A Comparison of Compressible and Anelastic Models of Deep Dry Convection

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ABSTRACT

The response to an instantaneous diabatic warming and the resulting hydrostatic and geostrophic adjustment in compressible and anelastic models is examined. The comparison of the models includes examining the initial conditions, time evolution, potential vorticity, and both the traditional and available energetics. Between the two models, the buoyancy flow fields and potential vorticity perturbations are qualitatively and quantitatively similar. Traditional and available energetics can both be accurately conserved within the models. There are some short-lived (e.g., several minutes) differences in the model solutions as the compressible model undergoes an acoustic adjustment that contains vertically propagating acoustic waves and horizontally propagating Lamb waves. The acoustic waves are effectively eliminated in an upper-level numerical sponge layer using Rayleigh damping. Moreover, the relative computational efficiency and accuracy of the two models are assessed.

1. Introduction

Atmospheric convection is nonhydrostatic. For example, for deep convective phenomena such as thunderstorms, the vertical pressure gradient, and buoyancy forces are not in balance. Because the depth scale of deep convection is comparable to the density-scale height, the use of compressible and/or anelastic models is warranted. The compressible model contains the acoustic, Lamb, and buoyancy waves; the anelastic model retains only the buoyancy waves.

The presence of acoustic and Lamb waves in the compressible model places severe limits on the time step in the numerical integration of the model because of their large speed, hence, decreasing computational stability and efficiency. The Courant–Friedrichs–Lewy (CFL) stability criterion (e.g., Holton 2004) places an upper bound on the time step \( \delta t \), in a model using centered differencing: namely,

\[
\delta t \leq \frac{\delta x}{c},
\]

(1.1)

where \( \delta x \) is a spatial grid increment and \( c \) is the speed of the fastest wave. Qualitatively this criterion states that for a given increment \( \delta x \), a time step must be chosen so that the Lamb wave is advected a distance less than one \( \delta x \) per time step. This implementation of shorter time steps requires longer model run times. Hence, alternative time differencing methods such as implicit-differencing (e.g., Ooyama 2001) and time-splitting schemes (e.g., Klemp and Wilhelmson 1978) are often used to increase the time steps and shorten the run times in compressible models. These alternatives allow for larger time steps that ease the computational strain and enable the model to run faster.

To circumvent the computational burden of a compressible model, Ogura and Phillips (1962) introduced the anelastic approximation for studies of deep convection that filters acoustic and Lamb waves from the equations of motion. Their approximation is valid for flows whose time scale is the inverse of the buoyancy frequency or greater. Numerous alternative anelastic formulations have appeared in the literature (Dutton and Fichtl 1969; Gough 1969; Wilhelmson and Ogura 1972; Clark and Peltier 1977; Lipps and Hemler 1982, 1985; Durran 1989; Lipp 1990; Bannon 1996).

In this study, the compressible, nonhydrostatic, finite-difference cloud model of Bryan and Fritsch (2002) is compared to its anelastic formulation (Bannon et al. 2006). In each model, an instantaneous diabatic warming is used to excite convection in an initially resting standard atmosphere base state.

Section 2 outlines the numerical framework of the
compressible and anelastic models. It also gives the quantitative structure of the diabatic warming, the models’ equations, and the numerical techniques used. Section 3 provides a comparison of the compressible and anelastic model responses to the instantaneous diabatic warming. The study examines the model responses at the initial conditions and at 1 and 10 min. Moreover, the structure of the Lamb wave packet excited in the compressible model is examined. This section also briefly discusses the source terms in the anelastic diagnostic pressure equation and gives a description of the evolution of the vertical velocity at the center of the warm bubble. Sections 4 and 5 analyze the responses at the initial conditions and at 1 and 10 min.

2. Numerical models

The compressible and anelastic models are nonlinear, dry, inviscid, and rotating on an f plane in Cartesian coordinates. This section discusses their governing equations, the structure of the diabatic warming, the model’s base state, the numerical techniques, and the boundary conditions.

a. Compressible equations

The compressible model equations are

\[
\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - c_p \theta \nabla \pi - f \mathbf{k} \times \mathbf{u} - g \mathbf{k},
\]

\[
\frac{\partial \theta}{\partial t} = -\mathbf{u} \cdot \nabla \theta + \Theta,
\]

\[
\frac{\partial \pi}{\partial t} = -\mathbf{u} \cdot \nabla \pi - \frac{R}{c_v} \mathbf{v} \cdot \mathbf{u} + \frac{R \pi}{c_v \theta},
\]

\[
T = \theta \pi, \quad \text{and}
\]

\[
p = \rho RT,
\]

where \( f \) is the Coriolis parameter, \( g \) is the acceleration due to gravity, \( \nabla \) is the three-dimensional del operator, and \( \mathbf{k} \) is the unit vector in the z direction. The variables \( \mathbf{u}, \rho, p, \theta, \) and \( T \) are the three-dimensional velocity vector \( (u, v, w) \), the density, the pressure, the potential temperature, and the temperature, respectively. This study is two dimensional and all variables are assumed independent of the y direction. The constants \( R, c_p, \) and \( c_v \) are the ideal gas constant and the specific heats of dry air at constant pressure and constant volume, respectively. The parameter settings are \( f = 10^{-4} \text{s}^{-1}, g = 9.81 \text{ m s}^{-2}, R = 287 \text{ J kg}^{-1} \text{K}^{-1}, c_p = 1004 \text{ J kg}^{-1} \text{K}^{-1}, \) and \( c_v = 717 \text{ J kg}^{-1} \text{K}^{-1}. \) In the momentum Eq. (2.1), the nondimensional pressure \( \pi \) is substituted for pressure:

\[
\pi = \left( \frac{p}{p_{00}} \right)^{R/c_p},
\]

where \( p_{00} = 1000 \text{ hPa}. \) The last term in (2.2) is the prescribed diabatic warming rate. Equations (2.1)–(2.3) are prognostic equations for velocity, potential temperature, and the nondimensional pressure and form a closed system of equations.

b. Anelastic equations

The anelastic equations (Bannon 1996) are

\[
\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \left( \frac{p'}{\rho_s} \right) + \frac{g \theta'}{\theta_s} \mathbf{k} \times \mathbf{u},
\]

\[
0 = \nabla \cdot (\rho_s \mathbf{u}),
\]

\[
\frac{\partial \theta'}{\partial t} = -\mathbf{u} \cdot \nabla (\theta_s + \theta') + \Theta,
\]

\[
\frac{\theta'}{\theta_s} = \frac{p'}{\rho_s g H_{\nu}} - \frac{\rho'}{\rho_s}, \quad \text{and}
\]

\[
\frac{p'}{\rho_s} = \frac{\rho'}{\rho_s} + \frac{T'}{T_s},
\]

where \( \theta', p', \rho', \) and \( T' \) are the perturbations of potential temperature, pressure, density, and temperature, respectively; and \( \theta_s, \rho_s, \rho_v, \) and \( T_s \) are the base-state potential temperature, pressure, density, and temperature, respectively. The total field of each variable is the sum of its base state and perturbation. As articulated by Bannon (1996), the anelastic closure in (2.10) involves the base-state density-scale height \( H_{\nu} \) Bannon (1996) discusses the underlying assumptions made in deriving the anelastic equations.

Bannon et al. (2006) developed the formulation for the anelastic model used in this study. This study solves an elliptic diagnostic pressure equation that here takes the two-dimensional form:

\[
\frac{\partial}{\partial x} \left( \rho_s \frac{\partial \Phi_d}{\partial x} \right) + \frac{\partial}{\partial z} \left( \rho_s \frac{\partial \Phi_d}{\partial z} \right) = -\left[ \frac{\partial}{\partial x} (\rho_s \mathbf{u} \cdot \nabla \mathbf{u}) + \frac{\partial}{\partial z} (\rho_s \mathbf{u} \cdot \nabla \mathbf{w}) - \frac{\partial}{\partial x} (\rho_s g \frac{\theta'}{\theta_s}) \right],
\]

where \( \Phi_d = p'/\rho_s \) is the dynamic potential. The first three terms in the square brackets are dynamic source terms and the last term is the buoyancy source term of the potential. Because there is no variation in y, the y derivatives of these two terms are not present.
c. Diabatic warming

The diabatic warming on the right side of the potential temperature equations is

$$\Theta = \Delta \theta \cos^2 \left( \frac{\pi R}{2} \right) \delta(t), \quad \text{for} \quad R \leq 1, \quad (2.13)$$

where $\Delta \theta = 4$ K is the amplitude of the warming and $\delta(t)$ is the Dirac delta function in time $t$. This perturbation is instantaneously forced at the initial time $t = 0$. The cosine-squared function embodies the horizontal width and vertical depth of the diabatic warming. Here

$$R = \left[ \left( \frac{x - x_0}{x_r} \right)^2 + \left( \frac{z - z_0}{z_r} \right)^2 \right]^{1/2}, \quad (2.14)$$

where the parameters $x_0 = 0$ and $z_0 = 3.5$ km are the horizontal and vertical locations, respectively, of the center of the bubble. The parameters $x_r = 10$ km and $z_r = 2.5$ km are the horizontal and vertical radii of the diabatic warming, respectively.

d. Standard atmosphere base state

The base state is a standard atmosphere at rest with a surface pressure $p_\ast = 1000$ hPa. The surface pressure $p_\ast$ is 1000 hPa.

e. Numerical techniques

These finite-difference models are integrated with third-order Runge–Kutta time-differencing and fifth-order spatial derivatives. The advection terms are written as the sum of a flux term and a divergence term. A more thorough discussion of these techniques is given in Bryan and Fritsch (2002). Here the time step is 0.30 s and the grid increment is 200 m in both the horizontal and vertical. The effects of time splitting with explicit and implicit differencing of the compressible model are discussed in section 6. Both models were run for a 20-min simulation and the fields were saved every 15 s.

The two-dimensional numerical model domain is 30 km in height $z$ and 400 km in horizontal distance $x$. The lateral boundary conditions in the compressible model are open radiative at $x = 400$ km, but are symmetric about the boundary at $x = 0$. In the anelastic model the lateral boundary conditions are open radiative at $x = -200$ km and $x = 200$ km. In all simulations, there is no variation in the $y$ direction. Rigid boundary conditions are set at $z = 0$ and 30 km in the vertical. In both models, a Rayleigh damping layer (Bryan et al. 2003) is applied above 15 km. The damping rate increases sinusoidally from a minimum of zero at and below 15 km to a maximum value of $0.2 \, \text{s}^{-1}$ at $z = 30$ km. This layer effectively damps vertically propagating acoustic modes that enter the model stratosphere. Because this damping is a computation artifact, results for the fields presented here are restricted to the lower 15 km of the model domain. The effect of the Rayleigh damping is discussed in section 6.

3. Temporal evolution of the fields

a. Initial conditions

The initial conditions are defined as a snapshot of the thermodynamic fields before the flow begins; hence, the three-dimensional wind vector perturbations ($u^\prime, v^\prime, w^\prime$) are zero (not shown). The perturbations within the initial conditions are produced through the instantaneous diabatic warming (defined in section 2c). Subsequently, the process of hydrostatic and geostrophic adjustment occurs. It is important to note that the initial conditions in the two models differ.

Figure 1 presents the perturbation flow field initial conditions for the compressible model. Here perturbation is defined as the departure from the base-state value before the warming. By (2.2), the instantaneous diabatic warming, $\Theta$, produces a potential temperature perturbation of 4 K (Fig. 1a) and, by (2.3) and (2.6), a nondimensional pressure perturbation (Fig. 1b). Together, by (2.2) and (2.4), the potential temperature and nondimensional pressure perturbations produce a temperature perturbation (Fig. 1c). Because there is no motion in the model domain, there is no advection of mass. Hence, there is no perturbation in density (Fig. 1d). Therefore, the cloud model is indeed nonhydrostatic. All of the compressible perturbations are confined to the region of the warming where $x = 10$ km and $1 \leq z \leq 6$ km.

Figure 2 presents the anelastic model initial conditions. By (2.9), the diabatic warming again produces a 4-K potential temperature perturbation (Fig. 2a). This perturbation induces an anelastic pressure perturbation (Fig. 2b) through the thermal source term in (2.12) that differs from its compressible counterpart (Fig. 1b). (To aid comparison with the compressible case, the anelastic thermodynamic perturbations are displayed here and elsewhere without the zero horizontal wavelength far-field solution of Bannon et al. 2006). By (2.10), the arithmetic difference between the potential temperature and the pressure perturbation yields the density perturbation (Fig. 2d) and then (2.11) yields the tem-
FIG. 1. Compressible initial perturbations of (a) potential temperature with a contour interval of 0.5 K, (b) pressure with an interval of 250 Pa, (c) temperature with an interval of 0.5 K, and (d) density with an interval of 0.5 g m$^{-3}$. Here and elsewhere the solid contours denote positive values and the zero contours are omitted.

FIG. 2. As in Fig. 1, but for the anelastic initial perturbations. Intervals are (a) 0.5 K, (b) 15 Pa, (c) 0.5 K, and (d) 0.30 g m$^{-3}$. Here and elsewhere the dashed contours denote negative values.
perature perturbation field (Fig. 2c). The anelastic temperature perturbation has a maximum of 3.51 K that is less than the 4.96-K maximum of the compressible model, but both fields are confined to the region of the warming.

The anelastic pressure and density perturbations depict dipole structures (Figs. 2b,d) that are not confined to the region of the diabatic warming. These features reflect the effects of the instantaneous acoustic adjustment occurring only in the anelastic model. The structure of the anelastic pressure perturbation is considerably different from its compressible counterpart (Fig. 1b). By (2.13), the vertical gradient of the potential temperature perturbation gradient changes sign at \( z = 3.5 \) km. This change induces the dipole in the pressure perturbation and, by (2.10), the density perturbation.

There are also differences in the magnitudes of the thermodynamic fields. For example, the compressible pressure perturbation has a maximum value of 1228 Pa (Fig. 1a), which is much larger than the anelastic perturbation extremum of \(-170\) Pa (Fig. 2a). There are salient differences in the density perturbation (Fig. 2b) where it is virtually zero in the compressible case but it remains small but nonzero (a \(-12\, \text{g m}^{-3}\) extremum) for the anelastic case.

b. Temporal evolution at 1 min

Figures 3 and 4 present the flow perturbations after 1 min for the compressible and anelastic models, respectively. Below 10 km and within 15 km of the origin, the fields for the two cases are similar, qualitatively and quantitatively. A dipole pressure structure has developed in the compressible case (Fig. 3a) with a maximum of over 160 Pa and a minimum of \(-185\) Pa that is similar to that of the anelastic solution (Fig. 4a) with a maximum of 77 Pa and a minimum of \(-159\) Pa. The other thermodynamic fields (cf. Figs. 3b,d,f and 4b,d,f) are also in good agreement. The buoyancy driven circulations (cf. Figs. 3c,e and 4c,e) both show updrafts near the origin and subsidence around \( x = 8 \) km with outflow aloft and low-level inflow.

Above 10 km and beyond 15 km of the origin, the fields for the two cases are strikingly different. The anelastic solution (Fig. 4) shows little response in this outer region. In contrast, the largest pressure perturbation occurs there for the compressible solution (Fig. 3a). The arching positive perturbation in this field is composed of a horizontally propagating Lamb wave packet near \( x = 22 \) km with outflow (Fig. 3c) and vertically propagating acoustic modes near the origin with dipole structures in the pressure, temperature, and vertical velocity perturbations.

c. Temporal evolution at 10 min

Figures 5 and 6 present the flow perturbations within 50 km of the origin after 10 min for the compressible and anelastic solutions, respectively. Again the agreement is striking. Each panel pair (e.g., Figs. 5a and 6a) display the same major features and exhibit the same number of contours around each extremum. The values of the extrema differ by only a few percent or less. Each thermodynamic pair shows similar distortion around \( z = 11 \) km because of the tropopause.

Alternating patterns of rising and sinking motion (Figs. 5c and 6c) are intermittent throughout the two fields. The rising motion is correlated with the elongated positive pressure perturbations (Figs. 5a and 6a) and the sinking motion is correlated with the negative pressure perturbation. These and the other perturbation fields (i.e., density, horizontal velocity, temperature, and potential temperature) are more strongly modulated by the buoyancy waves. Indeed, the excellent agreement between the fields indicates that the acoustic signal is virtually nonexistent in this region in the compressible solution. The agreement of the density and temperature perturbation pairs support the validity of the closure [see (2.10)] of Bannon (1996).

As an objective measure of the agreement in the solutions, Fig. 7 prevents the difference between the fields of the anelastic solution in Fig. 6 and the compressible fields of Fig. 5. The contour interval (CINT) in each panel is an order of magnitude less than that in Figs. 5 and 6. Also presented above each panel is the rms difference (rms) between the two fields and their signal-to-noise ratio (SNR). The small rms and large SNR values (evaluated over the domain of the figure) indicate the close agreement in the solutions overall. The pressure and density differences (Figs. 7a,b) indicate a stronger near-surface low with reduced density for the anelastic solution. Figures 7c,e suggest a distortion in the \( n = 1 \) buoyancy mode with a slightly larger updraft for the anelastic solution. The temperature and potential fields (Figs. 7d,f) exhibit the greatest distortion above the tropopause.

1) Lamb wave

The compressible and anelastic solutions differ in the region beyond 100 km from the origin. Figure 8 presents the contour plots of the Lamb wave packet perturbations at 10 min for the compressible model. [There is no Lamb wave packet in the anelastic case because the anelastic continuity Eq. (2.8) has filtered them out.] This double maximum Lamb packet has propagated a considerable distance (~200 km) from the diabatic warming, implying a speed of about 333 m s\(^{-1}\). As in
FIG. 3. Compressible perturbations at time $t = 1$ min of (a) pressure with an interval of 25 Pa, (b) density with an interval of 1 g m$^{-3}$, (c) horizontal velocity in the $x$ direction with an interval of 0.25 m s$^{-1}$, (d) temperature with an interval of 0.25 K, (e) vertical velocity with an interval of 0.25 m s$^{-1}$, and (f) potential temperature with an interval of 0.25 K.

FIG. 4. As in Fig. 3, but for the anelastic perturbations at time $t = 1$ min. Intervals are (a) 25 Pa, (b) 1 g m$^{-3}$, (c) 0.25 m s$^{-1}$, (d) 0.25 K, (e) 0.25 m s$^{-1}$, and (f) 0.25 K.
FIG. 5. As in Fig. 3, but for the compressible perturbations at time $t = 10$ min. Intervals are (a) 10 Pa, (b) 1 g m$^{-3}$, (c) 0.5 m s$^{-1}$, (d) 0.25 K, (e) 0.25 m s$^{-1}$, and (f) 0.25 K.

FIG. 6. As in Fig. 3, but for the anelastic perturbations at time $t = 10$ min. Intervals are (a) 10 Pa, (b) 1 g m$^{-3}$, (c) 0.5 m s$^{-1}$, (d) 0.25 K, (e) 0.25 m s$^{-1}$, and (f) 0.25 K.
FIG. 7. As in Fig. 6, but for the difference (anelastic minus compressible) fields at time $t = 10$ min. Intervals are a tenth of that in Fig. 6. The contour interval, rms difference, and SNR are indicated.

FIG. 8. Lamb wave packet perturbations at $t = 10$ min of (a) pressure with an interval of 10 Pa, (b) density with an interval of 0.05 g m$^{-3}$, (c) horizontal velocity in the $x$ direction with an interval of 0.025 m s$^{-1}$, (d) temperature with an interval of 0.01 K, (e) vertical velocity with an interval of 0.01 m s$^{-1}$, and (f) potential with an interval of 0.005 K.
Fanelli and Bannon (2005), the highest pressure (Fig. 8a) and density (Fig. 8b) perturbations are at the lower boundary and the maximum horizontal wind exists aloft (Fig. 8c). Because the base state is nonisothermal, the vertical velocity (Fig. 8e) is nonzero and the upward motion produces a weak negative potential temperature perturbation (Fig. 8f).

2) Elliptic Diagnostic Pressure Equation

To examine the relative importance of dynamical versus buoyancy processes in the flow fields of Figs. 5 and 6, the anelastic diagnostic pressure in Eq. (2.12) is evaluated. Figure 9 presents the x and z gradients of the convective dynamical source terms (Figs. 9a,b) and the z-gradient thermal source term (Fig. 9c) and the Coriolis term (Fig. 9d). The sum of these terms (Fig. 9e) contributes to the pressure perturbation (Fig. 9f). The most significant contribution to the pressure perturbation is the thermal or buoyancy term. The dynamic source terms have maximum values on the order of $10^{-6}$ kg m$^{-3}$ s$^{-2}$, but the maximum values of the thermal source term are on the order of $10^{-3}$ kg m$^{-3}$ s$^{-2}$. The positive total source (Fig. 9e) at $x = 10$ km is collocated with a positive perturbation. There is a negative total source directly above at $z = 2.5$ km that is in the same position as a negative pressure perturbation. The sign continues to switch with increasing height with a negative total source near the tropopause above the center of the diabatic warming.

To gain insight into the relationship between the source and the response, (2.12) can be approximated by

$$\nabla^2 p' = -4\pi S,$$

where the response is $p'$, the pressure perturbation diagnosed from the total source $S$. Assuming a sinusoidal solution of the form $p' \propto \exp[i(kx + mz)]$, where $k$ and $m$ are the horizontal and vertical wavenumbers, respectively, then the pressure perturbation can be approximated by $p' = 4\pi S/k^2$, where $\kappa^2 = k^2 + m^2$. Furthermore, let $\kappa = 2\pi/L$ then $p' = L^2S/\pi$, where $L$ is a measure of the length scale of the source. Thus, the response of the source terms projects more strongly onto the larger scales of the pressure perturbation. In this case, the response is the pressure perturbation (Fig. 9f) and its horizontal extent is to $x = 40$ km but the $L$ of the total source (Fig. 9e) is approximately 25 km.

![Fig. 9. Contributions of the anelastic elliptic diagnostic pressure in (2.12) at t = 10 min. The (a) horizontal and (b) vertical components of the dynamical source terms with intervals of $2.5 \times 10^{-7}$ kg m$^{-3}$ s$^{-2}$, (c) the thermal source term with an interval of $5.0 \times 10^{-6}$ kg m$^{-3}$ s$^{-2}$, (d) the Coriolis term with an interval of $1.0 \times 10^{-6}$ kg m$^{-3}$ s$^{-2}$, and (e) the sum of the source terms with an interval of $5.0 \times 10^{-6}$ kg m$^{-3}$ s$^{-2}$, and (f) the pressure perturbation with an interval of 10 Pa.](image)
**d. Vertical velocity at the center of the diabatic warming**

Comparisons of Figs. 3 and 4 and Figs. 5 and 6 indicate that an acoustic adjustment is present at $t = 1$ min, but vanishes within 50 km of the origin by $t = 10$ min. To assess the time scale of this adjustment, Figs. 10 and 11 present the time evolution of the vertical velocity at the center of the diabatic warming for the compressible

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**FIG. 10.** Compressible vertical velocity at the horizontal center of the diabatic warming as a function of time and vertical location. The contour interval is 0.5 m s$^{-1}$.

**FIG. 11.** As in Fig. 10, but for the anelastic vertical velocity at the horizontal center of the diabatic warming.
and anelastic solutions, respectively. Beyond \( t = 4 \) min, the solutions are in excellent agreement. Maxima in the vertical velocity occur at \( t = 4 \) and \( 15 \) min. Each of these extrema increases with height and decrease in magnitude. The period of these oscillations is about 12 min and slightly longer than the tropospheric buoyancy period of 9.3 min, indicating that the extrema are associated with buoyancy waves and are not pure oscillations.

The solutions differ for times before 4 min. The compressible solution exhibits the signature of hydrostatic adjustment (Bannon 1995) of two dipoles of vertically propagating acoustic waves. The periods of 1 min or less confirm that these waves are acoustic. This period is much shorter than the maximum buoyancy period. Model runs without the sponge layer (not shown) show these acoustic modes reflecting off the vertical boundary. Their absence at times later than 4 min indicate they are effectively absorbed in the spongy stratosphere of the model.

4. Potential vorticity

The potential vorticity (PV) perturbations are analyzed in this section at 0, 5, 10, and 20 min into the simulation. The PV in the two-dimensional compressible model is

\[
PV_c = \frac{1}{\rho} \left[ \left( f + \frac{\partial v}{\partial x} \right) \frac{\partial \theta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial \theta}{\partial x} \right]. \tag{4.1}
\]

but the anelastic PV is only a function of the model base-state density

\[
PV_a = \frac{1}{\rho_s} \left[ \left( f + \frac{\partial v}{\partial x} \right) \frac{\partial \theta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial \theta}{\partial x} \right]. \tag{4.2}
\]

It is important to note that the vertical gradient in the potential temperature perturbation dominates the PV. Because \( v \) and the horizontal gradient in potential temperature are small, the last terms in (4.1) and (4.2) are small.

Figure 12 presents the compressible PV perturbations. The diabatic warming generates a vertical dipole of PV perturbation with positive (negative) values below (above) the maximum warming at \( t = 0 \) (initial conditions). The strength of this dipole weakens slightly with time. The magnitudes for the compressible PV within the dipole are approximately 3.0 K m\(^2\) s\(^{-1}\) kg\(^{-1}\) (negative) and 2.0 K m\(^2\) s\(^{-1}\) kg\(^{-1}\) (positive) at 10 min. In addition, the dipole is distorted in shape by advection.
and the negative (positive) perturbation is lengthened (shortened) by the high-level outflow (low-level inflow) from the region of the warming.

There are localized PV perturbations at the tropopause around \( z = 11 \) km, which are the result of undulations produced by buoyancy waves that evolve with time. For example, at 5 min there is a negative perturbation that extends from the origin to \( x = 12 \) km and a positive perturbation beyond 15 km. By 10 min, the perturbations have changed sign and size. Especially at 20 min (Fig. 12d) there are positive and negative perturbations extending into the stratosphere.

Inspection of the anelastic PV perturbations (not shown) at the same times as those in Fig. 12 indicates a close correspondence. The two perturbations are identical at the initial time because the potential temperature fields are identical and the anelastic density anomaly does not enter into the expression for the anelastic PV in (4.2), but the compressible density anomaly is zero (Fig. 1d). The anelastic model captures all the same structures as those in the compressible model. This agreement reflects the fact that the potential temperature fields are similar and that the density perturbations in the compressible model are small. This agreement further implies that the velocity fields in the y direction (not shown) are similar in the two models.

The primary differences in the PV are isolated to the immediate region (within 200 m) of the tropopause where the differences in the potential temperature (see Fig. 7f) are largest. For experiments with an isothermal atmosphere for which the base-state static stability is uniform, the tropopause undulations are absent, but still the compressible and anelastic solutions are essentially identical.

5. Energetics

This section describes the traditional and available energetics for each model. Traditional energetics includes potential energy (PE) and internal energy (IE). The available energetics includes available potential energy (APE) and available elastic energy (AEE). Kinetic energy (KE) is present in both energy types. For each model, the total energy (TE) of either type is conserved to within \( 10^{-5}\% \) in the absence of the sponge layer.

a. Traditional energetics

The TE for the compressible model is

\[
\text{Total Compressible Energy} = \int \rho \left[ \frac{\mathbf{u} \cdot \mathbf{u}}{2} + gz + c_p \theta \left( \frac{\rho}{\rho_0} \right)^{\frac{R}{c_p}} \right] \, dx \, dz, \tag{5.1}
\]

and that for the anelastic model is

\[
\text{Total Anelastic Energy} = \int \rho_s \left[ \frac{\mathbf{u} \cdot \mathbf{u}}{2} + gz + c_p T_s (\theta_s + \theta') \right] \, dx \, dz, \tag{5.2}
\]

where the integrals are over the model domain. In each case, the traditional TE is the sum of the KE, PE, and IE.

Figures 13a and 14a present the compressible and anelastic traditional energy perturbations as a function of time. In both cases, the KE is initially zero and the TE generated by the diabatic warming resides in the IE. The compressible TE (\( 36 \times 10^9 \) J m\(^{-1}\)) is comparable to that of the anelastic TE (35 \( \times 10^9 \) J m\(^{-1}\)). The compressible TE drops during the first several minutes, but the anelastic TE is conserved. This difference is attributed to the dissipation of the vertically propagating acoustic modes by the sponge layer in the compressible model while the anelastic model contains no such waves.

In the compressible case, the initial IE is converted into PE as the bubble ascends with a smaller conversion into KE. This case also displays a damped oscillation, with a period of about 2 min that exists in the IE and PE. However, these two phenomena are 180° out of phase. When there is a rise in the PE, the IE decreases. Rising motion leads to an increase in mass aloft and adiabatic cooling, hence, increasing the PE and decreasing the IE. The opposite holds for sinking motion. These fluctuations initially occur on the order of 10 \( \times 10^9 \) J m\(^{-1}\). A similar oscillatory behavior was also noted in Edson and Bannon (2008). In model runs without the sponge layer (not shown), the oscillation persisted for the 20 min duration of the run; implying that the sponge leads to the decay of the oscillation in Fig. 13a.

The oscillatory motion is not present in the anelastic case (Fig. 14a) indicating that the oscillation is associated with horizontally long wavelength acoustic waves (Fig. 13a) only present in the compressible model. Most
FIG. 13. Domain integrals of the compressible perturbation (a) traditional and (b) available energy in \(10^9 \text{ J m}^{-1}\) as functions of time. In (a) the curves for IE and PE are dashed–dotted and dashed, respectively. In (b) the curves for APE and AEE are dashed and dashed–dotted, respectively. In both (a) and (b) the curves for TE and KE are solid and dotted, respectively.

FIG. 14. As in Fig. 13, but for the domain integrals of the anelastic perturbation (a) traditional and (b) available energy. The anelastic potential and available elastic energy perturbations are identically zero.
The available energies for the anelastic model are

\[
\text{APE} = \int \rho c_p \left[ \frac{T}{T_r} - \frac{\theta}{\theta_r} \right] dx \, dz, \quad (5.5)
\]

and

\[
\text{AEE} = 0. \quad (5.6)
\]

There is no available elastic energy because acoustic waves do not exist in the anelastic approximation. The reference temperature \( T_r \) is taken to be 216.5 K, corresponding to the coldest temperature in the standard atmosphere base state. In each case, the total available energy is the sum of the KE, APE, and AEE, and is conserved in the absence of the sponge layer. It is noted that the available energetics presented here is valid for nonhydrostatic conditions and is thus distinctly different from the hydrostatic theory of Lorenz (1955).

Figure 13b presents the evolution of the compressible available energetics. The diabatic warming generates both AEE and APE. The TE exhibits only a weak decrease with time. The AEE and APE exhibit a damped oscillation analogous to, but weaker than, that in the traditional energetics (Fig. 13a). There is a net decrease in the AEE as the APE and KE increase in the first several minutes.

Figure 14b presents the anelastic available energetics. The TE is conserved and KE increases at the expense of the APE as the AEE is identically zero. The total available energy is well conserved for both the compressible and anelastic cases and is attributed mostly to buoyancy waves. The compressible TE is \( 6.8 \times 10^9 \text{ J m}^{-1} \) and the anelastic is \( 6.6 \times 10^9 \text{ J m}^{-1} \).
To examine the computational efficiency adequately, it is important to consider the compressible model with the alternative time-splitting integration technique (Klemp and Wilhelmson 1978) in which the acoustic terms of the compressible model are integrated at a smaller time step than the remaining terms. These acoustic terms include the local acceleration and vertical pressure gradient force of the momentum pressure in (2.3). Two versions of the time-splitting scheme are examined: one with explicit and one with implicit time differencing. The split scheme enables the compressible model to be run with 1.2-s time step with 10 smaller acoustic time steps in 551 s with identical results as the benchmark (i.e., SNRs of $10^6$). Similar results are found with an implicit time-splitting scheme. Larger time steps, say 1.8 s, produced floating-point exceptions with either 10 or 20 smaller acoustic time steps. This computational instability may be associated with the fixed Rayleigh damping time of 5 s used here. Nonetheless, both split schemes are slower than the anelastic model (Table 1).

Another advantage of the anelastic model is that its computational domain need not be as large as that of a compressible model because there is no need for an upper sponge layer or a large horizontal domain to help mitigate contamination of the solution with Lamb and acoustic modes that are reflected off the lateral boundaries.

### Table 1. Anelastic model run time comparisons at $t = 10$ min for various time steps with and without Rayleigh damping aloft. Accuracy is given in terms of the rms and SNR (in parentheses) for the vertical velocity and potential temperature perturbation. The floating-point exception (FPE) error within the model run is also shown.

<table>
<thead>
<tr>
<th>Time step (s)</th>
<th>Rayleigh damping</th>
<th>Run time (s)</th>
<th>Vertical velocity rms (m s$^{-1}$) SNR</th>
<th>Potential temperature rms (K) SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>Yes</td>
<td>1783</td>
<td>0.015 (407)</td>
<td>0.030 (138)</td>
</tr>
<tr>
<td>10</td>
<td>Yes</td>
<td>54</td>
<td>0.015 (408)</td>
<td>0.030 (140)</td>
</tr>
<tr>
<td>15</td>
<td>Yes</td>
<td>FPE</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>15</td>
<td>No</td>
<td>36</td>
<td>0.026 (134)</td>
<td>0.029 (143)</td>
</tr>
<tr>
<td>30</td>
<td>No</td>
<td>21</td>
<td>0.026 (135)</td>
<td>0.029 (142)</td>
</tr>
<tr>
<td>60</td>
<td>No</td>
<td>11</td>
<td>0.026 (134)</td>
<td>0.029 (142)</td>
</tr>
<tr>
<td>120</td>
<td>No</td>
<td>5</td>
<td>0.059 (26)</td>
<td>0.031 (132)</td>
</tr>
</tbody>
</table>
7. Conclusions

This paper provides a complete comparison of compressible and anelastic numerical models of deep, dry convection in a standard atmosphere base state triggered by an instantaneous diabatic warming. Initially, the warming generates no motion and the potential temperature field is the same in both models but the other thermodynamic fields differ. In the compressible model, the pressure and temperature perturbations are confined to the region of the warming and there is no density perturbation. In the anelastic model, pressure and density perturbations extend beyond the warming region as a result of an instantaneous acoustic adjustment because the pressure is diagnosed from the anelastic elliptic pressure in (2.12) rather than the prognostic compressible pressure in (2.3). Both model initial conditions are nonhydrostatic. The compressible model takes several minutes to complete its acoustic adjustment that consists of vertically propagating acoustic waves and a horizontally propagating Lamb wave packet. Once these waves leave the warming region, the two models exhibit similar structures and evolutions of the buoyancy and the potential vorticity fields. This agreement supports the anelastic closure (2.10) of Bannon (1995). In addition, the similar potential vorticity fields suggest that the steady-state geostrophic response of the two models to the warming will be the same.

The anelastic pressure perturbation is diagnosed with the elliptic diagnostic pressure in (2.12). The thermal source is shown to be the dominant contributor to the pressure. It is important to note that the larger the horizontal dimensions of the source term, the greater the amplitude of the pressure perturbation response.

For the traditional energetics, the internal energy perturbation contains all of the energy perturbation initially in both models. However, the compressible model case is interesting because the internal and potential energies contain an oscillatory behavior due to the long wavelength vertically propagating acoustic modes. Rising motion increases the mass aloft, hence, raising the potential energy but decreasing the internal energy through adiabatic cooling. A small amount of kinetic energy exists in the model domain. It is shown that the Lamb wave packet dominates the internal and potential energies. In the anelastic model, the potential energy perturbation is nonexistent because the base-state density is constant in time. This results in a monotonic conversion of internal energy to kinetic energy with time.

The available energetics contain available elastic and available potential energy, but each is distributed differently according to the model. In the anelastic model, the available elastic energy is zero; hence, the total energy perturbation is composed of available potential energy initially, which is monotonically converted to kinetic energy. In contrast, the presence of vertically propagating acoustic waves obscures the view of the conversions in the compressible model. The oscillatory motion is again due to the increased amount of vertical ascent and descent. For example, available elastic energy is converted to available potential energy in regions of rising motion. Hence, when descent dominates the model domain, there is a maximum in available elastic energy and a minimum in available potential energy and vice versa.

Model efficiency is defined as how quickly the model can complete its numerical integration. Based on this definition, the anelastic model (Table 1) is significantly more efficient. It runs most efficiently because the acoustic and Lamb waves are filtered out and a larger time step (Table 1) can be used without violating the CFL stability criterion. The efficiency of the compressible model can be improved slightly by the implementation of alternative time-splitting schemes.

Overall the anelastic model has the most advantages for this case. These advantages include the filtering of the acoustic and Lamb waves, an instantaneous acoustic adjustment, an accurate depiction of the buoyancy and potential vorticity dynamics, simplified energetics, and greater computational efficiency. These advantages of the anelastic model do not necessarily hold for all cases. For the weak winds seen here $[O \left(10 \text{ m s}^{-1}\right)]$, the Mach number (Ma) is very small (Ma $\sim 0.001$). In contrast, Bryan and Rotunno (2008) simulate anelastic and compressible gravity currents with some flow speeds approaching 100 m s$^{-1}$ implying Ma $\sim 0.10$. Their anelastic simulations required 50% more (less) computer time than the compressible simulations (G. H. Bryan 2008, personal communication) for the high (low) speed cases but produced essentially identical results. These results for the relative computational efficiencies of the models may change for three-dimensional simulations.

An important extension of this work would be to compare the compressible and anelastic models with the implementation of moisture and phase changes. Such a comparison is complicated by the fact that the equations of state for moist air differ significantly for the two models.

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REFERENCES