An Efficient and Accurate Method for Computing the Wet-Bulb Temperature along Pseudoadiabats

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ABSTRACT

A new technique for computing the wet-bulb potential temperature of a parcel and its temperature after pseudoadiabatic ascent or descent to a new pressure level is presented. It is based on inverting Bolton’s most accurate formula for equivalent potential temperature to obtain the adiabatic wet-bulb temperature on a given pseudoadiabat at a given pressure by an iterative technique. It is found that is a linear function of equivalent temperature raised to the power, where is the Poisson constant for dry air, in a significant region of a thermodynamic diagram. Consequently, Bolton’s formula is raised to the power prior to the solving. A good “initial-guess” formula for is devised. In the pressure range , this guess is within K of the converged solution for wet-bulb potential temperatures . Just one iteration reduces this relative error to less than K for . The upper bound on the overall error in the computed after one iteration is K owing to an inherent uncertainty in Bolton’s formula. With a few changes, the method also works for finding the temperature on water- or ice-saturation reversible adiabats.

The new technique is far more accurate and efficient than the Wobus method, which, although little known, is widely used in a software package. It is shown that, although the Wobus function, on which the Wobus method is based, is supposedly only a function of temperature, it has in fact a slight pressure dependence, which results in errors of up to K in the temperature of a lifted parcel. This intrinsic inaccuracy makes the Wobus method far inferior to a new algorithm presented herein.

1. Introduction

Numerical analyses of actual and model-output upper-air soundings [e.g., Prosser and Foster 1966; Stackpole 1967; Doswell et al. 1982] are used to determine several weather forecast parameters [e.g., convective available potential energy (CAPE), CAPE in the lowest km of the sounding, convective inhibition, level of free convection, height of the wet-bulb zero, bulk Richardson number, energy–helicity index, the height to which penetrative convection can reach, etc.] that identify environments that support various types of severe weather [e.g., Rasmussen 2003; Thompson et al. 2003] and that may factor in the forecast likelihood that a thunderstorm will produce a significant tornado in probabilistic models (e.g., Hamill and Church 2000). These parameters all require the computation of adiabatic wet-bulb temperature, along water-saturation pseudoadiabats. They should be calculated as accurately as possible because errors affect statistical measures of their forecast skill and also conditional tornado probabilities.

Given the initial state of a parcel, there is no simple way to compute its temperature during undiluted pseudoadiabatic ascent. In contrast, there are precise explicit formulas for equivalent potential temperature (EPT) so we can easily calculate the parcel’s equivalent temperature during its ascent. Inconveniently, the equivalent temperature of a saturated parcel is a complicated function of both explicitly and implicitly through the dependence of the parcel’s saturation mixing ratio on its temperature. This has discouraged meteorologists from trying to invert a formula for to get an explicit expression for . The general view has been that the problem is mathematically intractable, and that solutions for can be obtained.

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only through numerical integration, using small vertical steps, of the differential equation governing the pseudoadiabat or through iterative numerical techniques (e.g., Doswell et al. 1982). This paper demonstrates that there is in fact an explicit solution if errors up to 0.34 K relative to a converged solution are permitted. If greater accuracy is desired, this solution is an excellent first guess for an iterative method.

A variety of numerical techniques have been used to derive the temperature of a parcel lifted adiabatically (if initially unsaturated), then pseudoadiabatically (i.e., with all condensate instantly falling out) to some lower pressure, \( p \), (e.g., Prosser and Foster 1966; Stackpole 1967; Doswell et al. 1982). In these procedures for the automated analyses of soundings, condensation temperature, \( T_L \), which is needed for computation of \( \theta_e \) if the parcel is unsaturated initially, was determined by either a search technique (Prosser and Foster), by iteration (Stackpole), or by curve fitting (Doswell et al.). To compute the temperature along pseudoadiabats, Prosser and Foster used a computationally fast, but error-prone, scheme. First, they approximated the temperatures along three specific pseudoadiabats (the ones with wet-bulb potential temperatures of 10°, 20°, and 30°C) by third-order polynomials. Then they obtained the temperature of the lifted parcel by linear interpolation, after computing its wet-bulb potential temperature (WBPT) \( \theta_w \) from a crude empirical formula. Stackpole computed the difference between the EPT (via the imprecise Rossby formula) of the pseudoadiabat and that of a parcel at pressure \( p \) with temperature given by the latest iterative solution. He also used the inefficient interval-halving numerical procedure (Gerald and Wheatley 1984). Doswell et al. reported a similar technique, due to Hermann Wobus, with some important differences. The Wobus method employs the much faster secant method (Gerald and Wheatley 1984), which converges within a few iterations. It also uses the Wobus function, which was devised by Wobus in 1968. At the time of its invention, the Wobus method was much more efficient and faster than other methods.

The Wobus function, \( W_p(T_K) \), of absolute temperature \( T_K \) only.

The Wobus method is little known because it has never been documented previously in the formal literature. However, it is widely used because it is utilized unseen in the National Centers Skew–T/Hodograph Analysis and Research Program (NSHARP; Hart et al. 1999), which is the interactive software for upper-air profiles in the National Centers’ Advanced Weather Interactive Processing System/General Meteorological Package (N-AWIPS/GEMPAK; J. Hart 2007, personal communication). Since it is in wide use, its errors should be evaluated. Despite the advent of more recent empirical data and Bolton’s creation in 1980 of a highly accurate formula for EPT, the Wobus method has never been upgraded since its invention.

Bolton (1980) first obtained new empirical formulas for saturation vapor pressure and condensation temperature. With the use of these formulas, he accurately determined EPT as a function of condensation temperature and pressure by numerically integrating the differential equation for the pseudoadiabatic process from the saturation point to a great height. He then used the numerical results to obtain accurate formulas for EPT, of which his Eq. (39) is the most precise. Apart from the more accurate, but more complicated, formula for condensation temperature developed by Davies-Jones (1983), Bolton’s formulas are the most exact of their type. For initially saturated air, Bolton’s Eq. (39) is accurate to within 0.2 K in \( \theta_p \) with this error mostly owing to variation of \( \gamma \), the specific heat of dry air at constant pressure, with temperature and pressure (List 1971, his Table 88). Note that, although \( \gamma \) is treated as a constant, the variation of the specific heat of moist air at constant pressure, \( \gamma_p \), with mixing ratio \( r \) is parameterized.

This paper devises a new accurate method for computing temperature along pseudoadiabats, and hence for reducing the errors involved in evaluating the above forecast parameters. First an efficient algorithm for inverting Bolton’s Eq. (39) to obtain the wet-bulb temperature along a given pseudoadiabat at a given pressure \( p \) is formulated (section 2). The output at 1000 mb from this algorithm is then used to determine empirical formulas for \( \theta_w \) as a function of \( \theta_p \) (section 3). Over most of the atmospheric range of \( \theta_w \) (\( -19^\circ < \theta_p < 29^\circ \)), a linear relationship is discovered between \( \theta_w \) and the \( -\lambda \) power of \( \theta_E \), where \( \lambda = 1/\kappa_d (=3.504) \) and \( \kappa_d = R_d/\gamma pd (=0.2854) \) is the Poisson constant for dry air. In section 4, highly accurate initial guesses for the computation of wet-bulb temperature along pseudoadiabats are derived. A new linear relationship between \( T_w \) and the \( -\lambda \) power of equivalent temperature \( T_E (K) \) is found in a significant region of a thermodynamic diagram. One iteration of the algorithm then gives a highly accurate solution for \( T_w \). Next the algorithm is modified slightly for computation of temperature along reversible adiabats (section 5). The Wobus method is described in section 6 and its intrinsic errors are evaluated and found to be quite large. Although the Wobus function is supposedly only a function of temperature, it in fact has a slight dependence on pressure. The linear
relationship discovered in section 3 is used in section 7 to formulate a new more accurate Wobus function of both temperature and pressure. The modified Wobus method thus obtained is shown to be simply a convoluted version of the new method. The reason why the original Wobus method works fairly well is addressed in section 8 where it is shown that the error in Bolton’s Eq. (39) is the intrinsic error of the Wobus function is independent of pressure is less than 1 K.

Before proceeding, we explain our terminology of errors. “Relative error” denotes the error relative to the converged solution of Bolton’s Eq. (39). “Absolute error” also includes the error inherent in Bolton’s Eq. (39) itself. The “intrinsic error” of the Wobus method refers to the error resulting from the assumption that the Wobus function is a function of just temperature.

Table 1. Parameters in (2.1) and (2.2) as they apply to Bolton’s Eqs. (28), (35), (38), and (39) for water-saturation pseudoadiabats and to Saunders’s Eq. (3) for water-saturation reversible adiabats. The latent heat of vaporization is given by \( L = L_0 = L_1 T \) where the constants \( L_0 = 2.501 \times 10^6 \) J kg\(^{-1}\) and \( L_1 = 2.37 \times 10^3 \) J kg\(^{-1}\) K\(^{-1}\). In the last column, \( c_w \) is the specific heat of water and \( Q \) is the mixing ratio of total water to dry air.

<table>
<thead>
<tr>
<th>( k_0 )</th>
<th>Bolton’s Eq. (28)</th>
<th>Bolton’s Eq. (35)</th>
<th>Bolton’s Eq. (38)</th>
<th>Bolton’s Eq. (39)</th>
<th>Saunders’s Eq. (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((L_0 + L_1 C)/c_{pd})</td>
<td>2675 K</td>
<td>3376 K</td>
<td>3036 K</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(L_1/c_{pd})</td>
<td>0</td>
<td>2.54</td>
<td>1.78</td>
<td>(L_1/c_{pd} + c_w Q)</td>
<td></td>
</tr>
<tr>
<td>(k_2)</td>
<td>0</td>
<td>0.810</td>
<td>0.448</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(k_3)</td>
<td>0</td>
<td>0.28</td>
<td>0.28</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(\nu)</td>
<td>(c_W)</td>
<td>0</td>
<td>0</td>
<td>(R_d/c_{pd} + c_w Q)</td>
<td></td>
</tr>
</tbody>
</table>

2. Mathematical formulation of the new method

We start by developing the new method. We use Bolton’s nomenclature here, including his convention that a temperature with a capital subscript is in kelvins and one with a small subscript is in degrees Celsius. Temperatures with the same letter subscript but different case are the same variable in different units. The only departures from these rules are \( T \), the temperature in kelvins, \( T_K \), the absolute temperature (= \( T + C \), where \( C = 273.15 \) K), and \( \theta \), the potential temperature in kelvins. The one exception to Bolton’s nomenclature in this paper is the unit of mixing ratio, which is grams per gram instead of grams per kilogram. Any variable that can be determined uniquely from a thermodynamic diagram is a function of just two independent variables, chosen in the following analyses to be temperature and the nondimensional pressure \( \pi = (p/p_0)\lambda^2 \).

In his Eqs. (28), (33), (35), (38), and (39), Bolton (1980) gives five formulas for the EPT. These are, respectively, the traditional but inaccurate Rossby formula, Bolton’s adjustment of the Simpson (1978) formula, his adaptation of the Betts and Dugan (1973) formula, and two new formulas. The formulas are written for an unsaturated parcel at the point \( (T_K, \pi) \) and involve its mixing ratio \( r \), its vapor pressure \( e \) and \( T_L \). Bolton’s Eq. (39) is the most accurate. Simpson’s formula does not fit the same mathematical mould as the other formulas and so is not considered further here. We can apply the formulas to any saturated parcel at \( (T_W, \pi) \) simply by replacing \( T_L \) by \( T_W \), \( r \) by saturation mixing ratio \( r_s(T_W, \pi) \), and \( e \) by saturation vapor pressure \( e_s(T_W) \). When this is done, the remaining formulas all have the following form:

\[
T_E = T_W \left[ 1 - \frac{e_s(T_W)}{p_0} \right]^{-\nu} k_{3} r_s(T_W, \pi) \exp[G(T_W, \pi)],
\]

(2.1)

where

\[
G(T_W, \pi) = \left( \frac{k_0}{T_W} - k_1 \right) \left[ r_s(T_W, \pi) + k_2 r_s^2(T_W, \pi) \right],
\]

(2.2)

and the constants are listed in Table 1. Note that we have used Bolton’s Eqs. (24) or (7) and the relationship \( T_E = \theta_E \pi \) to write the formulas in the form in (2.1). We can find the temperature at pressure \( p \) along a given pseudoadiabat with EPT \( \theta_E \) by solving for \( T_W \). However, for the reason given below, it is generally more advantageous to solve (2.1) raised to the \(-\lambda\) power (i.e., to solve for given \( \pi \) and \( \theta_E \)):

\[
\left( \frac{C}{T_E} \right)^{\lambda} = f(T_W, \pi)
\]

\[
= \left( \frac{C}{T_W} \right)^{\lambda} \left[ 1 - \frac{e_s(T_W)}{p_0} \right]^{-\pi} \exp[-\lambda G(T_W, \pi)],
\]

(2.3)

where \( \lambda = 1/\kappa_d = c_{pd}/R_d = 3.504 \). In (2.3) \( T_W \) and \( T_E \) have been scaled by \( C \) simply to avoid large numbers.
By Taylor series expansion about a temperature \( \tau \) (K) at constant \( \pi \),

\[
\begin{align*}
    f(T_W; \pi) &= f(\tau; \pi) + f'(\tau; \pi)(T_W - \tau) \\
    &\quad + f''(\tau + \beta(T_W - \tau); \pi)(T_W - \tau)^2/2!,
\end{align*}
\]

\[ 0 < \beta < 1 \quad (2.4) \]

[The notation \( f(\tau; \pi) \) indicates that \( f \) is a function of \( \tau \) with \( \pi \) fixed.] At each pressure \( \geq 300 \) mb, we can choose \( \tau = \tau^* \) such that the remainder (the last term) is much smaller than the first-order term for \( T_W \in [\tau^* - 20^\circ C, \tau^* + 20^\circ C] \) (this can be deduced from Fig. 1). In other words, there is the almost linear relationship between \( (C/T_E)^\lambda \) and \( T_W \) in this interval,

\[
T_W = \tau^* - \frac{f(\tau^*; \pi) - (C/T_E)^\lambda}{f'(\tau^*; \pi)},
\]

where an expression for \( f'(\tau; \pi) \) is provided in the appendix for the reader’s convenience. This anticipates our later finding that, with a good initial guess \( \tau_0 \), one or two iterations of the algorithm,

\[
\tau_{n+1} = \tau_n - \frac{f(\tau_n; \pi) - (C/T_E)^\lambda}{f'(\tau_n; \pi)},
\]

Fig. 1. The second derivative \( f'' \) of \( f(T_W; \pi) \) w.r.t. \( T_W \) vs \( \tau \) at pressures of 1000, 700, 500, and 300 mb. (At 1000 mb, \( \tau \) is equal to \( \theta_w \).) The Taylor series expansion to first order of \( f(T_W; \pi) \) at constant pressure, evaluated at the temperature \( \tau^* = -50(1 - \pi) \) (marked by the asterisk), is written at the upper left of each grid window. Throughout the figure \(|f''|\) is less than 0.0002 within 20°C of \( \tau \). At each pressure therefore, the upper bound for the remainder term in (2.4) within the temperature interval \((\tau^* - 20^\circ C, \tau^* + 20^\circ C)\) is an order of magnitude smaller than the first-order term in the written expansion, proving that \( f(T_W; \pi) \) is almost a linear function of \( T_W \) in this interval.
in Newton’s method always provide a precise numerical solution, $T_W$, of (2.3). We can accelerate the convergence by retaining the second-order term in the series expansion and solving the resulting quadratic equation:

$$
\frac{f''(\tau_n; \pi)}{2} (T_W - \tau_n)^2 + f'(\tau_n; \pi) (T_W - \tau_n) + f(\tau_n; \pi) - (C/\theta_E)^{\lambda} = 0, \tag{2.7}
$$

in a form that is accurate for a small second-order term (see Henrici 1964, p. 199). This gives

$$
\tau_{n+1} = \tau_n - \frac{2 [f(\tau_n; \pi) - (C/\theta_E)^{\lambda}]}{f'(\tau_n; \pi) \pm \{f'(\tau_n; \pi) - 2f''(\tau_n; \pi) [f(\tau_n; \pi) - (C/\theta_E)^{\lambda}]^{0.5}\}}, \tag{2.8}
$$

where $f''(\tau; \pi)$ is given in the appendix. We choose the root that is closest to the linear $\tau_{n+1}$ provided by (2.6).

In this paper, we use Bolton’s Eq. (39) as the basis for the computation of $T_W$. However, we can compute $T_W$ from Bolton’s Eqs. (28), (35), or (38), or even compute temperature along water- or ice-saturation reversible adiabats from Saunders’s (1957) Eqs. (3) or (4), simply by changing a few parameters in the computer code as dictated by Table 1.

A first guess $\tau_0$ that is accurate to within 10 K is given by

$$
\tau_0 = \min[T_E, \theta_W - 150(1 - \pi)], \tag{2.9}
$$

where $\theta_W$ is obtained from $\theta_E$ via a formula obtained in section 3. This is sufficient for convergence of the algorithm, but not optimal. A far more accurate initial estimate is based on results presented below and so is supplied later in Eqs. (4.8)–(4.11).

### 3. Computing $\theta_w$ from $\theta_E$

We first look at the problem of computing WBPT from EPT to gain insight into the more general problem of computing temperature on a given pseudoadiabat at a given pressure. Using Bolton’s Eq. (39) for $\theta_E$, we find from (2.3) applied at 1000 mb that $\theta_W$ is the solution of

$$
f(\theta_W; 1) = \left(\frac{C}{\theta_W}\right)^{\lambda} \left[1 - e_s(\theta_W/p_0) \exp[-\lambda G(\theta_W, 1)]\right]
= \left(\frac{C}{\theta_E}\right)^{\lambda}, \tag{3.1}
$$

where

$$
G(\theta_W, 1) = \left(\frac{3036}{\theta_W} - 1.78\right)\left[e_s(\theta_W, 1) + 0.448r^2_s(\theta_W, 1)\right].
$$

One linear iteration of Newton’s method with a first guess of $C$ provides the following solution:

$$
\theta_w = 45.674 - 52.091(\theta_E)^{\lambda}, \tag{3.3}
$$

which is valid in some interval around $C$. This interval turns out to be fairly large owing to the small second derivative of $f(\theta_w; 1)$. A plot (Fig. 2) of the actual (i.e., converged iterative) solution for $\theta_w$ as a function of $(\theta_E/\theta)^{\lambda}$ shows that the linear solution in (3.3) is approximately valid in the interval $-19^\circ < \theta_w < 29^\circ$. The minimax-polynomial approximation method (Scheid 1989) was used to obtain the minimax line in this interval,

$$
\theta_w = 45.114 - 51.489(\theta_E/C)^{-\lambda}, \tag{3.4}
$$

which fits the solution to 0.1°C. Note that (3.3) and (3.4) are very similar.

Is there a better linear relationship than (3.4) between $\theta_w$ and another power of $(\theta_E/C)$, say $(\theta_E/C)^{-\mu}$? To answer this question, the standard error of the $\theta_w$ predicted by linear regression over the interval $-19^\circ < \theta_w < 29^\circ$ was computed for different values of $\mu$. The minimum standard error (0.06 K) occurred for $\mu = 3.5$, thus confirming that $\mu = \lambda$ produces the most linearity.

The linear fit naturally breaks down at large values of $\theta_E$ because $\theta_w$ cannot remain finite as $\theta_E$ tends to infinity and at cold values of $\theta_E$ because $\theta_W$ tends to $\theta_w$ as the saturation mixing ratio becomes small. For $377 \leq \theta_E < 674$ K ($28.2^\circ \leq \theta_w < 50^\circ$), a minimax polynomial was fitted to the difference between the actual and the linear solutions. For $\theta_E \leq 257$ K ($\theta_w \leq -18.6^\circ$), one iteration of Newton’s method applied directly to Bolton’s Eq. (35) version of (2.1)–(2.2) with $T_E = \theta_E, T_w = \theta_w, and \pi = 1$ suffices. After some minor approximations, the resulting solution is

$$
\theta_w = C - \frac{A'r_s(\theta_E, 1)}{1 + A'r_s(\theta_E, 1)d \ln e_s(\theta_E)/d\theta_E} \text{ for }$

$$
\theta_E \leq 257 \text{ K}, \tag{3.5}
$$

$$
\theta_w = 45.114 - 51.489(\theta_E)^{\lambda} \text{ for }$

$$
257 < \theta_E < 377 \text{ K}. \tag{3.6}
$$
\[ \theta_w = 43.380 - 51.489 \left( \frac{C}{\theta_E} \right)^{3.504} + 0.6069 \left( \frac{\theta_E}{C} \right)^{3.504} - 0.01005 \left( \frac{\theta_E}{C} \right)^{3.504} \]

for \[377 \leq \theta_E < 674 \text{ K},\] where a small term has been neglected in (3.5) and \(A = 2675 \text{ K}, \ C = 273.15 \text{ K}, \ \lambda = \pi \rho_p \rho_p r_s(T_K, 1) = ee_s(T_K) \]

\[ [p_0 \pi^\lambda - e_s(T_K)], \ e_s(T_K) = e_s(C) \exp[a(T_K - C)/(T_K - C + b)], \ d \ln e_s(T_K)/dT_K = ab/(T_K - C + b)^2, \ v = 0.6220, \ e_s(C) = 6.112 \text{ mb}, \ a = 17.67, \text{ and } b = 243.5 \text{ K}. \]

Remarkably, this solution fits the actual solution to 0.1°C. It should be emphasized, however, that the accuracy of Bolton’s Eq. (39) is not known beyond \(\theta_w = 40°C \ (\theta_E = 478.4 \text{ K}).\)

For even greater overall accuracy a rational function was collocated to the actual solution of (3.1) at \(30^\circ, 20^\circ, \ldots, 40^\circ, 50^\circ \text{C}.\) The resulting approximate solution is

\[ \theta_w = \begin{cases} \theta_E - C & -\exp \left[ a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 \right] \bigg|_{1 + b_1 X + b_2 X^2 + b_3 X^3 + b_4 X^4} \quad \text{for } \theta_E \geq 173.15 \text{ K} \\ \theta_E - C & \text{for } \theta_E < 173.15 \text{ K} \end{cases} \]
where \( X = \theta_e/C \) and \( a_0 = 7.101574 \), \( a_1 = -20.68208 \), \( a_2 = 16.11182 \), \( a_3 = 2.574631 \), \( a_4 = -5.205688 \), \( b_1 = -3.552497 \), \( b_2 = 3.781782 \), \( b_3 = -0.6899655 \), \( b_4 = -0.5929340 \). For \( \theta_w \leq 50^\circ \text{C} \), its maximum deviation from the actual solution (the relative error) is 0.02 K, and it is within 0.005 K for \( \theta_w \in (-20^\circ, 40^\circ) \), the range of \( \theta_w \) tabulated in the Smithsonian tables (List 1971, see his Table 78). The magnitude of the argument in the exponential in (3.8) becomes large at temperatures less than \(-100^\circ\text{C} \), and the rational-function approximation fails. However, at these temperatures we can assume that \( \theta_w = \theta_e \) with negligible error (\( \leq 10^{-4} \text{K} \)).

Associated with the absolute error \( \delta \theta_e \) of up to 0.2 K in the value of \( \theta_e \) provided by Bolton’s Eq. (39), there is a corresponding error in \( \theta_w \) given, from (3.6), by

\[
\delta \theta_w = 51.5 \lambda (C/\theta_e)^3 \theta_e^{-1} \delta \theta_e \quad \text{for} \quad 257 < \theta_e < 377 \text{K}.
\]

For \( \delta \theta_e = 0.2 \text{K} \), \( \delta \theta_w \) varies from 0.17 K at \( \theta_e = 257 \) to 0.05 K at \( \theta_e = 335 \) to 0.03 K at \( \theta_e = 377 \text{K} \). The absolute error in \( \theta_w \) becomes quite small at the EPTs most often associated with atmospheric convection.

4. The solution for \( T_w \) as a function of \( T_E \) and \( p \)

The first step toward finding an efficient and accurate algorithm for computing temperature along pseudoadiabats is to obtain the “true” values that are produced by precisely inverting Bolton’s Eq. (39). We apply this formula initially to saturated parcels at 1000 mb to acquire the \( \theta_e \) values corresponding to \( \theta_w \in \{-20^\circ, -18^\circ, \ldots, 40^\circ\} \). We then obtain the wet-bulb temperatures \( T_w(\theta_w, p) \) along these 31 pseudoadiabats at 25-mb intervals from 1050 to 100 mb (39 different pressures) to a tolerance of \( 5 \times 10^{-5} \text{K} \) by Newton’s algorithm (2.6) applied to Bolton’s Eq. (39) raised to the \(-1/k_d\) power. We refer to the array of 31 \( \times 39 \) (=1209) points in \( (\theta_w, \ p) \) space as the “grid.”

Plots (Fig. 3) of the actual solution of (2.3) as a function of \((C/T_E)^k\) at selected pressures have some of the same characteristics as Fig. 2. In each plot, there is a range in which the solution (marked by Xs) is almost linear and an immediately adjacent range at cold equivalent temperatures, where \( T_w \) approaches \( T_E \), and the solution becomes asymptotic to the curve \( T_w = Cx^{-1/\lambda} \), where \( x \) is the abscissa \((C/T_E)^k\). The slopes and intercepts of the linear parts, and the transition points \( S \) where the solutions depart from linearity toward their asymptotes all vary with pressure. The solution also becomes nonlinear at the warmest equivalent temperatures \([(C/T_E)^k < 0.4]\). This is clearly visible at 1000 mb (Fig. 2), but hardly evident at 850 mb (Fig. 3).

From (2.3) and (2.4) it is evident that the equations of the straight-line portions are given by

\[
T_w = k_1(\pi) - k_2(\pi)(C/T_E)^k, \tag{4.1}
\]

where

\[
k_1(\pi) = \tau^* - f(\tau^*; \pi)/f'(\tau^*; \pi), \quad k_2(\pi) = -1/f'(\tau^*; \pi),
\]

and \( \tau^* \) is any value of \( T_w \) such that the horizontal line \( T_w = \tau^* \) intersects the linear portion of the curve. By inspection \( \tau^* = -50 \) (\( 1 - \pi \)) always lies on the linear part of the solution curve (Figs. 4 and 1) and is used here as the points E in Figs. 2 and 3, where \( k_1(\pi) \) and \( k_2(\pi) \) are evaluated for the 39 different pressures. The coefficient \(-50 \) was chosen because it is nearly optimal for minimizing the maximum absolute error over the grid. Fitting quadratic regression curves to the results yields the following expressions:

\[
k_1(\pi) = -38.5 \pi^2 + 137.81 \pi - 53.737, \tag{4.3}
\]

\[
k_2(\pi) = -4.392 \pi^2 + 56.831 \pi - 0.384, \tag{4.4}
\]

with correlation coefficients, \( r \), of 1 (Figs. 5 and 6). Remarkably, \( k_2 \) is almost linear in \( \pi \) with the regression line:

\[
k_2(\pi) = 49.896 \pi + 2.2648. \tag{4.5}
\]

A similar procedure to that used in obtaining (3.5) gives the following good initial estimate for the solution at cold temperatures:

\[
T_w = T_E - C - \frac{A_r(T_E, \pi)}{1 + A_r(T_E, \pi)d \ln(\tau_0/T_E)dT_E}.
\]

The transition point \( S \) at each of the 39 pressure levels was located by evaluating the errors in the two approximate solutions \([(4.1)-(4.2) \text{ and } (4.6)] \) as a function of \((C/T_E)^k\), and determining by linear interpolation the value \( D(p) \) of \((C/T_E)^k\) where the magnitudes of the errors are equal. Then a regression line was fitted to the reciprocal of the data for \( D(p) \). The resulting equation,

\[
D(p) = (0.1859p/p_0 + 0.6512)^{-1}, \tag{4.7}
\]

fits the data adequately enough (Fig. 4). The transition points at particular pressure levels are plotted in Figs. 2 and 3.

Two empirical corrections were applied to (4.1). For \( T_E > C \), the coefficients were adjusted slightly, and at warm equivalent temperatures \((T_E > 355 \text{K})\), an additional term in \((C/T_E)^k\) was added and the constant term adjusted to describe the “warm-side” nonlinearity.

The resulting initial guess for \( T_w \) is
The two empirical corrections reduce the maximum relative error at any grid point in the initial guess from 1.8 to 0.34 K (Fig. 7). One ordinary (accelerated) iteration of the initial guess was found to be sufficient for convergence. The EPT was recomputed from Bolton’s Eq. (39) using the one-iteration solution for Tw. The largest difference between recomputed and original values of EPT was less than 0.002 K.

where $A = 2675$ K, $C = 273.15$ K, $\lambda = c_{pd}/R_d$, and $k_1(\pi), k_2(\pi),$ and $D(p)$ are given by (4.2) and (4.7). The regions in which the different parts of the initial solution apply are shown in Fig. 4.

The relative errors in the initial guess and that after one iteration were computed. Finally, as a check the EPT was computed again from Bolton’s Eq. (39) using the one-iteration solution for Tw. The largest difference between recomputed and original values of EPT was less than 0.002 K.

The two empirical corrections reduce the maximum relative error at any grid point in the initial guess from 1.8 to 0.34 K (Fig. 7). One ordinary (accelerated) itera-
tion then reduces this relative error to less than 0.002 K (0.001 K), which is more than sufficient. When \( k_1(p) \) and \( k_2(p) \) are approximated by the quadratic regression curves in (4.3) and (4.4), the maximum relative error in the initial solution increases to 0.47 K. When linear regression is used for \( k_2 \) [i.e., when (4.5) is used instead of (4.4)], the initial relative error is slightly larger (0.52 K).

The overall accuracy of the algorithm is determined almost entirely by its absolute error. The absolute error of up to 0.2 K in \( \theta_E \) in Bolton’s Eq. (39) is caused mostly by variation of \( c_{pd} \) with temperature and pressure. The effect of a 0.2-K error in \( \theta_E \) on \( T_w \) is shown in Fig. 8. Clearly the upper bound on the corresponding absolute error in \( T_w \) is also 0.2 K. Since the algorithm’s relative error after one iteration is much smaller, its overall error is \( \approx 0.2 \) K.

5. Computation of temperature along reversible adiabats

The equation governing the temperature \( T_R \) along reversible water-saturation adiabats is

\[
c_{pd} d\{T_R[\pi^\lambda - e_s(T_R/p_0)^{-k_d}] + c_w Q d \ln T_R \} = 0 \tag{5.1}
\]

(Saunders 1957). We make the usual assumption that \( c_w \), the specific heat of liquid water, is 4190 J kg\(^{-1} \) K\(^{-1} \) (e.g., Saunders 1957; Bolton 1980; Emanuel 1994), even though it increases by 8%, 14%, and 30% over this value at temperatures of -30°, -40°, -50°C, respectively (List 1971, see his Table 92). Equation (5.1) also fits the mold in (2.1) and (2.2) with the constants listed in the last column of Table 1. Since the mixing ratio of all phases of water to dry air, \( Q \), is conserved, this equation has the following integral:

\[
T_R[\pi^\lambda - e_s(T_R/p_0)^{-k_d}] - R_{pd} e^{c_{pd} + c_w Q} \exp \left\{ \frac{L(T_R)r_s(T_R, \pi)}{(c_{pd} + c_w Q)T_R} \right\} = A_1, \tag{5.2}
\]
where $A_1$ is a constant along a given reversible adiabat. The constant $A_1$ is evaluated at a parcel’s saturation point. It depends on $Q$ so that through each point on a thermodynamic diagram there passes a unique pseudoadiabat and an infinity of reversible adiabats, one for each value of $Q$ (Saunders 1957). Raising (5.2) to the $\frac{1}{\lambda}$ power gives the following version of (2.3):

$$f(T_R, \pi) = \left( \frac{C}{T_R} \right)^{\lambda} \left[ 1 - \frac{e_\pi(T_R)}{p_0^{\lambda}} \right]^{\lambda \nu} \times \exp \left[ - \frac{\lambda \nu L(T_R) \rho(T_R, \pi)}{R_d} \right] = \left( \frac{C}{A_1 \pi} \right)^{\lambda},$$

where $\nu = R_d/(c_{pd} + c_W Q)$.

---

**FIG. 5.** Quadratic regression for $k_1(\pi)$.

**FIG. 6.** Quadratic and linear regression for $k_2(\pi)$. The regression line and regression curve are nearly collocated.
It should be possible to find good initial guesses and identify near-linear relationships between $T_R$ and $(A_1 \pi)^{a/b}$ in a region of the parameter space by the procedures used above for pseudoadiabatic ascent. Because of the complication of dealing with an additional parameter ($Q$), this has been left for future work. Instead, we used the same initial guess for $T_r$ as the one for $T_w$ [see (4.8)–(4.11)], even though it is no longer accurate because retention of liquid water during ascent from low levels can make a parcel warmer by as much as $6 \text{ K}$ at $100 \text{ mb}$ (Emanuel 1994, p. 133). Initially and after one and two accelerated iterations, the maximum error relative to the converged solution is $6.25$ and $0.034 \text{ K}$, respectively. With two ordinary iterations, the maximum relative error reduces from the initial $6.25$ to $0.36 \text{ K}$ and then $0.001 \text{ K}$. Figure 9 shows the difference $T_r - T_w$ between the temperatures attained in reversible adiabatic expansion and pseudoadiabatic expansion from the same initial state at $900 \text{ mb}$. This figure is qualitatively similar to Fig. 2 in Saunders (1957), but there are quantitative differences owing to the use of up-to-date data.

The method also works for computation of the temperature along ice-saturation reversible adiabats with trivial substitutions. The specific heat of ice replaces that of water, the latent heat of sublimation supplants that of vaporization (Saunders 1957), the saturation mixing ratio is now with respect to ice instead of water, and $a$ and $b$ become Tetens’s (1930) ice coefficients ($a = 21.87$, $b = 265.5 \text{ K}$).

### 6. The Wobus method

We now show that the Wobus method has an intrinsic error, owing to it being supposedly a function of just temperature, that makes it inferior to the new method described above. The Wobus function is defined as fol-
At any point \((T, \pi)\) on a pseudoadiabatic (Stüve) diagram, one can consider two hypothetical wet-bulb potential temperatures \(\theta_S\) and \(\theta_A\) (Fig. 10). The saturated WBPT, \(\theta_S\), is reached by saturating a parcel at \((T, \pi)\), then bringing it down to 1000 mb \((\pi = 1)\) pseudoadiabatically. [Just enough rain is assumed to fall into and evaporate in the descending parcel to keep it saturated without leftover liquid water.] The dry WBPT, \(\theta_A\), is attained by desiccating a parcel at \((T, \pi)\), lifting it dry adiabatically to a great height and then bringing it down pseudoadiabatically to 1000 mb. The original Wobus function \(W_F\) of just temperature is the difference between these two temperatures, that is, \(W_F(T, \pi) = \theta_S(T, \pi) - \theta_A(T, \pi)\). It is evaluated using Wobus’s numerical fit to the data.

Incidentally, \(\theta_S\) is an important variable in its own right. The distribution of \(\theta_W\) and \(\theta_S\) with height, \(z\), determine atmospheric stability. A layer is potentially unstable if \(\partial \theta_W/\partial z < 0\) and conditionally unstable if \(\partial \theta_S/\partial z < 0\). The atmosphere is latently unstable if the \(\theta_W\) of any parcel lifted pseudoadiabatically exceeds the \(\theta_S\) of the unmodified environment at a higher level (see Fig. 12 in Browning and Donaldson 1963).

To test the claim that the Wobus function is a function solely of temperature, precise values of \(T_w\) and \(\theta_W - \theta_A\) at the 1209 grid points were computed and plotted on a scatter diagram (Fig. 11). The points tend to lie on the curve of the Wobus function \(W_F(T_w)\), but there is some scatter, which indicates minor dependence on pressure. The pressure dependency is also evident in Fig. 12, which shows the slight misalignment of the contours of \(T_w\) and \(\theta_W - \theta_A\) on a \((\theta_w, p)\) thermodynamic diagram, and maxi-
mum variation at constant temperature in $\theta_W - \theta_A$ of almost 1 K.

To include the pressure variation, we define a generalized Wobus function of two arguments, $W^*$, as

$$W^*(T_K, \pi) = \theta_S(T_K, \pi) - \theta_A(T_K, \pi). \quad (6.1)$$

For a saturated parcel, $T_K$ is the same as its adiabatic wet-bulb temperature $T_W$ and $\theta_S$ is its WBPT, $\theta_W$. Therefore,

$$W^*(T_W, \pi) = \theta_W(T_W, \pi) - \theta_A(T_W, \pi). \quad (6.2)$$

If this parcel descends dry adiabatically to 1000 mb, it still has the same potential temperature $\theta$ ($=T_W/\pi$ here because Wobus imprecisely used the Poisson constant for dry air) so its $\theta_A$ is unchanged. However, its $\theta_S$ is now $\theta$ and its $T_K$ becomes $\theta$. Hence, by (6.1),

$$W^*(T_W/\pi, 1) = T_W/\pi - \theta_A(T_W/\pi, 1)$$
$$= T_W/\pi - \theta_A(T_W, \pi). \quad (6.3)$$

If on the other hand, the parcel is lifted pseudoadiabatically to a great height and then brought down dry adiabatically to 1000 mb, its $T_K$ becomes equal to its equivalent potential temperature $\theta_E(T_W, \pi)$, its new $\theta_S$ is $\theta_E$ and its new $\theta_A$ is $\theta_W$ (a bijective function of $\theta_E$). Thus, by (6.1),

$$W^*(\theta_E, 1) = \theta_E - \theta_W(\theta_E), \quad (6.4)$$

Fig. 9. The difference $T_r - T_w$ between the temperature attained in the water-saturation adiabatic expansion and that attained in the water-saturation pseudoadiabatic expansion from the same initial state at 900 mb. Contours of $T_r - T_w$ at levels listed at the bottom are plotted on a $\theta_w$ vs $p$ diagram. Also shown are the $-40^\circ$, $-20^\circ$, and $0^\circ$C contours of temperature. This is an updated version of Fig. 2 in Saunders (1957).
which indicates that at \( \pi = 1 \) the Wobus function maps the EPT of a parcel to the difference between its EPT and its WBPT. According to Doswell et al. (1982), Wobus evaluated the right side of (6.4) using the data that gives \( \theta_E \) as a function of \( \theta_W \) in the header of Table 78 of the *Smithsonian Meteorological Tables* (List 1971) and then essentially fitted a high-order polynomial to the reciprocal of the right side to obtain an approximation to the Wobus function. However, the values of \( \theta_W \) computed from (6.4) with List’s (Bolton’s) values of \( \theta_E \) have a maximum error of 0.66 K (0.53 K). There are similar errors in the computation of \( \theta_A \) from \( \theta \) using (6.3). Errors of these magnitudes suggest that Wobus did not seek an accurate fit to (6.4), but instead fitted his function to the more general equation in (6.2) using data from parcels at many different pressure levels (not just at 1000 mb).

Eliminating \( \theta_A \) from (6.2) and (6.3) gives a formula for \( \theta_W \):

\[
\theta_W(T_w, \pi) = T_w/\pi - W^*(T_w/\pi, 1) + W^*(T_w, \pi).
\]

The temperature \( T_w \) at a pressure \( p \) along a pseudo-adiabat with WBPT \( \theta_W \) can be found as the solution of (6.5). Note that for a saturated parcel at 1000 mb, \( \pi = 1 \) and (6.5) reduces to \( \theta_w = T_w \). The method gives the
correct answer at 1000 mb despite the errors in $\theta_W$ as a function of $\theta_E$ and in $\theta_A$ as a function of $\theta$ that exist because (6.4) and (6.3) are not satisfied exactly.

In the special case when the parcel is initially unsaturated, then (6.5) applies at its saturation point ($T_L, \pi_L$) instead of at its initial location ($T_W, \pi$). In this case (6.5) becomes

$$\theta_W(T_L, \pi_L) = T_L/\pi_L - W^*(T_L/\pi_L, 1) + W^*(T_L, \pi_L).$$

(6.6)

If the pressure dependency of $W$ is disregarded and the original Wobus function is used in (6.5), we get the equation that Wobus solved for $T_W$,

$$\theta_W(T_W, \pi) = T_W/\pi - W_f(T_W/\pi) + W_f(T_W).$$

(6.7)

To solve (6.7), Wobus used the secant method (Gerald and Wheatley 1984), which, starting from his initial guess $T_W = \theta_W/\pi$, achieved convergence within a few iterations.

The error in the Wobus estimate of $\theta_W$ at a grid point is defined as $T_W/\pi - W_f(T_W/\pi) - W_f(T_W) - \theta_W$, where $T_W$ is the accurate value determined by the new method [not the one computed as the solution of (6.5) by the Wobus routines]. This error was computed at each grid point and plotted on a ($\theta_w, p$) diagram (Fig. 13). The maximum error is 0.58 K over the whole of the domain, and 0.53 K over the region defined by $\theta_w \leq 28^\circ C$. The associated error in the temperature of a parcel lifted from 1000 mb adiabatically to its lifted condensation level (LCL) and then pseudoadiabatically to 200 mb ranges up to 1.2 K (Fig. 14).

7. The generalized Wobus method reduces to the new method

We now show that the generalized Wobus method, which is obtained above by allowing the Wobus function to have some pressure dependency, reduces to the new method. First, note that Eqs. (3.5)–(3.7) or (3.8) can be written as $\theta_W(T_W, \pi) = X[\theta_f(T_W, \pi)]$, where $X
is a function that maps EPT into the corresponding WBPT. Hence, for a parcel that is initially saturated at \((T_w, \pi)\),

\[ W^*(T_w, \pi) = X[\theta_E(T_w, \pi)] - \theta_A(T_w, \pi). \]  

(7.1)

Since \(\theta_A(T_w, \pi)\) is by definition the WBPT associated with an EPT \(T_w/\pi\) (the parcel’s potential temperature), we also have \(\theta_A(T_w, \pi) = X(T_w/\pi)\). Therefore, the relationship between the generalized Wobus function and the \(X\) function is

\[ W^*(T_w, \pi) = X[\theta_E(T_w, \pi)] - X(T_w/\pi). \]  

(7.2)

If this parcel is brought down dry adiabatically to 1000 mb, its new temperature is \(T_w/\pi\) and its desiccated WBPT remains \(\theta_A(T_w, \pi)\). However, its saturated WBPT becomes \(T_w/\pi\). Therefore, at the new location \((T_w/\pi, 1)\) on the pseudoadiabatic diagram,

\[ W^*(T_w/\pi, 1) = T_w/\pi - \theta_A(T_w, \pi). \]  

(7.3)

Substituting (7.1) and (7.3) into (6.5) yields

\[ \theta_w = X[\theta_E(T_w, \pi)], \]  

(7.4)

which is the basis of the new method. Thus, the generalized Wobus method is equivalent to, but more convoluted than, the new method.

8. How the original Wobus method works to a degree

Despite its widespread use, no one seems to know why the Wobus function \(W\) is primarily a function of

\[ (-140.8, -140, 40, 41.3, 0.10) \quad (-0.00, 1, 27, 27.84, 0.1) \]

Fig. 12. Contour lines of \(T_w\) (solid for positive and long dashes for negative values) and of \(\theta_w - \theta_A\) (short dashes) on a \((\theta_w, p)\) diagram. The parentheses at bottom left and right are for \(T_w\) and \((\theta_w - \theta_A)\), respectively. The quantities inside the parentheses are the same as in Fig. 7.
temperature and consequently why the Wobus method works to about 1-K precision. Doswell et al. (1982) claim that \( W \) is a function only of temperature because “the amount of water vapor needed to saturate a parcel is dependent only upon its temperature,” forgetting that it is saturation vapor pressure \( e_s \), not saturation mixing ratio \( r_s \), that is a function of temperature alone. Although Wobus is the last author on the Doswell et al. paper, he apparently was aware that \( W \) had a slight dependence on pressure because, in a letter to Dr. Joseph Schaefer dated 3 November 1975, he stated that “A more accurate approximation of \( \theta_W \) is possible by using two slightly different functions for the two arguments [i.e., \( \theta_W = \theta - W_1(\theta) + W_2(T_W) \)]. This would permit the function to be tuned in favor of the lower value arguments as used for \( T_W \) and the other to be tuned in favor of the higher arguments appearing as \( \theta \).” The Wobus method works reasonably well only if the pressure dependency of the Wobus function is small. We can show this over the WBPTs most likely to occur in the atmosphere (\( \theta_w = 28.2^\circ \text{C} \)) as follows. From (3.6) the function \( X \) is given to a very good approximation by

\[
X(\theta_E) = K_1 + C - K_2 \left( \frac{C}{\theta_E} \right)^{\frac{1}{2}} \left\{ \begin{array}{ll}
\text{for} & 257 \leq \theta_E \leq 377 \text{ K} \\
\text{or} & -18.6 \leq \theta_w \leq 28.2^\circ \text{C}.
\end{array} \right.
\]

(8.1)

where \( K_1 = 45.114 \) and \( K_2 = 51.489 \) K. We progress further by using the simplest formula for \( \theta_E \) that is still quite precise. This is Bolton’s (1980) Eq. (35), which is

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**Fig. 13.** The error in the unmodified Wobus-method determination of \( \theta_W \) from (6.7) as a function of \( \theta_w \) and \( p \). Contour lines with positive (negative) values are solid (dashed). The quantities inside the parentheses are the same as in Fig. 7.
a slight modification of Betts and Dugan’s (1973) formula. Applying it to any saturated parcel yields

\[ \theta_E = \theta \exp[A_s(T_W, p)/T_W], \quad (8.2) \]

where \( A \) (a surrogate for \( L/c_{pd} \)) = 2675 K when \( r_s \) is in units of grams per gram. Raising (8.2) to the \(-\lambda\) power and substituting the usual approximate expression for \( r_s \) gives us

\[ (C/\theta_E)^\lambda \approx (C\pi/T_W)^\lambda \exp[-\lambda A e c(T_W)/p T_W], \quad (8.3) \]

where \( \varepsilon = 0.6220 \) is the ratio of the gas constants for dry air and water vapor.

Substituting (8.1) and (8.3) into (7.2) provides us with a generalized Wobus function:

\[ W^*_G(T_W, \pi) = K_2 \left( \frac{C}{T_W} \right)^\lambda \pi^\lambda \times \left\{ 1 - \exp\left[ \frac{-\lambda A e c(T_W)}{p_0 T_W} \pi^{-\lambda} \right] \right\} \quad \text{for} \]

\[ \theta_o \geq -18.6^\circ C, \quad \theta_w \leq 28.2^\circ C. \quad (8.4) \]

Figure 15 shows that for EPTs less than 377 K, \( W^*_G \) varies only slightly with pressure.

It is easily verified by series expansion that

\[ \pi^\lambda \left\{ 1 - \exp\left[ \frac{-\lambda A e c(T_W)}{p_0 T_W} \pi^{-\lambda} \right] \right\} \approx 1 - \exp\left[ \frac{-\lambda A e c(T_W)}{p_0 T_W} \right] \quad (8.5) \]
to second order in the arguments of the exponentials. Inserting this approximation into (8.4) gives us an expression for the pressure-independent Wobus function:

\[
W_G(T_W) = K_2 \left( \frac{C}{T_W} \right)^{\lambda} \\
\times \left\{ 1 - \exp \left[ -\lambda A \varphi(T_W) \right] \right\} \\
\text{for} \theta_u \geq -18.6^\circ C, \quad \theta_w \leq 28.2^\circ C,
\]

(8.6)

which has an intrinsic error \( \varphi(T_W, \pi) \approx W_G^*(T_W, \pi) - W_G(T_W) \), where \( \varphi(T_W, 1) = 0 \) because \( W_G(T_W) = W_G^*(T_W, 1) \). A corrected version of (6.7) that uses the pressure-independent Wobus function (8.6) is therefore

\[
\theta_W(T_W, \pi) = T_W/\pi - W_G(T_W/\pi) + W_G(T_W) \\
+ \varphi(T_W, \pi).
\]

(8.7)

The inclusion of \( \varphi \) reduces the maximum relative error in the computed \( \theta_W \) from 0.87 to 0.25 K (Figs. 16 and 17) for \( \theta_w \leq 28.2^\circ C \). (Note that we can safely exclude the lower range limit \( \theta_u \geq -18.6^\circ C \) because \( \varphi \) is negligible at cold WBPTs.) Thus, the estimated pressure correction is indeed quite small, which explains why the original Wobus method works fairly well.

9. Conclusions

A new method for computing \( \theta_w \) and the adiabatic wet-bulb temperature along pseudoadiabats is presented. Currently Wobus’s method is widely used for these purposes. It is based on a Wobus function \( W \) that is supposedly a function only of temperature. However, \( W \) has a slight dependency on pressure, which gives rise to errors of over half a degree in \( \theta_w \) and to errors up to 1.2 K in the temperature of parcels that are lifted adiabatically and then pseudoadiabatically to 200 mb. Although a new Wobus function of both temperature and

\[\text{FIG. 15. The Wobus function } W_G^*(T_W, \pi) \text{ vs } T_w \text{ at constant pressures of } 1000, 900, \ldots, 300 \text{ mb. The curves are truncated at the data points marked by } | \text{ signs by excluding data from points where } \theta_E \geq 377 \text{ K (} \theta_w \geq 28.2^\circ C).\]
pressure is devised in this paper, the resulting modified Wobus method is then just a convoluted version of the new method. The new technique is based on Bolton’s (1980) formula for $\theta_w$. The temperature $T_w$, on a given pseudoadiabat at a given pressure is obtained from this formula by an iterative technique. A very good “initial-guess” formula for $T_w$ is devised. In the pressure range $100 \leq p \leq 1050$ mb and wet-bulb potential temperature range $\theta_w \leq 40^\circ$C, this formula is accurate to within 0.34 K of the iterated solution. With only one iteration, the relative error is reduced to less than 0.02 K. There is an absolute error of up to 0.2 K in $\theta_w$ in Bolton’s formula that is caused mostly by variation of $c_{pd}$ with temperature and pressure. It is shown that the upper bound on the corresponding absolute error in $T_w$ is also 0.2 K. Since the algorithm has a relative error after one iteration that is much smaller, its overall error is $\leq 0.2$ K. With a few minor changes, the procedure also finds the temperature on water- or ice-saturation reversible adiabats.

Part of the initial solution is a linear relationship (4.9) or (4.10) between wet-bulb temperature and equivalent temperature raised to the $1/k_d$ power in a significant region of a thermodynamic diagram. This appears to be an interesting new discovery.

**Acknowledgments.** I acknowledge the ingenuity of the Wobus method, which was ahead of its time. My sporadic attempts over the years to discover how it worked enabled me to invent the new method. Valuable suggestions from the two anonymous reviewers led

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**FIG. 16.** The error in kelvins in the determination of $\theta_w$ from (6.7) using the pressure-independent Wobus function $W_c$ as a function of $\theta_w$ and $p$. The vertical line marks $\theta_w = 28^\circ$C to facilitate comparison with Fig. 17.
to significant improvements in the paper. This work was supported in part by NSF Grant ATM-0340693.

APPENDIX

The Derivatives of $f$

The first derivative of $f(\tau; \pi)$ at fixed pressure is given by

$$f'(\tau; \pi) = (\partial f/\partial \tau)_\pi = f(\partial \ln f/\partial \tau)_\pi. \quad (A.1)$$

where $\tau$ is in kelvins and

$$\left( \frac{\partial \ln f}{\partial \tau} \right)_\pi = -\Lambda \left[ \frac{1}{\tau} + \frac{\nu}{p - e_s} \frac{d e_s}{d \tau} + k_3 \kappa \ln \left( \frac{p}{p_0} \right) \left( \frac{\partial r_s}{\partial \tau} \right)_\pi \right] + \left( \frac{\partial G}{\partial \tau} \right)_\pi. \quad (A.2)$$

The second derivative of $f(\tau; \pi)$ at fixed $\pi$ is given by

$$f''(\tau; \pi) = (\partial^2 f/\partial \tau^2)_\pi = f'(\partial \ln f/\partial \tau)_\pi + f(\partial^2 \ln f/\partial \tau^2)_\pi. \quad (A.6)$$

Fig. 17. Same as in Fig. 16, but the error has been reduced by the derived correction $\varphi$ in the region of the grid for which the correction is either valid or negligible ($\theta \approx 28^\circ$).
where

\[
\left( \frac{\partial^2 \ln f}{\partial \tau^2} \right)_a = \lambda \left[ \frac{1}{\tau^2} - \frac{\nu}{(p - e_s)^2} \left( \frac{d e_s}{d \tau} \right)^2 - \frac{\nu}{p - e_s} \frac{d^2 e_s}{d \tau^2} \right. \\
\left. - k_3 k_d \ln \left( \frac{p}{p_0} \right) \left( \frac{\partial^2 r_s}{\partial \tau^2} \right)_a - \left( \frac{\partial^2 G}{\partial \tau^2} \right)_a \right].
\]  

(A.7)

\[
\left( \frac{\partial^2 G}{\partial \tau^2} \right)_a = \frac{2k_0}{\tau^3} \left( r_s + k_3 r_s^2 \right) - \frac{2k_0}{\tau^2} (1 + 2k_2 r_s) \left( \frac{\partial r_s}{\partial \tau} \right)_a \right.
\\

+ \left( \frac{k_0}{\tau} - k_1 \right) 2k_2 \left( \frac{\partial r_s}{\partial \tau} \right)_a \right. + \left( \frac{k_0}{\tau} - k_1 \right)
\\
\times (1 + 2k_2 r_s) \left( \frac{\partial^2 r_s}{\partial \tau^2} \right)_a. 
\]

(A.8)

\[
\left( \frac{\partial^2 r_s}{\partial \tau^2} \right)_a = \frac{e p}{(p - e_s)^2} \left[ \frac{d^2 e_s}{d \tau^2} - \frac{2}{p - e_s} \left( \frac{d e_s}{d \tau} \right)^2 \right].
\]

(A.9)

\[
\frac{d^2 e_s}{d \tau^2} = \frac{ab}{(\tau - C + b)^2} \left( \frac{d e_s}{d \tau} - \frac{2e_s}{\tau - C + b} \right).
\]

(A.10)

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