An Upwind-Biased Conservative Advection Scheme for Spherical Hexagonal–Pentagonal Grids

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ABSTRACT

A discrete form of the flux-divergence operator is developed to compute advection of tracers on spherical hexagonal–pentagonal grids. An upwind-biased advection scheme based on a piecewise linear approximation for one-dimensional regular grids is extended simply for spherical hexagonal–pentagonal grids. The distribution of a tracer over the upwind side of a cell face is linearly approximated using a nodal value and a gradient at a computational node on the upwind side. A piecewise linear approximation is relaxed to a local linear approximation, and the relaxation precludes the complicated conditional branching present in remapping schemes. Results from a cosine bell advection test show that the new scheme compares favorably with other upwind-biased schemes for spherical hexagonal–pentagonal grids.

1. Introduction

Several atmospheric general circulation models (AGCMs) use spherical hexagonal–pentagonal grids, which are characterized by a quasi-homogeneous grid cell distribution (Ringler et al. 2000; Majewski et al. 2002; Tomita et al. 2005). The efficient computational performance of such models on massively parallel computers has enabled global cloud-resolving simulations (Tomita et al. 2005; Miura et al. 2005, 2007). In AGCMs, water substances and wind fields interact through latent heating and advection, and aerosols affect atmospheric radiation and cloud formation. Conservation is one requirement in the advection of such tracers, and a second requirement is to eliminate physically impossible negative values.

Conservation of tracers can be achieved by using the finite-volume (FV) method to derive the discrete form of operators. Second-order-centered schemes have been commonly used (e.g., Tomita et al. 2001). However, second-order-centered schemes are influenced by numerical dispersion (e.g., Durran 1999), which can produce unphysical undershoots and overshoots.


This paper derives an upwind-biased advection scheme using a concept similar to that described in Thuburn (1997). The flux limiter of Thuburn (1995) ensures monotonicity, but unlike Thuburn (1997), regular hexagonal–pentagonal grids are not assumed because hexagonal grids are somewhat distorted on spherical hexagonal–pentagonal grids (Miura and Kimoto 2005). A linear function approximates the profile on the upwind side of a cell face, although Thuburn (1997) used a quadratic function.

The scheme developed in this study is a simple extension of scheme I (van Leer 1977) from one-dimensional grids to two-dimensional unstructured grids. This scheme is regarded as a simplification of the remapping schemes of Lipscomb and Ringler (2005) and Yeh (2007). A piecewise linear approximation in
the remapping schemes is relaxed to a “local linear approximation” in this study. This relaxation removes any complicated conditional branching, and only a determination of the upwind and downwind sides is required for each edge. Numerical results are equivalent to those of remapping schemes despite the simplicity, as shown in section 3.

2. An upwind-biased scheme

The time evolution of a tracer mixing ratio \( q \) is given in flux form by

\[
\frac{\partial}{\partial t} (\rho q) + \nabla \cdot (\rho q \mathbf{v}) = 0,
\]

where \( \rho \) and \( \mathbf{v} \) are fluid density and velocity, respectively. Figure 1a shows a hexagonal grid. Nodal values \( \rho_i \) and \( q_i \) are defined at computational nodes \( \mathbf{P}_i \). In the ZM-grid arrangement (Ringler and Randall 2002), vectors such as the fluid velocity \( \mathbf{v}_i \) are defined at cell corners \( \mathbf{Q}_i \).

The FV method is used to derive a discrete form of Eq. (1). If scalar and vector variables are approximated at cell face centers \( \mathbf{R}_i \), the time evolution of a tracer mixing ratio at a node \( \mathbf{P}_i \) can be approximated as

\[
\frac{\partial}{\partial t} (\rho_i q_i) = -\frac{1}{A_0} \sum_{i=1}^{N_i} (l_{ij}^i \rho_{ij} q_{ij} \mathbf{v}_{ij} \cdot \mathbf{n}_i),
\]

where \( A_0 \) is the area of the hexagonal or pentagonal cell with the node \( \mathbf{P}_i \), and \( N_i \) is the number of cells surrounding the cell. Cell face vectors are given by a midpoint rule as

\[
\mathbf{v}_{ij} = \frac{\mathbf{v}_i + \mathbf{v}_{i-1}}{2}.
\]

Approximating \( \rho_{ij} \) and \( q_{ij} \) is the focus of this study. Consider a distribution of \( q \) at a time level \( t \). The objective is to approximate the distribution of \( q \) at the next time level \( t + \Delta t \). Figure 1b assumes that the arc \( \mathbf{Q}_i \mathbf{Q}'_{i+1} \) at time \( t \) moves with a constant velocity \( \mathbf{v}^{i+\Delta t/2} \) and coincides with the arc \( \mathbf{Q}_i \mathbf{Q}'_{i+1} \) at time \( t + \Delta t \). The total amount of flux through the edge \( \mathbf{Q}_i \mathbf{Q}'_{i+1} \) during the time interval \( \Delta t \) is approximated by the amount of a tracer inside the parallelogram \( \mathbf{Q}_i \mathbf{Q}'_{i+1} \mathbf{Q}_{i+1} \mathbf{Q}_i \) as

\[
(l_{ij} \rho_{ij} q_{ij} \mathbf{v}^{i+\Delta t/2} \cdot \mathbf{n}_i)\Delta t = \int_{S_i} \rho q \, dS,
\]

where \( S_i = l_{ij} \mathbf{v}^{i+\Delta t/2} \cdot \mathbf{n}_i \Delta t \) is the area of the parallelogram. Equation (4) can be rewritten as

\[
\frac{\rho_{ij}}{\rho_{ij}} = \frac{\int_{S_i} \rho q \, dS}{S_i}.
\]

Profiles of \( \rho \) and \( q \) inside the parallelogram \( \mathbf{Q}_i \mathbf{Q}'_{i+1} \mathbf{Q}_{i+1} \mathbf{Q}_i \) can be approximated by two-dimensional linear surfaces, even if the arc \( \mathbf{Q}_i \mathbf{Q}'_i \) or \( \mathbf{Q}_{i+1} \mathbf{Q}'_{i+1} \) intersects cell faces such as \( \mathbf{Q}_{i-1} \mathbf{Q}_i \) or \( \mathbf{Q}_{i+1} \mathbf{Q}_{i+2} \). This assumption is called a local linear approximation in this study, and the assumption is in contrast to a piecewise linear approximation. The local linear approximation allows the right-hand side of Eq. (5) to be approximated as

\[
\frac{\rho_{ij}}{\rho_{ij}} = \rho_{ij} \mathbf{c}_i \mathbf{g}_i.
\]
where \( \rho_C \) and \( q_C \) are the density and the tracer mixing ratio, respectively, at the mass centroid of the parallelogram \( Q_i^+Q_{i+1}^-Q_{i+1}^+Q_i^- \). Similarly, the continuity equation will yield

\[
\rho_{R_i} = \rho_{C_i} \tag{7}
\]

Thus, Eqs. (6) and (7) mean that

\[
q_{R_i} = q_{C_i}. \tag{8}
\]

The position of \( C_i \) can be computed as

\[
C_i = R_i - \frac{\mathbf{n_r} \cdot \Delta \mathbf{t}}{2}. \tag{9}
\]

Linear surfaces inside the parallelogram \( Q_i^+Q_{i+1}^-Q_{i+1}^+Q_i^- \) can be approximated by nodal values and gradients at a computational node that shares the cell face \( Q_i^+Q_{i+1}^- \) and is on the upwind side of the cell face. For the case of Fig. 1b, \( R_b, q_b, (\nabla p)_R, \) and \( (\nabla q)_P \) can be used to approximate \( \rho_C \) and \( q_C \) as

\[
\rho_{C_i} = \rho_0 + (\nabla p)_{P_0} \cdot (C_i - P_0), \tag{10}
\]

and

\[
q_{C_i} = q_0 + (\nabla q)_{P_0} \cdot (C_i - P_0). \tag{11}
\]

Substitution of Eqs. (10) and (11) into Eqs. (7) and (8) yields first guesses of the density and the mixing ratio of a tracer at cell face center \( R_i \).

Monotonicity is maintained by adjusting the first guesses by the flux limiter of Thuburn (1995). Such adjusted values are denoted \( \tilde{\rho}_{R_i} \) and \( \tilde{q}_{R_i} \). Description of the flux limiter in Thuburn (1995) was sufficiently concise and it was also given in Thuburn (1996). Thus, description of the flux limiter is not repeated here. Forward time differencing can be used to transform Eq. (2) into

\[
\rho_{R_i}^{t+\Delta t} = \rho_0^{t} - \frac{\Delta t}{A_0} \sum_{j=1}^{N_i} (l_j \tilde{p}_{R_i} \tilde{q}_{R_i}^{t+\Delta t} \cdot \mathbf{n_i}). \tag{12}
\]

Gradients used in Eqs. (10) and (11) are computed using the finite-difference (FD) method of Stuhne and Peltier (1999) because the convergence rate of a second order is obtained regardless of distortions of hexagonal cells (Majewski et al. 2002; Miura 2004). The FV method (e.g., Tomita et al. 2001) can also be used to compute gradients, although the accuracy of computed gradients is less than that given by the FD method due to distortions of the hexagonal cells (Miura 2004).

3. Numerical results

Test case 1 of Williamson et al. (1992) was performed to examine the behavior of the upwind-biased scheme. Thuburn (1997), Lipscomb and Ringler (2005), and Yeh (2007) also used this test case. The nondivergent wind field is

\[
u = \bar{u}_0 \cos \theta \cos \alpha + \cos \lambda \sin \theta \sin \alpha \quad \text{and,} \tag{13}
\]

\[
v = -\bar{u}_0 \sin \lambda \sin \alpha, \tag{14}
\]

where \( \lambda \) is longitude, \( \theta \) is latitude, and \( \alpha \) is the angle between the axis of solid body rotation and the pole axis of the spherical coordinate system. Only results with \( \alpha = 0 \) are shown because errors are hardly sensitive to \( \alpha \). The maximum wind speed is \( \bar{u}_0 = 2\pi \alpha/(12 \text{ days}) \), where \( \alpha \) is the earth radius. An initial cosine bell is given by

\[
h(\lambda, \theta) = \begin{cases} 
(h_0/2)[1 + \cos(\pi r/R)] & \text{if } r < R \\
0 & \text{if } r = R,
\end{cases} \tag{15}
\]

where \( h_0 = 1000 \text{ m}, R = a/3, \) and \( r \) is the great circle distance between an arbitrary point and the center of the bell initially at \((\lambda_c, \theta_c) = (3\pi/2, 0)\). The integration duration is 12 days, during which time the cosine bell rotates once around the earth. Three error norms are used to evaluate numerical errors:

\[
l_1(h) = \frac{\sum_{\text{cells}} A_i |h_i - h_i^{\text{true}}|}{\sum_{\text{cells}} A_i |h_i^{\text{true}}|}, \tag{16}
\]

\[
l_2(h) = \frac{\sqrt{\sum_{\text{cells}} A_i (h_i - h_i^{\text{true}})^2}}{\sqrt{\sum_{\text{cells}} A_i (h_i^{\text{true}})^2}}, \tag{17}
\]

and

\[
l_{\text{inf}}(h) = \frac{\max_{\text{cells}} |h_i - h_i^{\text{true}}|}{\max_{\text{cells}} |h_i^{\text{true}}|}, \tag{18}
\]

where \( h_i \) is a computed solution, and \( h_i^{\text{true}} \) is the exact solution.

The spherical hexagonal–pentagonal grids used in this study were described in Miura and Kimoto (2005). After recursive subdivisions starting from an icosahedron, the algorithm of Du et al. (2003) is used to optimize intersection points of a triangular grid. Then, hexagonal–pentagonal cells are generated by defining cell corners at barycenters of triangles. Computational nodes are defined at centroids of hexagonal–pentagonal cells. The spherical hexagonal–pentagonal grids constructed according to the procedure were called barycentric-type spherical-centroidal-Voronoi (BT SCV) grids in Miura and Kimoto (2005). BT SCV grids were selected because distortions of hexagonal cells are reasonably small and the ratio between the
maximum and the minimum cell areas is not large. Computations were performed on grids with $N_c = 2562, 10 242, 40 962,$ and 163 842; $N_c$ is the number of hexagonal-pentagonal cells over the globe. The grids can be called as glevel 4, 5, 6, and 7, respectively, following the notation rules in Tomita et al. (2001). Time intervals were $\Delta t = 7200, 3600, 1800,$ and 900 s, respectively. Variables were arranged as on the ZM grid (Ringler and Randall 2002).

Figure 2 shows the computed solutions, the exact solution, and the errors. The shape of the cosine bell is preserved in all directions and spurious overshoots or undershoots do not develop. Results are almost equivalent to those of Thuburn (1997), and are also comparable to those of Yeh (2007). The height with $N_c = 10 242$ is slightly less eroded than that with “grid 6” in Thuburn (1997). This change is mostly caused by a difference in the time intervals used rather than by differ-

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**Figure 2.** Advection test results. (a) $N_c = 10 242$. Computed solution (solid contours) and exact solution (dashed contours). (b) $N_c = 10 242$. Computed solution minus exact solution. (c) Same as in (a), but for $N_c = 40 962$. (d) Same as in (b), but for $N_c = 40 962$. 
ences in the grids used. Computation with \( N_x = 10242 \) and \( \Delta t = 1800 \) s, settings that are the same as in Thuburn (1997), mostly reproduced results shown in Thuburn (1997).

If wind speed is assumed constant on one-dimensional grids, as in Durran (1999), the phase speed and damping factor of scheme I (van Leer 1977) are

\[
\frac{\text{Re}(\omega)}{c} = \frac{\sin(k\Delta x)}{2\Delta x} \left[ (3 - \sigma) - (1 - \sigma) \cos(k\Delta x) \right],
\]

(19)

and

\[
\frac{\text{Im}(\omega)}{c} = -\frac{1}{\Delta x} \left[ \frac{1}{4} \frac{(3 + \sigma) + (1 - \sigma) \cos(2k\Delta x)}{4} - \cos(k\Delta x) \right],
\]

(20)

where \( k \) is the wavenumber, \( \omega \) is the frequency, \( c \) is a wind velocity, \( \Delta x \) is a grid interval, and \( \sigma = c\Delta t/\Delta x \) is a Courant number. Figure 3 compares phase speed error and amplitude error from scheme I with errors from the first- and third-order upwind schemes (Durran 1999). Both the phase error and the amplitude error of scheme I depend on time intervals as discussed in van Leer (1977).

Figure 4a shows that error norms compare well with those shown by Yeh (2007), and are better than those shown by Lipscomb and Ringler (2005). Numerical convergence rates of \( l_1 \) and \( l_2 \) errors are better than second order. If the flux limiter is not used, the upwind scheme is second-order accurate as shown in Fig. 4b. The faster numerical convergences are a benefit of the flux limiter.

Relatively larger errors in Lipscomb and Ringler (2005) may be due to shorter time intervals than those used in this study or by Yeh (2007), as inferred from the overestimated phase speed in Fig. 8a of Lipscomb and
Ringler (2005). Van Leer (1977) notes that scheme I has “zero-average phase error” for $\sigma = 0.5$. The inference can be confirmed by computations with the time interval $\Delta t = 50$ s, which is the same as that used by Lipscomb and Ringler (2005; T. D. Ringler 2006, personal communication). The phase speed is overestimated with $\Delta t = 50$ s (Fig. 5) and errors are larger than those shown in Fig. 4a (Fig. 6).

4. Summary

An upwind-biased tracer advection scheme was derived for spherical hexagonal–pentagonal grids. The method of scheme I (van Leer 1977) was applied to spherical hexagonal–pentagonal grids. Results from a numerical test compared well with those derived from more sophisticated algorithms. In the scheme developed here, a quadratic profile (Thuburn 1997) was simplified to a linear profile and a piecewise linear approximation in remapping schemes (Lipscomb and Ringler 2005; Yeh 2007) was relaxed to a local linear approximation. Slope limiters like those used in the remapping schemes cannot be used because of the local linear approximation. However, the flux limiter of Thuburn (1995) worked very well in preserving monotonicity.

The new scheme allows distorted hexagonal or pentagonal cells to be treated naturally and the local linear approximation eliminates the conditional branching in the computations required in remapping schemes. A further advantage may be extensibility to three-dimensional space without dimensional splitting. All that is additionally required is a computation of the vertical component of the gradient. The mass centroid of a parallelepiped on the upwind side can be estimated for each cell face using three-dimensional wind components. In this study, the ZM grid is assumed to define vector positions. However, the A grid is also available. Cell face wind components can be approximated using second-order accurate interpolations (e.g., Miura 2004).

On unstructured grids, monotone upstream-centered schemes for conservation laws (MUSCL)-type finite-volume schemes are widely used to solve systems of equations (e.g., Hubbard 1999). In addition, higher-resolution methods such as the weighted essentially nonoscillatory (WENO) scheme have been extended to unstructured grids (Friedrich 1998). These schemes can resolve shocks sharply; however, large computational costs are required for characteristic decompositions and Riemann solvers. It is attractive to apply such high-resolution methods for spherical hexagonal–pentagonal grids. Meanwhile, in meteorological models, it is uncertain whether discontinuities generated by subgrid-scale parameterizations should be resolved sharply at the expense of large computational costs. The present scheme is not so sophisticated, but it may be useful because of its simplicity.

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