Use of Multiquadric Interpolation for Meteorological Objective Analysis

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ABSTRACT

The method of multiquadric interpolation is described and compared to the Barnes and Cressman methods of meteorological objective analysis. The method of multiquadric interpolation uses hyperboloid radial basis functions to fit scattered data to a uniform grid. Results for an analytical function indicate that the method is more accurate than the Barnes or Cressman methods. Application to actual meteorological data indicates that multiquadric interpolation produces excellent analyses that retain small-scale features resolved by the observations in any subregion of the analysis.

1. Introduction

The problem of analyzing the scattered meteorological observations to produce values on a regular grid for modeling or diagnostic computations has presented meteorologists with a significant challenge for many years. Numerous methods have been devised to represent scattered observations on a uniform grid. Optimum interpolation (Gandin 1963; Schlatter 1976; and others) has become the standard method at most operational numerical forecast centers. The wide use of optimum interpolation in conjunction with a four-dimensional data assimilation scheme can be principally attributed to its ability through statistical methods to combine wind and mass field observations into a dynamically consistent analysis, which is extremely important for numerical modeling. However, the computational expense and the applicability of the method to mesoscale problems has limited its use for local analysis problems or especially for the analysis of special sets of observations from field experiments. The research community often uses Barnes (1973) or Cressman (1959) analysis techniques due to their ease of application to small datasets or local analysis problems. While optimum interpolation analysis with a four-dimensional data assimilation method has significant advantages, other easily applied methods are still needed for local analysis problems that arise in research and in local forecast situations as well as situations where the statistically defined structure functions are not known.

While the Barnes (1973) or Cressman (1959) methods produce generally acceptable analyses, other mathematical methods may produce more accurate analyses or lend themselves to direct grid-free diagnostic calculations as suggested by Caracena (1987). One such method developed by Hardy (1971), which has been referred to as multiquadric interpolation, has been widely applied to geodesy, geophysics, geography, and surveying and mapping problems. However, it has not received much attention in the meteorological community. Franke (1985) compared a variety of interpolation techniques, including univariate statistical interpolation, and found that thin-plate spline methods, which are comparable to the multiquadric method, produced results at least as accurate as the statistical interpolation method and definitively superior to the Barnes (1973) scheme. These results were for analytically generated observations, so the applicability of these methods to the analysis of actual meteorological observations containing significant errors is not known. The goal of this paper is to describe the technique of multiquadric interpolation (Hardy 1971) and to demonstrate its application to the analysis of actual meteorological observations.

The mathematical theory of multiquadric interpolation and its application as used in this study are described in section 2. Section 3 examines the accuracy and response characteristics of this analysis scheme as well as the specification of its free parameters. Section 4 provides several examples of application to actual data. Section 5 gives future extensions and a summary about this technique.

2. Multiquadric interpolation theory

The basic theory of multiquadric interpolation and some previous applications have been reviewed by Hardy (1990). The foundation for multiquadric as well as statistical interpolation and the method of Caracena...
(1987) lies within the general theory of interpolation using radial basis functions. The interpolation equation using radial basis functions is

$$H(X) = \sum_{i=1}^{N} \alpha_i Q(X - X_i),$$  \hspace{1cm} (1)$$

where $H(X)$ is a spatially varying field, such as pressure or temperature, and $Q(X - X_i)$ is a radial basis function, where the argument represents the vector between an observation point $X_i$ and any other point in the domain. The coefficients $\alpha_i$ are weighting factors that must be determined from the observations or specified in some manner. For statistical interpolation, the covariance functions between the field at observed points and other points in the domain serve as the basis functions. Caracena (1987) used Guassian functions as the radial basis functions. The multiquadric method uses hyperboloid functions as the basis functions in the form

$$Q(X - X_i) = -\left(\frac{||X - X_i||^2}{c^2} + 1.0\right)^{1/2},$$  \hspace{1cm} (2)$$

where $c$ is an arbitrary, and typically, small constant. This form of the multiquadric basis function differs from that introduced by Hardy (1971) but is a member of the general class of multiquadric functions described by Madych and Nelson (1990) and gives similar results to the original form. The constant $c$ makes the basis function infinitely differentiable by preventing the basis function from vanishing at the point of the observations and affects the condition number of the coefficient matrix by controlling the relative sizes of the diagonal and off-diagonal terms. This constant will be referred to as the multiquadric parameter. Here, $X$ may represent the position vector in one, two, or three dimensions. For example, the hyperboloid function in two dimensions becomes

$$Q_i(x, y) = -\left(\frac{|x - x_i|^2 + |y - y_i|^2}{c^2} + 1.0\right)^{1/2}.$$  \hspace{1cm} (3)$$

To determine the coefficients $\alpha_i$, a set of linear equations must be solved. Applying the interpolation equation to the field at every observation point $(x_j, y_j)$ results in the following set of equations:

$$H(x_j, y_j) = \sum_{i=1}^{N} \alpha_i Q_i(x_j, y_j),$$  \hspace{1cm} (4)$$

where

$$Q_i(x_j, y_j) = -\left(\frac{|x_j - x_i|^2 + |y_j - y_i|^2}{c^2} + 1.0\right)^{1/2}.$$  \hspace{1cm} (5)$$

Note that the observations $H(x_j, y_j)$ may represent ei-
Fig. 2. Observation (×) distributions for the (a) pseudouniform, (b) land-sea, (c) satellite, and (d) aircraft observation samples. The contours show analytical function amplitude in intervals of 0.1.

Fig. 3. Root-mean-square errors for the multiquadric (MQ), Cressman, and Barnes objective analyses of the analytical function.

Thus the raw observations or the deviation of the observations from some background field. In the applications presented in this paper, the deviations away from the mean of the observations are analyzed and then the mean is added back into the solution. Equation (4) holds at all N observation points. This results in a set of N equations with N unknown coefficients $\alpha_i$. In matrix notation,

$$H_j = Q_j \alpha_j,$$  \hspace{1cm} (6)

and the solution for the $\alpha_i$ in this set of equations is given mathematically as

$$Q_j^{-1} H_j = \alpha_i.$$  \hspace{1cm} (7)

In practice, the coefficients $\alpha_i$ as well as the inverse matrix $Q_j^{-1}$ are determined by solving the set of linear equations using a variety of computational techniques available in many software libraries. We use a routine called LINREG in the International Mathematics and Statistical Library (IMSL). Computational stability in solving the system of linear equations is a potential problem (see next section). The interpolation of the
Fig. 4. Absolute differences between the analytical function and the (a) multiquadric analysis and (b) Barnes analysis using 25 pseudouniformly distributed observations (case 5).
solution to any desired uniform grid $H_g$, represented by grid points $(x_g, y_g)$, is then given by

$$H_g = Q_g \cdot Q_g^{-1} \cdot H_j,$$

where each element in the matrix $Q_{gj}$ is given by

$$Q_{gj} = \left( \frac{|x_g - x_j|^2 + |y_g - y_j|^2}{c^2} + 1.0 \right)^{1/2}.$$  

Since the number of grid points is not necessarily equal to the number of observations, the matrix $Q_{gj}$ is not a square matrix. Note that, once the coefficients $\alpha_i$ are determined, the solution or approximated function can be determined on any arbitrary grid, such as a grid with 10- or 1000-km spacing. The resolved scales and accuracy of the approximated function are completely determined by the number and spacing of the observations that were used to determine the coefficients. However, the choice of output grid may limit the representation of these scales on the output grid. The relationship between the set of observations and the resolved scales in the approximated function will be explored more fully in the next section.

As just described, the multiquadric interpolation technique applies to observations without error and can be used as a very accurate interpolation method. Kansa (1990) has applied the multiquadric technique to the solution of partial differential equations and found that the accuracy of the solutions was greater than with standard finite-difference techniques. Franke (1982) shows that this method of interpolation is the most accurate of the 29 methods he tested on analytically generated observations. The problem in applying such techniques to meteorological observations is that errors as well as incomplete sampling of small-scale features may result in unrealistic analyses. In general, we need to smooth or filter these unresolved scales from the analysis. Optimum interpolation achieves this filtering by using spatially smooth covariance functions. To achieve the required filtering in this multiquadric interpolation, the technique of generalized cross validation by Wahba and Wendelberger (1980) can be used. Their method determined the smoothing parameter for a thin-plate spline interpolation method from the observations, and this approach can be applied to multiquadric interpolation as well.

To account for observational uncertainty, the interpolation equation is altered slightly. The resulting interpolation equation in matrix notation then becomes

$$H_j = [Q_{ij} + (N\sigma_i^2 \delta_{ij})] \cdot \alpha,$$  

where $N$ is the number of observations, $\sigma_i^2$ is the mean-squared observation error, $\lambda$ is a smoothing parameter, and $\delta_{ij}$ is the Kronecker delta. The observation error $\sigma_i^2$ can be varied for different observation sources with the result that the analysis will fit more closely to some observations than others. As described by Wahba and Wendelberger (1980), the parameter $\lambda$ governs the degree of smoothing in the approximated function and can be related to the half-power point of a spectral filter. Wahba and Wendelberger (1980) determined the smoothing parameter $\lambda$ from the observations using a generalized cross-validation equation, whereas we choose to leave this term as a free parameter that must be set.

The solution to this modified interpolation equation (10) is similar to that in (7) described above. The only difference is that the diagonal terms in the matrix $Q_{ij}$ are modified by the parameter $N\lambda \sigma_i^2$. Once the inverse to this modified matrix is obtained, the solution on the output grid is derived exactly as described above.

![Figure 5](image1.png)

**Figure 5.** Maximum absolute errors for the multiquadric (MQ), Cressman, and Barnes objective analyses of the analytical function.

![Figure 6](image2.png)

**Figure 6.** Mean rms errors as a function of the number of observations for the multiquadric (solid) and Barnes (dashed) analyses. The mean rms error is based on 100 realizations of a pseudouniform observation distribution. The 95% confidence limits are shown by lines above and below plotted data points.
3. Characteristics of multiquadric interpolation

Previous studies (Franke 1982; Kansa 1990a) have found the multiquadric interpolation technique to be highly accurate for the interpolation of scattered data and for the solution of partial differential equations (Kansa 1990b). For application of this interpolation technique to meteorological observations, it is useful to compare its performance to other familiar analysis techniques for both analytic fields and actual meteorological data. In addition, the dependence of the resultant analysis on the smoothing parameter, \( \lambda \) in (10), and the multiquadric parameter, \( c \) in (2), needs to be documented to fully understand the performance of multiquadric interpolation on meteorological data.

a. Tests with analytic function

To examine the sensitivity to the data distribution, the Barnes (1973), Cressman (1959), and multiquadric analysis techniques were compared with observations extracted from analytically specified fields. The analytical function consisted of a set of Guassian hills and valleys over a two-dimensional domain (Fig. 1), which were used by Franke (1982). The value of the analytic function varied between 0.0 and 1.25 over the domain. The observation set consisted of the function amplitudes between 0.0 and 1.25 at randomly selected points scattered over the domain. These scattered observations were supplied to the three techniques to generate an analysis of the analytical function on a regular grid. Root-mean-square (rms) errors between the analytical function and the three analyses were calculated over the domain at the analysis grid points.

The three analysis techniques contain several parameters that must be set by the user. To make the comparisons as objective as possible, consistent methods for setting the free parameters were used in all cases.

Fig. 7. Sea level pressure analysis for 0000 UTC 23 January 1993 based on the (a) multiquadric technique and (b) Barnes technique. The multiquadric smoothing parameter is 0.025, and the Barnes smoothing length scale is 250 km. Observing stations are plotted with sea level pressure to the upper right of the station marker. The contour interval is every 2 mb.
For the Barnes (1973) scheme, a two-pass technique was used with the smoothing length scale set to four-thirds of the mean observation spacing, the convergence parameter was set to 0.33, and the weighting function was set to 0 for distances greater than 25 grid units. For the sets of scattered observations used in these tests, the mean observation spacing varied from 7 to 27 grid units, where a grid unit was 0.01 of the analysis domain. For the Cressman (1959) scheme, a five-pass technique was used with the scan radius for each pass decreasing for each subsequent pass. The values of the scan radii were 25.0, 15.0, 10.0, 5.0, and 2.5 grid units. For the multiquadric scheme, the smoothing parameter $\lambda$ was set to 0.0 for these analytically generated observations and the multiquadric parameter $c$ was 0.05. All three techniques were applied in a univariate sense to analyze the amplitude of the function on a grid of $101 \times 101$ points.

Although the values of the free parameters for the three methods may not be optimal for a given data distribution and analysis method, experimentation showed that these values produced consistently accurate results that allowed direct comparison of the methods over a wide range of data distributions. For the Barnes (1973) and multiquadric methods, tests were done to determine "optimal" values of the free parameters for some data distributions in order to compare the absolute accuracy for the best that a given method could achieve. For the Barnes (1973) method, the smoothing length scale was typically increased above the four-thirds of the mean observation spacing, which was based on the results of Pauley and Wu (1990) for a uniform mesh of observations. The increase in the smoothing length scale as the scatter increases is in accord with the results of Smith et al. (1986). For the multiquadric method, the value of the multiquadric parameter was typically...
increased until just before the coefficient matrix became ill-conditioned. Examples of this optimization for both schemes are presented later in this section.

To test the interpolation accuracy and its sensitivity to observation distribution, sample sets were used with distributions (Fig. 2) that replicate typical observation sets in meteorology. Extreme sensitivity to the distribution of observations would render an analysis scheme worthless for wide application to meteorological data analysis. The observation sets consisted of four basic types: 1) 150 (case 1) and 25 (case 5) randomly scattered observations with pseudouniform density over the domain; 2) a land–sea distribution (case 2) in which 150 observations were well scattered with dense coverage over half the domain and with sparse coverage over the other half; 3) a "satellite distribution" (case 3) with gaps between multiple swaths of 150 densely spaced observations; and 4) a set of 150 aircraft observations (case 4) that crisscross the center of the domain with sparsely scattered points off the aircraft tracks.

The results of these tests using the analytically generated observations are compared in Fig. 3. For all schemes, the rms errors are relatively small (0.15 or less), with the smallest error associated with the pseudouniform distribution of 150 observations (case 1), which is expected as this case represents the most complete spatial sample of the analytic function. The larger errors for the land–sea (case 2), satellite (case 3), and aircraft (case 4) data distributions result from less complete spatial sampling of the analytic function for these cases. The multiquadric interpolation had lower rms errors than either the Barnes or Cressman schemes for all four data distributions. For example, the rms error for the Cressman (1959) and Barnes (1973) schemes with 150 observations were 1.5–5.0 times higher than for the multiquadric scheme using the selected data distributions and tuning parameters, except for the aircraft
data distribution (case 4) in which the Cressman (1959) scheme did nearly as well as the multiquadric scheme. The greatest difference between the multiquadric method and the other methods occurred when the sample size was decreased to 25 observations for the scattered distribution (case 5). The rms error for the multiquadric method was 0.025 compared to 0.067 for the Cressman method and 0.11 for the Barnes method. Although the rms error for the multiquadric method using only 25 observations is nearly triple that of the 150 observations sample, the increased error for the data-sparse set is less than found for the other methods. Although the absolute magnitudes of the rms error differences between the various schemes were small compared to the amplitude of the function, these differences represent significant differences in the ability of a given scheme to represent the function, as will be shown herein for a single case.

The results presented above were based on the observation samples shown in Fig. 2 and tuning parameters that were not optimally selected for a given observation sample, which potentially bias the results for one scheme versus the other, particularly for the land–sea (case 2), satellite (case 3), and aircraft (case 4) observation distributions. To determine whether the results were biased by the observation samples, each observation sample was varied 100 times to generate an unbiased estimate of the rms errors for the multiquadric and Barnes schemes. The 100 realizations of each observation sample were produced by varying the location of the data-dense region relative to the analytical function and taking a unique set of random points for each observation sample. For example, in case 2, the land–sea boundary was rotated with respect to the analytical function and the actual observation points relative to this boundary were different for each realiza-
tion. The rms error was calculated for each realization from which the mean rms error, its standard deviation, and a 95% confidence interval were calculated. The results (not shown) indicate that the rms errors shown in Fig. 3 are representative of the mean rms errors. For example, the mean rms error in case 2 was 0.048 for the multiquadric scheme and 0.095 for the Barnes scheme, as compared to the rms errors of 0.039 and 0.105 shown in Fig. 3. Although the mean rms error for the multiquadric scheme based on 100 realizations was somewhat higher than the single realization, the mean rms error is still less than half that of the mean rms error for the Barnes scheme. The mean rms errors for the multiquadric and Barnes schemes were found to be statistically different at the 95% confidence level for case 2 as well as the other cases. These results indicate that the multiquadric scheme was consistently more accurate than the Barnes scheme for these sets of observations and tuning parameters.

To provide a more definitive comparison of the techniques and to better understand the potential effects of the interpolation errors, optimal values of the tuning parameters for the Barnes and multiquadric methods were determined for the set of 25 scattered observations (case 5) and the rms errors computed for these optimal selections of parameters. For the multiquadric scheme, the optimal value for the multiquadric parameter was found to be 0.1 as compared to the original value of 0.05 used above. For the Barnes scheme, the smoothing length scale was increased from 27 (three-fourths of mean observation spacing) to 50 grid units, and the maximum radius of influence was set to 50 grid units as compared to 25 used in the other tests. Little sensitivity to the convergence parameter was found, and so
a value of 0.33 was used in these tests as well. The rms errors for these optimal selections of parameters did not change substantially but were reduced slightly in both cases—0.017 for the multiquadric and 0.105 for the Barnes as compared to 0.025 and 0.11, respectively. The largest impact of using optimal values was that the higher-frequency structure between observation points was removed and the magnitude of the largest deviation was reduced at the expense of increasing the smaller deviations elsewhere, particularly in the Barnes scheme.

The rms errors may be put into a meteorological perspective by considering the Gaussian hills to represent 500-mb waves with trough-to-ridge height variations of 300 m. Then the case 5 rms error of 0.11 for the Barnes scheme with nonoptimal parameters represents a 33-m height error, and the rms error of 0.025 for the multiquadric scheme with nonoptimal parameters is only a 7.5-m height error. Although this indicates a better analysis by the multiquadric scheme, even greater differences are evident if we examine the distribution of the errors (Fig. 4). Even though the distribution of the errors for each method is similar, the amplitudes of the largest errors for the Barnes scheme (Fig. 4b) are 3.5 times those of the multiquadric scheme (Fig. 4a). In the Barnes scheme, 0.35 (70 m) errors are evident, and in the multiquadric scheme the largest errors are only 0.10 (10 m). The magnitude and distribution of the errors changed very little when optimal parameters were used in the two schemes. These are significant differences because they contribute dramatically to increased errors in the height gradients, which may result in large errors in derived quantities (e.g., geostrophic winds) or as initial conditions in a numerical model. The relatively small maximum differences produced by the multiquadric analysis scheme for the 25 observations sample indicate that the multiquadric scheme works very well with sparse data.
ilar, but less dramatic, results occur in the other cases as well (Fig. 5), which compares the maximum errors between the three methods.

To further explore the performance of the multiquadric technique when the number of observations changes, a set of experiments was done using the scattered observation distribution. The experiments consisted of varying the number of observations from 10 to 150. For each experiment, 100 realizations were produced by choosing a unique set of random observations for each sample. Neither scheme was optimized for any individual realization of the observation distribution. The Barnes scheme used a smoothing length scale of four-thirds of the mean observation spacing, a maximum radius of influence of 25 grid units, and a convergence parameter of 0.33. The multiquadric scheme used a multiquadric parameter of 0.05 and no smoothing. The result (Fig. 6) indicates that the multiquadric technique has substantially less error than the Barnes scheme for any number of observations and the mean rms error for both methods increases as the number of observations decreases. The mean rms errors for the multiquadric technique and the Barnes scheme were statistically different at the 95% confidence limit (Fig. 6). The most important implication of this analysis is that the multiquadric method can produce a more accurate analysis than the Barnes scheme for considerably fewer observations. For example, the rms error for the multiquadric method using only 40 observations is lower than the rms error for the Barnes scheme using 150 observations. Although a particular distribution of 40 observations may produce greater error in the multiquadric analysis than any 150-point set of observations in the Barnes analysis, these results show that for a random distribution of observations of this particular function a 150-point Barnes analysis is not likely on average to be better than a 40-point multiquadric analysis. If these statistics can be extended to an arbitrary function, then this aspect of the multiquadric method is significant when analyzing data-sparse regions or doing mesoscale analyses in data-rich regions.

b. Tests with meteorological observations

To examine the performance of the multiquadric interpolation scheme on actual meteorological observations, routine surface observations from 0000 UTC 23 January 1993 covering the central United States were selected for analysis. Although the time of the analysis was arbitrarily selected, this period contained significant weather features. Analysis over the data-rich central United States is not as challenging as over the data-sparse regions, but this example is useful for highlighting the characteristics of the multiquadric interpolation applied to actual observations. The performance over data-sparse areas will be described by examples in the next section.

As with the analytical tests, a comparison was made between the Barnes (1973) and multiquadric schemes. Because the rms differences were calculated only at the observation points, in this case a lower rms difference does not necessarily imply the best meteorological analysis or lower actual error; it implies only that the observations have been closely fit. For the Barnes scheme, a convergence parameter of 0.33 was used and the smoothing length scale (185 km) was four-thirds of the mean observation spacing of about 140 km. Although this is a good choice for the smoothing length scale for uniformly spaced observations (Pauley and Wu 1990), results from the previous section as well as other studies (Smith et al. 1986) indicate that for scattered observations the smoothing length scale should be larger than four-thirds of mean observation spacing. Several other smoothing length scales were tested between 185 and 350 km, and a scale of 250 km seemed to produce an analysis in which the high-frequency structure was removed while maintaining most of the mesoscale structure evident in the observations. For the multiquadric scheme, the smoothing parameter was specified as 0.025, the observation rms error was assumed to be 1.0 mb for all observations, and the multiquadric parameter was set at 0.05. The effects of both the smoothing parameter and multiquadric parameter are examined in later sections.

The resulting analyses of sea level pressure using the multiquadric and Barnes (1973) methods are shown in Fig. 7. The major features, such as the low pressure center over South Dakota and the ridging up the Mississippi Valley, are captured without difficulty by both schemes because they are resolved by the available observations. The most notable differences between the analyses (Figs. 7a,b) are related to the structure of the smaller-scale features—for example, the low pressure
area in eastern Colorado. The multiquadric analysis (Fig. 7a) produces lower pressures in southeastern Colorado and a tighter pressure gradient through south-central Colorado than the Barnes analysis (Fig. 7b). The structure in the multiquadric analysis (Fig. 7a) appears to be more consistent with the 1002.7-mb pressure at La Junta (LHX) and the 10–12.5 m s\(^{-1}\) winds through central Colorado. The exact placement of the primary low pressure center also highlights a difference between the two analysis methods. The multiquadric analysis (Fig. 7a) places the low pressure center directly over Pierre, South Dakota, (PIR) with a pressure of 997.5 mb (the lowest observed pressure), while the Barnes (1973) analysis places the low center between Pierre and Valentine, Nebraska, (VTN) with a pressure of 998.9 mb. Although the exact placement of the low cannot be determined from the pressure observations alone, the low is probably not directly on top of Pierre as placed by the multiquadric scheme, given its 5 m s\(^{-1}\) east wind. However, the distribution of pressure and wind observations suggests that the low center is likely closer to Pierre than placed by the Barnes (1973) scheme. Neither method used any wind information, which indicates the need for multivariate analysis, but the two univariate methods can be assessed using this independent wind information. From this perspective, it appears that the multiquadric solution for the placement of the low for this particular set of observations may be slightly better than the Barnes solution.

These differences between the analyses indicate the characteristic of the multiquadric scheme to fit the observations closely, even with smoothing, without producing artificial features between the observations. The multiquadric analysis had an rms difference of 0.68 mb and the Barnes (1973) analysis had an rms difference of 0.75 mb. The closeness of the fit to the observations and rms differences can be reduced for both analyses by decreasing the smoothing—for example, by setting the smoothing length scale in the Barnes scheme to 140 km with a 1400-km radius of influence, and by setting the smoothing parameter in the multiquadric scheme to
The multiquadric scheme produces a credible analysis (Fig. 8a) with an rms difference of 0.54 mb. Theoretically, the multiquadric scheme should fit the observations exactly (rms difference of 0.0) when the smoothing is set to 0.0. This does not occur due primarily to inaccuracies associated with a bilinear interpolation of the analysis to the observation locations when computing the rms error. The Barnes (1973) scheme produces a rather noisy looking analysis (Fig. 8b) with an rms difference of 0.31 mb. The Barnes scheme does fit the observations closer than with a larger smoothing length scale, but a side product is unrealistic features and a mathematically unsmooth analysis, where second- or third-order derivatives may be unrealistically large.

c. Effect of the smoothing parameter

As discussed in section 2, the smoothing parameter in the multiquadric interpolation equation (10) can be thought of as either a measure of the least-squares fit or as a spectral low-pass filter (Wahba and Wendelberger 1980). To demonstrate the effect of the smoothing parameter λ on the analysis of meteorological observations, the 0000 UTC 23 January 1993 analysis was repeated using smoothing parameter values that ranged from 0.0 to 100.0. The analysis with a smoothing parameter of 0.1 is shown in Fig. 9a and the analysis with a smoothing parameter of 1.0 is shown in Fig. 9b. These analyses with smoothing can be compared to the unsmoothed analysis that appears in Fig. 8a. As expected, increasing the smoothing produces an analysis with fewer small-scale features, which is in accord with a low-pass filtering process where features with short horizontal wavelengths (large wavenumber) have been removed. In addition to removing the smaller-scale structure, the magnitudes of the large-scale highs and lows are also decreased by this process. For example, the primary low pressure center near Pierre (PIR)
increases in pressure from 997.5 mb in the unsmoothed analysis to 999.4 mb with smoothing of 0.1, and to 1004.9 mb with smoothing of 1.0.

To quantify the effect of the smoothing in a least-squares sense, the rms difference between each smoothed analysis and the observations was calculated at the observation points. This rms difference was then normalized by the variance of the raw observations used in the analysis, which allows the effect of the smoothing to be stated independent of the type of observations. In this case, the 245 sea level pressure observations had a variance of 5.78 mb about a mean of 1014.3 mb. Thus, the rms difference of 0.54 mb associated with no smoothing, as stated above, produces a normalized difference of 0.09 (Fig. 10). As discussed above, the nonzero value for no smoothing is an artifact of the technique used to interpolate the analysis to the observation points. As expected, the greater the smoothing, the greater the analysis difference relative to the observations. The parabolic shape for the smoothing parameter over the range of 0.001–100 in Fig. 10 indicates that the difference increases by about a factor of 2 for each order of magnitude increase in the smoothing parameter. The normalized error curve in Fig. 10 is approximately given by $R = 0.04(\log S + 3)^2$, where $S$ is the smoothing parameter multiplied by the observation error $\sigma^2$ and $R$ is the ratio of the rms error to the observational variance. Although Fig. 10 is based only on the 23 January analyses, tests using other datasets produced very similar results. The relationship between the smoothing and the difference implied by Fig. 10 provides some practical guidance in setting this free parameter. For example, if the multiquadric scheme with smoothing were applied to the analysis of 500-mb heights, then the expected rms error for a smoothing parameter
value of 0.01 would be 16 m, assuming an observational variance of 100 m and observational error of 10 m. These values were derived by substituting into the approximate equation describing the curve in Fig. 10 and may not strictly apply to arbitrary distributions of observations. Further testing of this relationship is being done to provide a more definitive relationship for setting this parameter.

To better understand the impact of the smoothing from the perspective of a low-pass filter, a one-dimensional analytical function that consisted of the sum of 15 sine waves of equal amplitude was used to define the spectral response for various values of smoothing. The function was then sampled at 131 uniformly spaced points to produce a set of observations to which multiquadric analysis was applied in one dimension. Fourier coefficients were calculated for the resultant analysis using a fast Fourier transform (FFT), and the ratio of the Fourier coefficient amplitude for the smoothed to the unsmoothed analysis was calculated (Fig. 11) to indicate the transfer function properties for various values of smoothing. As the smoothing increases from 0.01 to 100.0, the ratio of the smoothed to unsmoothed Fourier coefficient amplitude decreases for the larger wavenumbers. For example, a smoothing value of 1.0 reduces the amplitude for the smoothed wave at wavenumber 10 to about 70% of the value for the unsmoothed wave. Increased smoothing further reduces the wave amplitude and increases the roll-off of the transfer function. Although the transfer function was not explicitly derived for the multiquadric interpolation with smoothing, the curves in Fig. 11 suggest a transfer function that varies with wavenumber in a manner similar to that given by Wahba and Wendel-
berger (1980) for equispaced data and a one-dimensional spline.

d. Effect of multiquadric parameter

The other free parameter in the multiquadric interpolation equation (2) is the multiquadric parameter $c$. Mathematically, a nonzero value of this parameter is required to ensure that the multiquadric basis function has continuous derivatives. This parameter determines the curvature of the hyperboloids used in the interpolation. For small values of $c$, very sharp (large curvature) hyperboloids are generated, and so very tight gradients are easily represented. For larger values of $c$, flat hyperboloids are used and the interpolation cannot easily represent tight gradients or fit closely spaced observations. This parameter has the biggest impact on the computational stability when solving for the coefficients. If the value of the $(X - X_i)^2$ is small and the value of $c^2$ is large in the multiquadric function, then the matrix has nearly equal diagonal and off-diagonal elements, which results in an ill-conditioned matrix. Kansa (1990a) has suggested varying this parameter over the set of observations to maintain computational stability and increase interpolation accuracy. We have found for datasets with closely spaced observations that very small values of the interpolation constant will maintain computational stability. We typically use a value of 0.05 for analysis done on the unit square. For ease of application, we transform all analysis problems to a unit domain where $x$ and $y$ vary from 0.0 to 1.0. However, the analysis can be done in a dimensional domain, and the parameter $c$ is then in the same dimensional units, such as kilometers or meters.
While the choice of the interpolation constant is very important for maintaining computational stability in solving for the coefficients, it potentially influences the analysis as well. Hardy (1990) states that the interpolation is not sensitive to the exact value of the multiquadric parameter. To test this assertion, $c$ was varied from 0.000005 to 0.5 for the 23 January analysis (not shown). The value of 0.5 represented the upper limit on this parameter for this set of observations for which the coefficient matrix remained well conditioned and a solution for the coefficients could be obtained. Below this limit, the impact on the analysis was nearly imperceptible for all values tested when compared to Fig. 8a, in which the multiquadric parameter is equal to 0.005. Tests with the analytic function in section 3a suggest that if $c$ becomes too small, high-frequency features may arise around observations that are very close together. The relative insensitivity to the choice of this parameter suggests that excellent results can be obtained by routinely choosing a value that is sufficiently small to guarantee computational stability yet large enough to produce relatively flat hyperboloids.

4. Examples of application

The analysis in the preceding section indicates that the multiquadric analysis technique is quite robust and easily applied to meteorological analysis problems. Several additional examples are presented and compared to similar analyses using other techniques in this section. These examples were selected as typical, yet difficult, analysis problems. Two-dimensional, single variable analysis procedures have been used in each case. For comparison, the corresponding Barnes (1973) analysis, operational multivariate optimum interpolation analysis, and (if available) subjective hand analysis are presented.

The first example is taken from the Experiment on Rapidly Intensifying Cyclones over the Atlantic (ER-
ICA), which was designed to study extratropical cyclones over the ocean. The analysis of sea level pressure over the ocean is generally difficult due to the sparsity of observations. Sea level pressure observations taken at 1200 UTC 13 December 1988 during intensive observation period (IOP) 2 of ERICA are used for this example. Sanders (1991) has documented the subjective analyses of this cyclone in detail and indicated the difficulty of obtaining a true analysis. The subjective analysis by Sanders (1991) and the National Meteorological Center (NMC) Global Data Assimilation System (GDAS) final analysis for 1200 UTC 13 December 1988 are used for comparison in this case. Although the set of observations for the multiquadric analysis is similar to the subjective analyses by Sanders (1991), the NMC GDAS analysis did not have the benefit of some special observations taken during ERICA, such as the National Oceanic and Atmospheric Administration P-3 aircraft observations. However, no special ERICA observations were available at 1200 UTC 13 December, which should minimize the differences in datasets in this comparison.

The four analyses for 1200 UTC 13 December 1988 are compared in Fig. 12. Subjective filtering of the erroneous observations prior to the multiquadric and Barnes analyses discarded essentially the same observations that Sanders (1991) chose to ignore as well. If we accept that the analysis by Sanders (Fig. 12a) is the closest to the true sea level pressure field, then it is evident the multiquadric analysis (Fig. 12b) is closer to the true field than either the Barnes (Fig. 12c) or NMC GDAS (Fig. 12d) analyses. The NMC GDAS analysis misses most of the smaller-scale structure, which is to be expected as this analysis is tuned to analyze only the larger-scale structure. The Barnes and multiquadric analyses use identical datasets and produce similar large-scale structures.

The Barnes scheme has boundary problems and has difficulty with the data-void region in the south-central region of the domain. The boundary problem can be fixed by including observations from outside the analysis domain. However, data-void regions are generally problematic and can be fixed only by increasing the smoothing length scale to retain scales larger than the
data-void region. This increased smoothing length scale prevents the retention of smaller-scale structure in regions where the observations may support it. This behavior is evident (Fig. 12c) in the inverted trough to the east of the mid-Atlantic states, which is not as sharp as analyzed by Sanders (Fig. 12a) or by the multiquadric scheme (Fig. 12b) even with minimal smoothing. To improve the analysis of this feature, the smoothing length scale in the Barnes scheme was decreased. The resultant analysis (not shown) did somewhat better with the inverted trough but produced considerable unsupported small-scale structure elsewhere. Thus, the desire to analyze smaller-scale structure in the Barnes scheme must be balanced against the problems produced by data-void regions.

The multiquadric analysis (Fig. 12b) does not appear to have boundary problems and seems to do a reasonable job in the data-void regions. Because observations exist in this case within a few kilometers of the boundary around most of the domain, the analysis is well constrained at the boundary. When observations are a large distance from the boundary, the multiquadric scheme has a small tendency to continue the gradient defined by the nearest observations. However, this extrapolation effect is considerably less severe than for the Barnes scheme. The treatment of the data-void regions by the multiquadric scheme demonstrates an important property of the technique. The scheme smoothly analyzes the scales represented by the observations in a particular region of the domain while not producing undesired results elsewhere. Consequently, data-sparse regions retain only large-scale features, while data-dense regions produce small-scale features present in the data, which are retained for a given value of smoothing or filtering.

Similar results were found in another example taken from the Tropical Cyclone Motion Experiment 1990 (TCM-90) (Elsherry 1990). The winds at 200 mb were analyzed over a large 12 000 km x 9000 km domain on a 50-km grid using both operational and research datasets that included reprocessed cloud track winds and winds from the National Aeronautics and Space Administration DC8 aircraft. However, in this comparison, only a subregion near the outflow region of Typhoon Flo at 0600 UTC 17 September 1990 is examined. Both the multiquadric (Fig. 13a) and Barnes (Fig. 13b) analyses used smoothing parameters appropriate for analysis over the entire domain and not the typhoon subregion. The large-scale flow features are similar for both analyses. In the region near Typhoon Flo, the high winds on the southwest side of the typhoon that are evident in the cloud-track winds (CTW) and DC8 observations are analyzed quite differently. Whereas the multiquadric analysis has a banana-shaped maximum exceeding 20 m s⁻¹ to the southwest of the center, the Barnes scheme completely omits this feature and has a maximum exceeding 10 m s⁻¹ in the eastern quadrants. This difference can be traced to the over-smoothing by the Barnes scheme that is required to produce a credible analysis elsewhere in the region.

5. Future extensions and summary

The multiquadric method presented in this paper is a two-dimensional, single-variable analysis procedure. The results presented in the previous section indicate that high quality analyses can be produced using this method. As with any two-dimensional, single-variable analysis method, potential problems arise when it is applied independently to various vertical levels and/or to the mass and wind fields. The mathematical technique has a natural extension to three-dimensional analysis as indicated in section 2. The equations do not change in form as the third dimension is simply added to the position vector. Tests of this extension are under way by the authors and are reported in Nuss (1994).

The other problem of analyzing the mass and wind fields separately can also potentially be addressed by incorporating a dynamic constraint on the interpolation. The multiplaric basis function is continuously differentiable, which suggests that dynamic relationships can be applied to the interpolation. A dynamic constraint for midlatitude regions has been included in the analysis of the TCM-90 observations by Tittley (1994). The wind observations were taken to be a weak constraint on the slope of the height analysis (geostrophic constraint). Following Hardy (1990), the interpolation equation in this case can be written in matrix notation as

\[ H_j = Q_\alpha \alpha_j + \frac{\partial Q_{ij}}{\partial x} \beta_j + \frac{\partial Q_{ij}}{\partial y} \gamma_j, \]

\[ V_j = \frac{\partial Q_{ij}}{\partial x} \alpha_j + \frac{\partial^2 Q_{ij}}{\partial x^2} \beta_j + \frac{\partial^2 Q_{ij}}{\partial y \partial x} \gamma_j, \]

\[ U_j = \frac{\partial Q_{ij}}{\partial y} \alpha_j + \frac{\partial^2 Q_{ij}}{\partial x \partial y} \beta_j + \frac{\partial^2 Q_{ij}}{\partial y^2} \gamma_j, \]

where \( V \) and \( U \) represent the gradients of the height field \( H \), and the gradient of the multiquadric basis function is used in the interpolation equation. The extension of this approach to complete dynamic constraints may be possible and is under investigation. The size of the matrix to be solved increases substantially when three-dimensional constraints are to be applied. The computational stability has not been addressed in this case.

To summarize, a relatively simple application of the multiquadric interpolation method has been shown to be more accurate than either the Barnes (1973) or Cressman (1959) techniques, which is consistent with Franke (1982). Franke also found the multiquadric interpolation to be as accurate as statistical interpolation. A first guess or background can be incorporated into the scheme if desired. Application to meteorological observations that contain error can be done by includ-
ing a smoothing parameter in the interpolation equa-
tion. Meteorologically acceptable fields are obtained
with minimal smoothing. The analysis apparently re-
solves features on the smallest scales that are re-
presented by the observations in a particular region of the
analysis domain. Because the method is computa-
 tionally very efficient and well behaved in data-void
regions, the multiquadric technique is recommended
for local analysis of meteorological fields.

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