2 by means of an “equivalent” CO$_2$ concentration. The direct effect of sulfate aerosols is included by increasing the surface albedo as in Reader and Boer (1998). In transient climate change simulations, the change in greenhouse gas forcing is represented in the model as a perturbation relative to the equivalent CO$_2$ concentration used in the control simulation. Equivalent CO$_2$ concentrations from 1850 to 1990 are based on historical changes in greenhouse gas forcing, provided by the Hadley Centre. From 1990 onward three scenarios have been employed.

- The Intergovernmental Panel on Climate Change (IPCC) IS92a scenario (Leggett et al. 1992) specifies equivalent GHG concentrations increasing at 1% per year after 1990 and sulfate aerosol loadings from 1850 to 2100.
- The IPCC A2 scenario (Nakienovic and Swart 2001) envisions population growth to 15 billion by the year 2100 and rather slow economic and technological development. It projects slightly lower GHG emissions than the IS92a scenario, but also slightly lower aerosol loadings, such that the warming response differs little from that of the earlier scenario.
- The IPCC B2 scenario (Nakienovic and Swart 2001) envisions slower population growth (10.4 billion by 2100) with a more rapidly evolving economy and more emphasis on environmental protection. It therefore produces lower emissions and less future warming.

For each scenario, an ensemble of three integrations has been produced for the period from 1990 to 2100. The only difference between the ensemble members is that they are initiated from different initial conditions. The ensemble members are treated as independent climate realizations.

CGCM1 and CGCM2 have similar global mean temperature responses when forced with the IS92a scenario. The most notable difference in the pattern of the response between the two models is relatively larger warming in southern middle and higher latitudes in CGCM2 producing a more meridionally symmetric warming pattern, which is a consequence of the change in the ocean mixing parameterization (Flato and Boer 2001).

3. Methodology

As in KZ2000, we describe extreme events in terms of return values of annual extremes. The T-yr return value, or T-yr return level, is defined here as a thresh-
old that is exceeded by an annual extreme in any one year with probability \( P_r = 1/T \). The time \( T \) is often referred to as to the return period and is related to the exceedance probability as \( T = 1/P_r \). In this study, we analyze annual extremes of daily maximum and daily minimum near-surface (2 m) air temperature, denoted as \( T_{\text{max}} \) and \( T_{\text{min}} \), and of 24-h precipitation rates \((P)\).

For each quantity, we estimate 10-, 20-, and 50-yr return values at every grid point and for each year in the 1990–2100 simulation period.

The procedure for estimating return values is similar to that in KZ2000; that is, the return values are estimated in a two-step procedure. First, a generalized extreme value distribution is fitted to a sample of annual extremes. Then, return values are obtained by inverting the fitted GEV distribution. The main difference from the previous study is that the GEV parameters are treated as time-dependent quantities so that the transient nature of the climate change simulations is reflected in the estimating procedure.

In the present study, we assume that annual extremes of temperature and precipitation are distributed according to a GEV distribution with time-dependent parameters \( \xi_t, \alpha_t, \) and \( \kappa_t \) given by

\[
F(x; \xi_t, \alpha_t, \kappa_t) = \begin{cases} 
\exp \left[ -\exp \left( -\frac{x - \xi_t}{\alpha_t} \right) \right], & \kappa_t = 0, \\
\exp \left[ -\left( 1 - \frac{x - \xi_t}{\alpha_t} \right)^{1/\kappa_t} \right], & \kappa_t \neq 0, \quad 1 - \frac{x - \xi_t}{\alpha_t} > 0. 
\end{cases}
\]

Here \( \xi_t \) is the location parameter, \( \alpha_t \) is the (positive) scale parameter, and \( \kappa_t \) is the shape parameter. Note that the current practice in the statistical literature is to represent the shape parameter as \( \gamma_t = -\kappa_t \). Here, we keep the convention adopted by KZ2000, which is still common in the hydrology literature.

The GEV distribution is heavy tailed and its probability density function decreases at a slow rate when the shape parameter \( \kappa \) is negative. The GEV distribution has a finite upper tail for \( \kappa > 0 \). The Gumbel distribution \( (\kappa = 0) \) is obtained by taking the limit as \( \kappa \to 0 \). The “three kinds theorem” (Fisher and Tippett 1928; Gnedenko 1943; Gumbel 1958) states that these three types of extreme value distributions, Fréchet for \( \kappa < 0 \), Weibull for \( \kappa > 0 \), and Gumbel for \( \kappa = 0 \), are the only types that can arise as a limiting distribution of extremes in random samples. More recent treatments of the subject are given in Leadbetter et al. (1983) or Coles (2001). A review of recent developments in the statistical theory of extreme values and their applications in hydrology is given in Katz et al. (2002).

Note that the three extreme value distributions are asymptotic distributions, that is, they are obtained as the limits taken over samples of increasing size. The annual extreme is the maximum (or minimum) of a sample of size (typically) 365 daily observations. Serial correlation and the presence of an annual cycle may reduce the effective annual sample size to a number substantially smaller than 365.

Also, the rate of convergence to the asymptotic distribution depends on the underlying parent distribution from which annual extremes are drawn and may be very slow for some distribution types such as the normal distribution (e.g., Leadbetter et al. 1983). Thus, goodness-of-fit tests should be used to evaluate whether the GEV distribution provides a reasonable description of the behavior of a sample of observed annual extremes (KZ2000).

The GEV parameters are assumed to vary with time as:

\[
\begin{align*}
\hat{\xi}_t &= \xi_0 + \xi_1(t - t_0), \\
\ln \hat{\alpha}_t &= \ln \alpha_0 + \alpha_1(t - t_0), \\
\hat{\kappa}_t &= \kappa_0 + \kappa_1(t - t_0).
\end{align*}
\]

The intercept coefficients \( \xi_0, \alpha_0, \) and \( \kappa_0 \) are the parameter values at year \( t_0 \). The slope coefficients \( \xi_1, \alpha_1, \) and \( \kappa_1 \) characterize the rate of change in the GEV parameters. A log-linear trend in the scale parameter ensures that it remains positive. The trend in the scale parameter \( \alpha_t \) is approximately linear in the vicinity of \( t_0 \): that is,

\[\alpha_t = \alpha_0 [1 + \alpha_1(t - t_0)] + O[(t - t_0)^2].\]

A recent application of generalized extreme value analysis with time-dependent covariates can be found in Wang et al. (2004).

Having fitted the GEV distribution to a sample of annual extremes, the \( T \)-yr return values \( X_T \) for each year are estimated from the quantile function (inverse of the cumulative distribution function) as

\[
X_T(t) = \begin{cases} 
\hat{\xi}_t + \frac{\hat{\alpha}_t}{\hat{\kappa}_t} \left[ 1 - \left( -\ln \left( 1 - \frac{1}{T} \right) \right)^{\hat{\kappa}_t} \right], & \hat{\kappa}_t \neq 0, \\
\hat{\xi}_t - \hat{\alpha}_t \ln \left[ -\ln \left( 1 - \frac{1}{T} \right) \right], & \hat{\kappa}_t = 0.
\end{cases}
\]

Because the L moments are only defined for identically distributed random samples, the use of time-dependent GEV distribution parameters automatically precludes us from using the method of L moments for their estimation. Instead, the parameters (2) are estimated by the method of maximum likelihood (ML; see the appendix for details). The ML method (e.g., Cox and Hinkley 1974) possesses a number of attractive
properties such as asymptotic unbiasedness and efficiency. It is numerically somewhat more complex than the method of L moments as it involves a nonlinear optimization problem. However the latter does not represent a serious computational obstacle. A more serious objection to the ML method is that its performance for the GEV distribution may be erratic for small sample sizes \( N \) (e.g., \( N \leq 25 \) annual extremes for a GEV distribution with constant parameters), in which case it may occasionally result in rather unrealistic shape parameter estimates (e.g., Martins and Stedinger 2000). Despite potential pitfalls that occur only rarely with environmental data, the ML method is a very general and powerful technique for estimating distributional parameters. Some of the small-sample problems can be eliminated by incorporating information about the parameters that is supplementary to that provided by the data. For example, Martins and Stedinger apply a generalized maximum likelihood analysis to improve the small-sample efficiency of quantile estimators by specifying a prior distribution to restrict the shape parameter values to a statistically/physically reasonable range within a Bayesian framework. Coles and Dixon (1999) achieve the same goal by using the penalized maximum likelihood estimator. Sample sizes in the present study were large enough that the aforementioned small-sample problems did not occur.

Approximate standard errors of the estimated parameters can be obtained in the ML method by using the inverse of the observed information matrix, or through the profile likelihood (e.g., Coles 2001). However, these estimates can be quite unreliable for small sample sizes. “Resampling” approaches for estimating parameter uncertainties have become increasingly popular (Efron and Tibshirani 1993; Davison and Hinkley 1997). There is some evidence that resampling methods for extremes may produce confidence intervals that are too narrow (Dupuis and Field 1998). Nevertheless, the resampling techniques provide an idea of the nature of the uncertainty in the return values that results from sampling errors and internal variability as simulated by the model. The reader should be aware that none of the above uncertainty estimates take into account other sources of uncertainty such as “model uncertainty,” “forcing scenario uncertainty,” etc.

In the present study, confidence intervals for the return value estimates are obtained by a “bootstrap” technique in which “new” samples of annual extremes are generated from the original sample by random resampling with replacement. These new samples are used to reestimate the GEV distribution parameters and the corresponding return values. The 80% confidence interval is obtained as the lower and upper 10% percentiles of the resulting collection of return values. Since the GEV distribution parameters are treated as time-dependent quantities, the original samples must be transformed to samples of identically distributed random numbers before resampling. This is achieved by computing the residuals \( y_i, i = 1, \ldots, N \) as

\[
y_i = \begin{cases} 
-\frac{1}{\kappa_i} \ln \left( 1 - \frac{x_i - \xi_i}{\alpha_i} \right), & \kappa_i \neq 0, \\
\frac{x_i - \xi_i}{\alpha_i}, & \kappa_i = 0, 
\end{cases}
\]

where \( t_i \) is the time at which \( x_i \) is observed. The variable \( y_i \) is identically distributed according to the Gumbel distribution with the zero location parameter and unit scale parameter. New samples are generated by resampling the residuals \( y_i \) with replacement and converting them back as

\[
x_i = \begin{cases} 
\xi_i + \frac{\alpha_i}{\kappa_i} \left( 1 - \exp\left(-\kappa_i y_i\right) \right), & \kappa_i \neq 0, \\
\xi_i + \alpha_i y_i, & \kappa_i = 0.
\end{cases}
\]

In the following, the difference between two return value estimates is said to be statistically significant when their corresponding 80% confidence intervals do not overlap, which corresponds, approximately, to a 10% statistical significance level. The significance level was determined empirically by performing Monte Carlo simulations. This level is somewhat larger than the asymptotic rate of about 7% that holds when 80% confidence intervals derived from two samples of normally and identically distributed variables do not overlap.

4. Maximum likelihood method versus L-moment method

Hosking et al. (1985) and Hosking (1990) demonstrated that the method of L moments produces more efficient estimates of return values than the method of maximum likelihood when samples are small. His results were obtained for samples of identically distributed random variables. In this section, we examine sampling properties of return values estimated by the maximum likelihood and L-moment methods in the setting that approximates the typical transient behavior of the time series encountered in the CGCM2 climate change simulations. In particular, we are interested in the biases in estimated return values when the method of L moments is used.

As an example, we consider the ML estimates of time-dependent GEV parameters of daily temperature and precipitation annual extremes for the CGCM2 A2 integrations in the last half of the twenty-first century when the rate of change is large. We assume that all three GEV parameters have trends as described in Eq. (2). Typical parameter values are obtained by averaging their estimates over the oceans and landmasses separately and are summarized in Table 1. The intercept coefficients \( \xi_0, \alpha_0, \) and \( \kappa_0 \) are the spatially averaged...
parameter estimates for year $t_0 = 2075$. The location ($\xi_0$) and scale ($\alpha_0$) intercept parameters are expressed in the units of the corresponding variables (that is, °C for temperature extremes and mm day$^{-1}$ for precipitation extremes). The shape parameter $\kappa_0$ is dimensionless. The slope coefficients $\alpha_1$ and $\kappa_1$ are expressed in units of per year. For intercomparison purposes, the location slope coefficient $\xi_1$ in Table 1 is normalized by the corresponding scale parameter $\alpha_0$, and is also expressed in per year.

The interannual variability of annual temperature extremes, as characterized by the scale parameter $\alpha$, is generally smaller over the oceans than over land, amounting to more than 10% yr$^{-1}$ on average over the oceans as compared to about 4% yr$^{-1}$ over land. There is virtually no trend in the scale and shape parameters for temperature extremes, at least not in the spatially averaged parameters. Trends in precipitation extremes are substantially weaker, a few tenths of a percent per year, both for the location and scale parameters.

For each set of parameters in Table 1, Monte Carlo simulations are performed by generating $3 \times 1000$ random samples of different lengths centered at time $t_0$ from the corresponding GEV distributions. Every set of three samples (which emulate the three members in the CGCM2 ensembles) is used to estimate GEV parameters by the maximum likelihood and L-moment methods. Taking into account that there is virtually no trend in the shape parameter, it is assumed to be constant in the ML fitting algorithm. The estimated parameters are used to calculate 10-, 20-, and 50-yr return values for the year $t_0 = 2075$. The resulting collections of return values are used to compute mean return values and their confidence intervals.

Some results of the Monte Carlo simulations are illustrated in Fig. 1. The upper two panels are for $T_{\text{max}}$ extremes using the parameters averaged over the oceans. The lower two panels are for precipitation extremes over the oceans. The L-moment return value estimates are displayed in the left panels, and the ML estimates are on the right. The x axes indicate the length of samples centered at year $t_0$, ranging from 11 to 61 years. Thus, the analyses are based on combined ensembles of samples containing from 33 to 183 simulated annual extremes. Thick solid lines indicate the means of the estimated return values at time $t_0$. The true return values are indicated by horizontal dashed lines. Shaded bands indicate 80% confidence intervals for the true return values. The results for $T_{\text{min}}$ (not shown) are very similar to those for $T_{\text{max}}$, except that the biases are of the opposite sign.

To summarize the findings in Fig. 1:

- The return values of temperature extremes obtained by the method of L moments for the year in the center of the sampling windows are biased (overestimated), even for relatively short windows. For example, the mean bias for the 21-yr time window is so large that the estimated mean 20-yr return value is nearly equal to the true 50-yr return value. Clearly, the L-moment return value estimates suffer considerably from the violation of the stationarity requirement in the presence of such trends. Reducing the time window length does reduce the bias, but this is hardly advisable because the resulting return value estimates become very uncertain as indicated by a very wide confidence interval.

- Trends in precipitation extremes are much smaller and thus do not substantially affect the L-moment return value estimates in the middle of the considered time periods. However, the estimated extremes are biased when regarded as estimates of return values at the ends of the sampling windows. The solid points in the lower-left panel in Fig. 1 indicate the true 20-yr return values at the sample ends. For example, given a 51-yr window, say, for years 2050–2100, with properties similar to those of the precipitation annual extremes considered here, the L-moment return value estimates are nearly unbiased for the year 2075 but negatively biased for the year 2050 and positively biased for 2100. The ML method with time-varying GEV parameters largely avoids such biases.

- As expected, the ML return value estimates exhibit only modest biases. Because considerably larger samples can be used without introducing large biases, the sampling variability of return value estimates is reduced despite the fact that the number of unknown parameters that must be estimated from the data is increased. For example, 10-, 20-, and 50-yr return values estimated by the ML method from 51-yr windows in the middle of the sampling window are less biased and more reliably distinguished from each other than those estimated by the method of L moments for 21-yr samples. Note though that the uncertainty of the ML return value estimates is larger at the ends of sampling windows because these estimates are more sensitive to sampling errors in the slope coefficients than the estimates for times in the middle of the sampling windows.
Overall, the ML method appears to be a viable, and often preferred, alternative for estimating extreme values in transient climate change simulations.

5. Changes in the GEV distribution parameters

In selecting an appropriate model for the time dependence of the GEV parameters a decision must be made about whether all, or only some, parameters should be treated as varying in time. In principle, trends can be assumed in all three parameters $\xi$, $\alpha$, and $\kappa$, as in Eq. (2). However, if some of the parameters change only a little in the considered sample, it may be advantageous to reduce the number of free parameters by keeping these parameters constant. Generally speaking, when all assumptions are valid, the smaller the number of unknowns to be estimated from a sample, the more precise the estimates.

Here we further examine changes in the GEV distribution parameters as simulated in the CGCM2 A2 integrations in more details to facilitate our decision about the time dependence model for the GEV parameters. Examining changes in the distributional parameters may have merits on its own and offer some insights into the nature of the changes in the extremes that will be discussed in the next section. We are primarily interested in the “typical” temporal behavior of the parameters, which we grossly characterize by spatially averaging quantities over the oceans and landmasses. To allow for possible nonlinear changes in the 1990–2100 period covered by the integrations, the parameters are estimated from shorter, 51-yr moving time windows. Given the three-member ensembles (i.e., three annual extremes for each year), the total sample size for each 51-yr window is 51 × 3 = 153. The estimates for the year in the center of the moving time windows are used as the GEV parameter estimates for the corresponding year. The parameters for the first and last 25 years of the whole 1990–2100 record are obtained by extrapolating the GEV parameters estimated from the corresponding 51-yr windows according to Eq. (2).

Figure 2 displays the time evolution of the spatially averaged GEV parameters estimated using the ML method and assuming trends in all three parameters as in Eq. (2). The range in the location parameter (solid
lines) and scale parameter (dashed lines) is indicated on the left-hand vertical axes. The shape parameter (dotted lines) range is indicated on the right-hand vertical axes. The location parameter \( \xi \) is offset by its 1990 value (\( \xi_{1990} \)). To summarize the results in Fig. 2:

- Not unexpectedly, the location parameter representing the overall position of the GEV distribution of annual temperature extremes exhibits the largest change (top and middle panels). The response to the anomalous forcing imposed in the A2 integration is generally larger over land than over the oceans and is larger for \( T_{\text{min}} \) than for \( T_{\text{max}} \).
- For temperature extremes, the land- and ocean-averaged scale parameter \( \alpha \), and shape parameter \( \kappa \), remain virtually constant over the whole integration period. The scale parameter, which is a measure of interannual variability of annual temperature extremes, is smaller over the oceans where near-surface temperature is moderated by the ocean surface.
- The magnitude of the trend in the location parameter for precipitation extremes, as measured in terms of the corresponding scale parameter, is considerably smaller than that for the temperature extremes. Another important feature is that there is also a noticeable increasing trend in the scale parameter that is comparable to the trend in the location parameter. The shape parameter decreases slightly over land by the end of the twenty-first century, indicating that the upper tail of the distribution of annual precipitation extremes becomes slightly heavier.

The above findings indicate that global changes in temperature extremes are associated primarily with the
overall shift of the extreme value distribution of annual extremes toward a warmer mean climate with comparatively small changes in the shape of the distribution. However, there are areas with substantial local changes in the scale parameter (see next section). These are regions where land and ocean surface properties change dramatically under global warming. The latter is frequently associated with the retreat of sea ice in the oceans and snow cover over land. For precipitation extremes, there are comparable changes both in the location and scale parameters and some modest changes in the shape parameter, indicating that both the position and the shape of the distribution change.

To illustrate these points, Fig. 3 displays GEV distribution density functions for the ocean- and land-averaged parameters for the years 2000 (solid lines) and 2090 (dashed lines). The shift toward warmer temperatures in the distributions of temperature extremes, as compared to the width of the distributions, is quite substantial. The changes in the corresponding 20-yr return values, as indicated by the arrows on the x axis, are approximately equal to the changes in the location parameter.

![Fig. 3. GEV probability density functions obtained when parameters are set to average estimates over the (left) ocean and (right) land for annual extremes of (top) daily max surface temperature \( T_{\text{max}} \), (middle) daily minimum surface temperature \( T_{\text{min}} \), and (bottom) 24-h precipitation rate \( P \). Parameter estimates were obtained by fitting the GEV distribution to annual extremes simulated by CGCM2 for the A2 IPCC climate change scenario. Solid (dashed) curves display the estimated density functions in year 2000 (2090). The corresponding 20-yr return values are indicated by the arrows on the x axis.](image-url)
rameter indicated by the vertical dashed lines. There is a slight diminishing trend in the scale parameter for $T_{\min}$ over the oceans, which results in a narrower GEV distribution at the end of the integrations. This is mainly attributed to the reduction of sea ice cover under global warming. The atmosphere becomes exposed to open waters in these regions, which reduces the interannual variability of annual cold temperature extremes. There is also a slight increase in the variability of annual warm extremes over land. Global warming in CGCM2 is associated with a reduction of soil moisture content in some parts of the world so that daily maximum temperature extremes are less moderated by evaporative cooling from the land surface. The extreme precipitation distribution shifts modestly toward somewhat wetter and more variable annual extreme values. The corresponding changes in 20-yr return values are larger than those implied by the changes in the location parameter alone. The increase in the interannual variability of annual precipitation extremes also contributes to changes in size of long return period return values.

Competing sets of assumptions about $\xi$, $\alpha$, and $\kappa$, can be evaluated by performing likelihood ratio tests (see the appendix for details). In particular, it was found that a statistical model with a constant shape parameter is statistically indistinguishable from a model in which this parameter is treated as a linear function of time for all considered quantities. The likelihood ratio tests indicate that it is beneficial to treat the location and scale parameters for temperature and precipitation extremes as time-varying functions because the statistical models in which these parameters are assumed to be constant are rejected in many parts of the world. It is particularly important to treat both the location and scale parameters as time-varying functions for precipitation extremes because the changes in these parameters generally have comparable effects on the corresponding changes in return values.

The analysis described above was repeated with the IS92a and B2 integrations. Similar results were obtained, with changes in the IS92a ensemble being comparable to those in the A2 ensemble and more moderate changes in the B2 ensemble. In all cases, it appears reasonable to assume, for all quantities, that the shape parameter is constant within any one 51-yr sampling window. Therefore, the results presented in the following section are obtained assuming a linear trend in the location parameter, a log-linear trend in the scale parameter, and no trend in the shape parameter.

6. Changes in return values and return periods

Figure 4 displays the temporal evolution of 10-, 20-, and 50-yr return values of $T_{\max}$, $T_{\min}$, and $P$ spatially averaged over the oceans (left) and over landmasses (right) as simulated by CGCM2 for the IS92a (red lines), A2 (green lines), and B2 (blue lines) emission scenarios. Dashed lines indicate the corresponding changes in return values.

Changes in the A2 and IS92a integrations are comparable and tend to accelerate in the second half of the twenty-first century. Changes in the B2 runs are more moderate and more linear in time. The variability of annual temperature extremes is lower over the oceans than over the land. Thus the fitted extreme value distributions have narrower upper tails over the oceans. Consequently, return value estimates for annual temperature extremes over oceans increase only modestly as the return period increases from 10 yr to 20 and 50 yr. Changes in cold extremes are notably greater than changes in warm extremes. Near-surface air over landmasses warms faster than that over the oceans. Also, although not quite obvious from the graphs, changes in cold extremes are somewhat greater than the corresponding changes in the $T_{\min}$ annual mean, while changes in warm extremes are slightly smaller than the corresponding changes in the annual mean of $T_{\max}$. These results are in agreement with findings in previous studies (e.g., KZ2000).

Changes in extreme precipitation are larger than the corresponding changes in annual mean precipitation, in both absolute and relative terms. For example, the globally averaged mean precipitation rate increases by about 0.07 mm day$^{-1}$ (less than 3%) in the A2 simulations by the end of the twenty-first century. The corresponding increase in the globally averaged 20-yr return values of annual precipitation extremes is greater than 10 mm day$^{-1}$ (more than 12%). When plotted on the same scale as the return values, the annual mean appears to be nearly constant. Figure 5 displays the corresponding changes in the spatially averaged probability, $Pr$, of extreme precipitation events exceeding year 2000 10-, 20-, and 50-yr events, expressed in terms of the corresponding return periods $T = 1/Pr$. The return periods are reduced by a factor of 2 or more by the end of the twenty-first century.

Local changes in 20-yr return values of $T_{\min}$ and $T_{\max}$ annual extremes and the corresponding changes in the GEV location and scale parameters as simulated by CGCM2 in the A2 scenario integrations in year 2050 relative to year 2000 are displayed in Fig. 6. It is clearly evident that most of the changes in return values are due to changes in the location of the distribution of annual extremes. The largest changes in $T_{\min,20}$ occur over high-latitude ocean areas where sea ice has retreated and over land areas with reduced snow cover. The biggest changes in $T_{\max,20}$ occur inland in areas with reduced soil moisture. All of these results are similar to those obtained with CGCM1 with IS92a forcing (KZ2000). The pattern of the response in extreme temperatures is more symmetric about the equator in CGCM2 than that in CGCM1 (not shown), which is in accordance with the findings by Flato and Boer (2001) for the mean climate response.
Changes in the interannual variability of annual temperature extremes as indicated by the changes in the GEV scale parameter (Fig. 6, bottom) are largely associated with changes of the underlying surface in the warmer climate. In particular, the variability of $T_{\text{min}}$ extremes is greatly reduced in areas that are originally covered with snow and sea ice but that become snow and sea ice free in the warmer world. On the other hand, the variability of annual cold extremes is increased in some areas adjacent to the sea ice and snow cover lines because these areas are sea ice/snow covered in some years and snow free in other years. The increased variability of warm extremes in some regions over land is likely associated with a reduction of soil moisture content. Changes in the shape parameter of the GEV distribution of temperature extremes (not shown) are very noisy without clear large-scale structure and are not statistically significant.

The pattern of changes in estimated precipitation extremes suffers from a considerable amount of noisy sampling variability. The GEV parameter estimates for precipitation were therefore smoothed spatially using a simple procedure similar to that used in KZ2000; that is, the estimated GEV parameters at each grid point were replaced with the corresponding averages calculated from that grid point and its eight surrounding neighbors. These “regional” parameters were then used to calculate regional return values.

Figure 7 shows changes in the resulting 20-yr return values of precipitation ($P_{20}$) simulated in the CGCM2 A2 runs in year 2050 relative to 2000, the corresponding changes in the GEV location and scale parameters, and
the probability of precipitation events that are equal to, or larger than, year 2000 20-yr events, expressed in terms of the return periods $T = 1/Pr$. Only statistically significant changes in return values and periods, for which the 80% confidence intervals of the corresponding regional estimates are nonoverlapping, are displayed in color. Such changes are significant at about the 10% significance level. The confidence intervals were obtained with the bootstrap procedure outlined in section 3. The extreme precipitation increases almost everywhere over the globe with the largest changes occurring over the tropical Pacific. The shortening of return periods tends to be somewhat larger at high latitudes than elsewhere, except for the tropical Pacific where the return periods are reduced by more than half. The pattern of change in the Tropics is consistent with the El Niño–like warming of the surface that occurs in CGCM2 (Yu and Boer 2002).

The general increase in extreme precipitation is consistent with the greater moistureholding capacity of a warmer atmosphere. There are also a few areas in the Tropics where extreme precipitation weakens and the corresponding return periods become longer. In particular, the ascending branch of the Walker circulation over western Pacific weakens and shifts eastward in the warmer El Niño–like model climate, which is accompanied by a precipitation increase over the central-eastern Pacific and a decrease over the western Pacific. Changes in 20-yr return values of annual precipitation extremes are generally larger than the corresponding changes in the location of the annual extremes distribution, mainly due to the increased interannual variability as indicated by changes in the scale parameter shown in the lower-left panel in Fig. 7.

We also carried out a regional analysis of changes in extreme value statistics. To do so, we divided the landmasses into several subcontinental-scale regions as illustrated in Fig. 8: Africa (AFR); central Asia (ASI); South Asia (SAS); Australia (AUS); Europe (EUR); North America (NAM); South and Central America (SAM); and two polar regions, Arctic (ARC) and Antarctic (ANT). The region definitions loosely follow those used in recent climate change and climate change detection studies (Giorgi and Francisco 2000; Stott 2003). However, we used larger subcontinental-scale areas in consideration of the greater uncertainties in the evaluated extreme value statistics. The region coordinates are provided in Table 2.

The barplots displayed in Fig. 9 summarize the subcontinental changes in 20-yr return values of $T_{max}$, $T_{min}$, and $P$ in year 2090 relative to 2000 and return periods of the year 2000 $P_{20}$ in year 2090 as simulated by CGCM2 with the A2 (red bars) and B2 (blue bars) emission scenarios. The corresponding changes in the annual means are also shown in lighter red and blue colors. The changes in extreme and annual mean precipitation are expressed in relative terms as a percentage relative to the year-2000 level. The relative changes are obtained by dividing the regionally averaged absolute changes by the regional average of the corresponding quantity in year 2000.

As expected, the changes by the end of the twenty-first century in the B2 runs are more moderate than those in the A2 integrations in all regions. Although global and land-averaged changes in $T_{max}$ are comparable to the corresponding changes in the annual mean of $T_{max}$, there are substantial variations in their relative magnitudes from one region to another. Changes in the warm extremes exceed the corresponding changes in the annual means in many parts of the world such as Europe, North and South America, and southeastern Asia. Temperature increase in these regions is accompanied by a reduction of the soil moisture content so that daily maximum temperatures are less likely to be moderated by evaporative cooling from the surface. In polar regions, changes in the warm annual extremes, which typically occur in the respective summer seasons, are weaker than the changes in the annual

Fig. 5. Return periods for year 2000 10-, 20-, and 50-yr return values of annual precipitation extremes averaged over the (left) ocean and (right) landmasses as a function of time simulated by CGCM2 with the IS92a (red lines), A2 (green lines), and B2 (blue lines) emission scenarios. The vertical axes are plotted on a log scale.
The lower-left panel in Fig. 9 clearly illustrates the differences between the simulated changes in extreme and annual mean precipitation. While the mean precipitation increases only a little, or even decreases in many parts of the world, extreme precipitation as measured in terms of 20-yr return values of annual 24-h precipitation extremes increases in all considered regions. The associated return periods of year 2000 extreme events are reduced everywhere by a factor of about 2 (Fig. 9, bottom right).

mean of daily maximum temperature. The latter is enhanced by extensive sea ice melting in the adjacent oceans in winter, resulting in greater warming in this season. Changes in extremely cold temperatures are amplified by the surface albedo feedback in regions that are covered with snow in winter, such as Europe, North America, and the Arctic. With global warming, the snow cover retreats in these areas, exposing a lower albedo surface, which in turn accelerates warming at the surface.
The sampling variability of extreme value statistics is greatly reduced by averaging over large areas. All averaged changes in temperature extremes in the considered regions are highly statistically significant. This is also true, although to a lesser extent, for precipitation extremes. For example, Fig. 10 displays changes in \( P_{20} \) as simulated by CGCM2 in 2050 and 2090 relative to 2000 with the A2 emission scenario overlayed with the corresponding 80% confidence intervals. The confidence intervals for 2050 and 2090 generally do not overlap with those for 2000, indicating that regional mean extreme precipitation changes are statistically significant. Naturally, the smaller the region, the wider the corresponding confidence interval (e.g., for Australia, or the Southeast Asia region). Notice also that the confidence intervals for year 2050 are generally somewhat smaller than for years 2000 and 2090. This comes about because return values for year 2050 are estimated from the middle of the 2025–75 time window while years 2000 and 2090 occur near the ends of the windows that cover those years.

Fig. 7. (top left) Change in spatially smoothed 20-yr return values of annual extremes of 24-h precipitation amounts and the corresponding changes in the (top right) fitted GEV location parameter \( \xi \) and (bottom left) scale parameter \( \alpha \) as simulated by CGCM2 in 2050 relative to 2000 with the A2 emission scenario. (bottom right) Return periods for year 2000 20-yr return values as simulated by CGCM2 in year 2050 with the A2 emission scenario.

Fig. 8. Definition of subcontinent-scale regions.

Table 2. Coordinates of continental-scale regions.

<table>
<thead>
<tr>
<th>Region</th>
<th>Label</th>
<th>Latitude</th>
<th>Longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Africa</td>
<td>AFR</td>
<td>40°S–30°N</td>
<td>20°–60°W</td>
</tr>
<tr>
<td>Central Asia</td>
<td>ASI</td>
<td>30°–65°N</td>
<td>45°E–180°</td>
</tr>
<tr>
<td>Australia</td>
<td>AUS</td>
<td>45°–10°S</td>
<td>105°E–180°</td>
</tr>
<tr>
<td>Europe</td>
<td>EUR</td>
<td>30°–65°N</td>
<td>20°W–45°E</td>
</tr>
<tr>
<td>North America</td>
<td>NAM</td>
<td>25°–65°N</td>
<td>165°–30°W</td>
</tr>
<tr>
<td>South America</td>
<td>SAM</td>
<td>55°–25°N</td>
<td>115°–30°W</td>
</tr>
<tr>
<td>South Asia</td>
<td>SAS</td>
<td>10°–30°N</td>
<td>60°–160°E</td>
</tr>
<tr>
<td>Arctic</td>
<td>ARC</td>
<td>65°–90°N</td>
<td>180°W–180°E</td>
</tr>
<tr>
<td>Antarctic</td>
<td>ANT</td>
<td>90°–65°S</td>
<td>180°W–180°E</td>
</tr>
</tbody>
</table>

Fig. 7 live 4/C
Note that caution is needed in interpreting model-simulated changes, particularly on smaller regional scales. It is widely believed that global climate models are less reliable in simulating regional aspects of climate variability. The confidence intervals discussed above indicate only uncertainty resulting from sampling errors and from the CGCM2-generated internal variability and do not take into account other sources of uncertainty such as model uncertainty and forcing scenario uncertainty. The uncertainty associated with the latter sources may significantly reduce our confidence in the projected future changes.

7. Summary

Changes in extreme temperatures and precipitation are examined in three 3-member ensembles of transient climate change simulations performed with the second-generation coupled global climate model of the Canadian Centre for Climate Modelling and Analysis. The ensembles cover the time period 1990–2100 and use the IPCC IS92a, A2, and B2 emission scenarios. The first two scenarios employ similar radiative forcings, while the B2 scenario produces less future warming.

Extreme events are described in terms of return values of annual extremes, that is, the thresholds that are exceeded by annual extremes in any given year with probability \( \frac{1}{T} \), where \( T \) is the return period in years. The return values are estimated from a GEV distribution with time-dependent parameters fitted to annual extremes obtained from 51-yr moving windows. Each sample of annual extremes contains 153 (3 x 51) values.

The GEV distribution was fitted by the method of maximum likelihood.

To summarize:

- The ML method for fitting a GEV distribution with time-dependent parameters appears to be a more appropriate choice for transient climate change simulations than the method of L moments. The latter may produce biased return value estimates for the climate change rates that are typical in the considered simulations by the end of the twenty-first century, particu-
larly for temperature extremes. The biases become larger with larger sampling windows and are the consequence of the violation of the stationarity assumption implicit in the L-moments method. The biases in L-moment return values obtained for 21-yr sampling windows are not very large in absolute terms so that the conclusions in KZ2000 based on the L-moment method are not seriously affected by this finding. However, the biases may be substantial when expressed in terms of the corresponding return periods.

- Globally averaged changes in temperature extremes are primarily associated with changes in the location of the extreme value distribution of annual extremes. There are no substantial changes in the shape of the distribution in most locations. Exceptions from this general rule occur in regions where land and ocean surface properties undergo drastic changes under global warming such as the regions of sea ice and snow cover retreat. In particular, the interannual variability of annual cold extremes (and therefore the scale parameter of the corresponding GEV distribution) is greatly reduced in sea ice and snow-covered areas that become generally ice and snow free in the warmer climate. Changes in cold extremes in these areas are much greater than changes in the annual mean daily minimum temperature. Wintertime warming in these regions is amplified by the surface albedo feedback. On the other hand, the variability of annual cold extremes is increased in areas adjacent to the sea ice and snow cover lines in the warmer world because these areas are snow covered in some years and snow free in other years.

The variability of annual warm extremes (and the corresponding scale parameter) increases over land areas where reductions in soil moisture reduce the moderating effect due to evaporative cooling. Consequently, these regions experience greater changes in warm extremes as compared to changes in the annual temperature mean. Conversely, changes in warm extremes in the polar land areas are smaller than changes in the annual mean of daily maximum temperature. This is because the greatest temperature changes in these areas occur in winter as a result of extensive melting of sea ice in the adjacent oceans, while warm temperature extremes tend to occur in summer.

- Changes in precipitation extremes are associated with changes in both the location and scale of the distribution. The distribution of annual extremes shifts to the right and becomes wider under global warming. Changes in the extreme daily precipitation rate are substantially larger than the changes in the annual mean precipitation rate. The latter is reduced in some parts of the world while extreme precipitation increases almost everywhere. The probability of precipitation events that are considered extreme in the year 2000 is increased by a factor about of 2 by the end of the twenty-first century in the considered transient climate change simulations.

Acknowledgments. We thank our colleagues at CCCma who helped to produce the climate simulations analyzed in this paper and Dr. Badal Pal for extracting daily datasets from the archived model output. We thank Drs. Vivek Arora and Xuebin Zhang and two anonymous reviewers for providing useful comments on an earlier version of the manuscript.

APPENDIX

The Maximum Likelihood Estimator

Let \( f_X(x; \theta) \) be the probability density function of a random variable \( X \) with parameters \( \theta = [\theta_1, \ldots, \theta_p] \).

Let \( x = \{x_i, i = 1, \ldots, n\} \) be \( n \) independent realizations of the random variable \( X \). The log-likelihood function for \( \theta \) based on data \( x \) is given by

\[
l_{x_1 \ldots x_n}(\theta) = \sum_{i=1}^{n} \ln f(x_i; \theta).
\]

The maximum likelihood estimator \( \hat{\theta} \) is the value of \( \theta \) that maximizes \( l_{x_1 \ldots x_n}(\theta) \).

The log-likelihood function for the GEV distribution (1) with constant location, scale, and shape parameters \( \xi, \alpha, \text{ and } \kappa \) is given by

\[
l_{x_1 \ldots x_n}(\xi, \alpha, \kappa) = \sum_{i=1}^{n} \{-\ln \alpha - (1 - \kappa)y_i - e^{-\gamma_i}\},
\]

\[
y_i = \begin{cases} 
  - \frac{1}{\kappa} \ln \left(1 - \frac{x_i - \xi}{\alpha}\right), & \kappa \neq 0, \\
  \frac{x_i - \xi}{\alpha}, & \kappa = 0,
\end{cases}
\]

provided \( 1 - \kappa(x_i - \xi)/\alpha > 0 \) for each \( i = 1, \ldots, n \).

Otherwise, the log-likelihood is undefined and assumed to be \( -\infty \). The extension to the case of time-dependent parameters (2) is trivial.

The maximization of the log-likelihood is formally unconstrained, but in practice it is advisable to test explicitly for violations of practical constraints such as \( \alpha > 0 \) and \( 1 - \kappa(x_i - \xi)/\alpha > 0 \) for each \( i \) by setting the log-likelihood to some very large value when these conditions are violated in a maximization algorithm. The peculiarity of the log-likelihood for the GEV distribution is that there is a singularity for short-tailed distributions with \( \kappa > 1 \), as \( (\xi + \alpha/\kappa) \to \max_i \{x_i\} \). In most practical situations, there is a local maximum some distance from the singularity. The correct procedure is to ignore the singularity and use the local maximum. One possibility, which occurs very infrequently with environmental data when samples are small, is that no local maximum exists and the singularity dominates. In this case some other solution must be sought. The other
theoretical possibility is that there may be several local maxima. Such phenomena are extremely rare in the case of GEV distribution with constant parameters. However, in more complex problems with many covariates the possibility of multiple local maxima increases. When in doubt, the maximization algorithm should be rerun from different initial conditions.

We employed a simplex function minimization procedure owing to Nelder and Mead (1965), as implemented by O’Neill (1971). The maximization algorithm requires starting values. Simple guesses, such as the sample mean for $\xi$, sample standard deviation for $\alpha$, and $\kappa = 0$, usually suffice for a GEV distribution with constant parameters. Another possible strategy as employed in the present study is to use L-moment estimates as the initial conditions. However, it is important to ensure that the initial parameter estimates are feasible. This can be achieved by using a hybrid estimator of Dupuis and Tsao (1998). Once the constant ML parameter estimates are obtained, they are used as the initial conditions in the likelihood maximization for the GEV distribution with the linearly varying location parameter. The outcome of that step is used in turn as a starting point for fitting the GEV distribution with two time-varying parameters, and so on.

The advantage of this stepwise approach is that the so called likelihood ratio test can be used to compare two competing nested statistical models. Let $M_0$ be a model that is obtained from another statistical model $M_1$ by keeping $q$ of its parameters constant. Let $l_0^{(1)}(\theta)$ and $l_1^{(1)}(\theta)$ be the corresponding log likelihoods, and let $\hat{\theta}^{(0)}$ and $\hat{\theta}^{(1)}$ be the respective ML estimates of $\theta$. Then the statistic

$$D = 2[l_1^{(1)}(\hat{\theta}^{(1)}) - l_1^{(0)}(\hat{\theta}^{(0)})]$$

is called the log-likelihood ratio statistic ($D$ is also known as the deviance). If $M_0$ is the true model then, approximately, $D \approx \chi^2_q$. For example, suppose we want to test the null hypothesis of a GEV distribution with all constant parameters against an alternative GEV distribution with a linear varying location parameter. In this case, the distribution of $D$ is approximately $\chi^2_2$, and we would reject the null hypothesis at about the 5% significance level when $D > 3.84$.

For the data analyzed in the present study, the null hypothesis that the shape parameter $\kappa$ is constant could not be rejected at any appreciable significance level. In contrast, the hypothesis that the location parameter $\xi$ is constant in the GEV distribution of annual temperature extremes is rejected almost everywhere. The null hypotheses that the location parameter for precipitation extremes is constant and that the scale parameter for temperature and precipitation extremes is constant are also rejected in many parts of the world. Therefore, we use the GEV distribution with a constant shape parameter and time-varying location and scale parameters for all considered quantities.

Some difficulties were encountered in fitting a GEV distribution to annual temperature extremes in some regions. The problem has been traced to the locations where the annual extremes appear to be drawn from populations with distinctly different statistical properties (see also KZ2000). For example, Fig. A1 shows annual minima of near-surface temperature at two grid points in the central North American continent (left) and in the Labrador Sea (right). Clearly, there are two populations of annual extremes, one of which is clustered around the freezing point of water and another that exhibits much more severe cold extremes. Over land, the problem is related to the one-layer “bucket” type land surface model used in CGCM1 and CGCM2. The ground temperature in this model does not deviate substantially from 0°C until all of the latent heat of fusion contained by the moisture in the bucket has been released to (or taken from) the atmosphere. This in turn places a stringent constraint on the diagnosed 2-m
near-surface air temperature in CGCM1 and CGCM2. Over the ocean, cold extremes are substantially larger over sea ice than over open water. Sea ice retreats in high latitudes under global warming, exposing the near-surface air to open waters and resulting in more moderate cold extremes.

We dealt with these effects by fitting a mixture of two GEV distributions to the population of annual extremes clustered around the freezing point and the population of the remaining extremes and estimating a time-dependent mixing coefficient. The return values are then estimated as follows: Let $F_1(x, t)$ and $F_2(x, t)$ be the distribution functions with time-dependent parameters for the two populations, and let $Pr(t)$ be the probability that the annual extreme comes from the first population. The overall distribution function for the annual extremes is given by the mixture,

$$F(x, t) = Pr(t)F_1(x, t) + [1 - Pr(t)]F_2(x, t).$$

The probability $Pr(t)$ is estimated by the proportion of the extremes in the population one in a sampling window. This probability is indicated by dashed lines in Fig. A1 for the examples displayed. Return values are calculated by inverting the estimate of the distribution function $F(x, t)$.

REFERENCES


