Numerical Dispersion of Gravity Waves

GUIDO SCHROEDER* AND K. HEINKE SCHLÜNZEN

Meteorological Institute, University of Hamburg, Hamburg, Germany

(Manuscript received 1 December 2008, in final form 6 June 2009)

ABSTRACT

When atmospheric gravity waves are simulated in numerical models, they are not only dispersive for physical but also for numerical reasons. Their wave properties (e.g., damping or propagation speed and direction) can depend on grid spacing as well as on the numerical schemes. In this work numerical dispersion relations for atmospheric gravity waves are theoretically derived as well as experimentally measured using the anelastic Mesoscale Transport and Stream model (METRAS). Both the theoretical solution and the numerical model show a retardation of gravity waves with decreasing grid resolution. Furthermore, the influence of a Shapiro seven-point filter is analyzed. The Shapiro seven-point filter causes damping of the shorter waves. Therefore, shorter waves can better be simulated without the seven-point filter. The influence of different advection schemes is analyzed by prescribing a background wind. A first-order upstream scheme and second- and third-order flux integrated essentially nonoscillatory (FIENO) schemes are used. As expected, the damping is the smaller the higher the order of the scheme. The numerical dispersion has severe consequences, when nonuniform grid spacing is used. Waves moving from the fine grid to the coarse are reflected because of numerical dispersion if they are only poorly resolved on the coarse grid. In tests with different refinement factors and wave lengths the reflection is found to be the larger the greater the refinement factor. The results show that refinement factors larger than 3 should not be used with nonuniform grid spacing or two-way nested grids.

1. Introduction

Solutions of the Navier–Stokes equations can include several types of waves. For the atmospheric flow internal gravity waves are especially important. These waves can be found on different scales throughout the atmosphere wherever it is stably stratified. They can be excited by mountains, convection, shear instability, and many more processes (Holton 1992; Zhang 2004). In atmospheric models, however, gravity waves can also be generated by dynamic and thermodynamic imbalances (e.g., caused by numerical noise or unbalanced initial conditions). Their group velocity depends on their wavelength and the buoyancy frequency. When gravity waves propagate upward, their amplitude increases with decreasing density (e.g., Zhang and Yi 2004). In the troposphere the wave energy density is highest close to the tropopause (Allen and Vincent 1995). They also play an important role in the middle atmosphere for energy and momentum transport (e.g., in the quasi-biennial oscillation; Dunkerton 1997), and they are a dominant source for turbulence due to gravity wave breaking and dissipation [cf. Fritts and Alexander (2003) for a review of middle atmospheric gravity waves].

As gravity waves influence the mesoscale flow (e.g., Zhang 2004), their accurate simulation is of great interest. One of the problems associated with the simulation of gravity waves is their dispersive property. The physical dispersion is mainly influenced by the static stability of the air. However, in numerical models the type of grid spacing as well as grid size and the numerical methods used additionally influence the properties of the waves. If different grid sizes are used within one domain, gravity waves that cannot be resolved on the coarse grid cannot leave the fine grid and are reflected at the internal boundary. In the most extreme case in which the wave cannot be resolved on the coarse grid it is trapped within the fine grid. Consequently, the stretching factor between fine and coarse grid is usually kept
smaller than 4 (Zhang et al. 1986), which still leaves some reflection. While in many publications factors of 4 or smaller are used for grid refinement, to our knowledge an actual thorough analysis of numerical gravity wave dispersion and reflection and the influence of the refinement factor thereupon has never been published. Wave reflection due to grid refinement has only been analyzed for simple wave problems like the advection of a wave structure (e.g., Vichnevetsky 1987), which is relevant for atmospheric models. In the field of general relativity theory this effect has also been investigated for gravitational waves by Choi et al. (2004).

Since high-resolution atmospheric simulations on large domains are still not feasible in terms of computational cost, two-way nesting and adaptive grid methods are frequently used. They are accompanied by nonuniform grid spacing (e.g., Lorenz and Jacob 2005; Bacon et al. 2007). Consequently, a justification of the acceptable refinement factors is needed, which can be achieved by quantifying gravity wave reflection. This is the motivation for this work, in which the numerical dispersion properties of gravity waves are analyzed. Numerical simulations of gravity waves on different grids are presented for different numerical methods. Section 2 describes the nonhydrostatic Mesoscale Transport and Stream model (METRAS) used for the simulations. Section 3 shortly describes analytical properties of gravity waves as found in text books and then presents a new theoretical derivation of the numerical dispersion related to grid refinement. Section 4 analyzes the numerical dispersion using numerical simulations. Furthermore, the influence of different numerical advection schemes is investigated. The consequences of grid refinement, namely reflection, are shown in section 5. The intensities of reflected waves are measured for different resolutions. Conclusions are drawn in section 6.

2. METRAS model

The METRAS model (Schlünzen 1990; Dierer et al. 2005) is used for testing the properties of gravity waves under different numerical conditions. METRAS is a three-dimensional, nonhydrostatic model in which the three wind components u, v, and w and the potential temperature \( \theta = T(1000 \text{ hPa}/p)^{R_C/\rho} \) (with the temperature \( T \), the specific gas constant \( R \) for dry air, and the specific heat capacity of dry air at constant pressure \( c_p \)) as well as humidity, cloud liquid water content, and rainwater content are calculated from prognostic equations. The equations are solved in flux form. The pressure \( p \) is calculated diagnostically. The perfect gas law relates density \( \rho \) to the potential temperature \( \theta \) and pressure. Turbulence, radiation, and cloud processes are parameterized.

Since for this work only the dry, dynamical core of the model is applied without frictional effects, Coriolis force, and the parameterization of turbulence, in the following only the simplified model equations are stated without the overbar commonly used to denote Reynolds-averaged quantities. Furthermore, all simulations presented here are two-dimensional, so that the equation for \( u \) and dependencies on \( y \) are omitted. In the equations of motion the Boussinesq approximation and the anelastic approximation are applied. It is assumed that \( \rho = \rho_0 \) except for the buoyancy term, which nearly cancels out with the vertical pressure gradient. The anelastic approximation assumes that density changes slowly with time. To make use of major balances in the atmosphere, the thermodynamic quantities are separated into a background part (subscript 0, only depending on \( z \)) and a deviation part (denoted by a tilde):

\[
\rho(x, z) = \rho_0(z) + \tilde{\rho}(x, z),
\]
\[
\theta(x, z) = \theta_0(z) + \tilde{\theta}(x, z), \quad \text{and}
\]
\[
\rho(x, z) = \rho_0(z) + \tilde{\rho}(x, z),
\]

with \( \tilde{p}(x, z) = p_1(x, z) + p_2(x, z) \). Here \( p_0 \) and \( \rho_0 \) as well as \( p_1 \) and \( \tilde{\rho} \) satisfy the hydrostatic equation:

\[
\frac{\partial \rho_0}{\partial z} = -g \rho_0 \quad \text{and} \quad \frac{\partial p_1}{\partial z} = -g \tilde{\rho},
\]

with the gravity acceleration \( g \); \( p_2 \) is the dynamic pressure calculated diagnostically from a Poisson equation to keep the momentum field divergence free.

Using the simplifications stated above quite simple equations remain—the two momentum equations:

\[
\frac{\partial \rho_0 u}{\partial t} = -\frac{\partial}{\partial x} (u \rho_0 u) - \frac{\partial}{\partial z} (w \rho_0 u) - \frac{\partial (p_1 + p_2)}{\partial x},
\]
\[
\frac{\partial \rho_0 w}{\partial t} = -\frac{\partial}{\partial x} (u \rho_0 w) - \frac{\partial}{\partial z} (w \rho_0 w) - \frac{\partial p_2}{\partial z},
\]
a thermodynamic energy equation:

\[
\frac{\partial \rho_0 \theta}{\partial t} = -\frac{\partial}{\partial x} (u \rho_0 \theta) - \frac{\partial}{\partial z} (w \rho_0 \theta),
\]

the continuity equation:

\[
0 = \frac{\partial}{\partial x} (u \rho_0) + \frac{\partial}{\partial z} (w \rho_0),
\]

and the linearized perfect gas law:

\[
\tilde{\rho} = \frac{\tilde{\theta}}{\theta_0} + c_p \frac{p_1 + p_2}{\rho_0}.
\]
Here \( \sigma \) is the specific heat capacity of dry air at constant volume.

For the results presented here advection of potential temperature and momentum is solved with first-order upstream (UP1) or second- (FIENO2) or third-order (FIENO3) flux-integrated essentially nonoscillatory schemes (Schroeder 2007). Pressure \( p_2 \) is solved from a hyperbolic Poisson equation by an ILU(1) preconditioned BiCGSTAB algorithm (Van der Vorst 1992).

### 3. Theoretical analysis—Properties of gravity waves on grids of different resolution

The linearized model equations yield gravity waves as solutions. The derivation of the physical dispersion relation of linear gravity waves can be found in many textbooks (e.g., Holton 1992). In this section, the results are briefly stated. The linearization of the velocities \( u = \bar{u} + u' \) and \( w = w' \) leads to a wave equation for the vertical deviation velocity \( w' \):

\[
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \left( \frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} \right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0. \tag{1}
\]

Note that in this case \( w' \) is not associated with turbulence, but with the deviation part that arises from the linearization. This is a common notation in dynamic meteorology. Equation (1) assumes \( \bar{u} \) is a constant, which makes the following derivations simpler and requires free-slip conditions at the surface. However, since all numerical simulations in this study are performed under the same conditions and with \( u' \ll \bar{u} \), all comparisons between numerical and analytical solution in the remainder of this paper are valid.

It can be shown that the group velocity depends on the vertical \( (\lambda_z) \) and horizontal \( (\lambda_x) \) wavelength as well as on the static stability (buoyancy frequency) \( N = \sqrt{g^0 \ln \theta_j / \partial z} \). The horizontal group velocity for a solution to this equation is given by

\[
c_x = \pm N \left( \frac{c_k}{c_k + c_m} \right)^{0.5} \tag{2}
\]

with the horizontal wavenumber \( k = 2\pi \lambda_x^{-1} \) and the vertical wavenumber \( m = 2\pi \lambda_z^{-1} \).

The derivations above can be found in standard textbooks, and there has been early work (e.g., Mesinger and Arakawa 1976) in which it was shown that the C-grid has the best properties in terms of numerical dispersion of shallow-water gravity waves. However, to our knowledge a theoretical derivation of the numerical dispersion relation of internal gravity waves with a dependence on grid spacing has never been published. For simplicity and independence of the advection scheme, we initially assume \( \bar{u} = 0 \). The numerical dispersion relation can be calculated in two steps. First, Eq. (3) is discretized spatially with the horizontal and vertical grid spacing \( \Delta x \) and \( \Delta z \), respectively:

\[
\left( \frac{\partial^2}{\partial t^2} + N^2 \right) \left( \frac{w_{j+1,l} - 2w_{j,l} + w_{j-1,l}}{\Delta x^2} \right) + \frac{\partial^2}{\partial t^2} \left( \frac{w_{j,l+1} - 2w_{j,l} + w_{j,l-1}}{\Delta z^2} \right) = 0. \tag{3}
\]

Second, a wave function \( w_{j,l}(t) \) is inserted in Eq. (3) (similarly to a von Neumann stability analysis):

\[
w_{j,l}(t) = \bar{w} \exp[i(k \Delta x + ml \Delta z - \nu t)], \tag{4}
\]

with the frequency \( \nu \) and the amplitude \( \bar{w} \). This yields

\[
(-\nu^2 + N^2) \exp(ik \Delta x) - 2 + \exp(-ik \Delta x)
\]

\[
- \nu^2 \exp(im \Delta z) - 2 + \exp(-im \Delta z) = 0.
\]

Solving this for \( \nu \) leads to

\[
\nu = \pm N \left( \frac{c_k}{c_k + c_m} \right)^{0.5} \tag{5}
\]

with the abbreviations

\[
c_k = \frac{1 - \cos(k \Delta x)}{\Delta x^2}
\]

and

\[
c_m = \frac{1 - \cos(m \Delta z)}{\Delta z^2}.
\]

The numerical horizontal group velocity in the \( x \) direction is

\[
c_{xm} = \frac{\partial \nu}{\partial k}
\]

\[
= \pm 0.5 N \sin(k \Delta x) \left( \frac{c_k + c_m}{c_k + c_m} \right)^{0.5}
\]

\[
= \pm 0.5 N \frac{s_k c_m}{c_k + c_m}^{0.5} \tag{6}
\]

where the abbreviation

\[
s_k = \frac{\sin(k \Delta x)}{\Delta x}
\]

has been used. From (6) it is clear that for \( \Delta x \approx \lambda_x / 2 \) the numerical horizontal group velocity tends against zero. In this case the waves are stationary.
When writing sine and cosine as Taylor series, it can be shown that for \( \Delta x \to 0 \) and \( \Delta z \to 0 \): \( c_k \to 0.5 k^2 \), \( c_m \to 0.5 m^2 \), and \( s_k \to k \). In this case (6) \to (2), which means that the numerical dispersion vanishes.

When a background wind \( \overline{w} \) is included into the derivation of numerical dispersion, \( \partial \overline{w} / \partial t \) needs to be replaced by \( \partial \overline{w} / \partial t + \overline{w}(\delta t_{i} - \delta t_{i-1})/\Delta x \). For a first-order upstream discretization and \( \overline{w} > 0 \), this would be equivalent to \( \partial \overline{w} / \partial t + \overline{w}(\delta t_{i} - \delta t_{i-1})/\Delta x \), where \( \delta t_{i-1} w_{i,i} = w_{i-1,i} \). Applying this to a wave function like (4) leads to

\[
\frac{\partial}{\partial t} + \frac{\delta t_{i} - \delta t_{i-1}}{\Delta x} w_{i,i}'(t) = \left\{ -i \nu + \frac{\overline{w}}{\Delta x} \right\} w_{i,i}'(t) \times (1 - \exp(-i k \Delta x)) \]

(7)

instead of

\[
\frac{\partial}{\partial t} w_{i,i}'(t) = (-i \nu) w_{i,i}'(t)
\]

in the original equation. Here, \( \nu \) indicates the frequency for the case with \( \overline{w} \not= 0 \). This shows that \( \nu \) in (5) needs to be replaced by \( \nu + i(\overline{w}/\Delta x)[1 - \exp(-i k \Delta x)] \), which leads to the dispersion relation:

\[
\nu = \pm N \left( \frac{c_k}{c_k + c_m} \right)^{0.5} - i \frac{\overline{w}}{\Delta x} \left[ 1 - \exp(-i k \Delta x) \right].
\]

(8)

This shows that an imaginary part of \( \nu \) is introduced, \( \Im(\nu) = -\pi/\Delta x[1 - \cos(k \Delta x)] \). For \( \overline{w} < 0 \) the upstream scheme would become a downstream scheme and \( w_{i,i}'(t) \) would grow exponentially. A downstream scheme would be unconditionally unstable, which is well known. However, for a first-order upstream scheme, the theoretical \( e \)-folding time of damping is then given by

\[
\tau = \frac{\Delta x}{|\overline{w}|[1 - \cos(k \Delta x)]}.
\]

(9)

For the real part we get

\[
\Re(\nu) = \pm N \left( \frac{c_k}{c_k + c_m} \right)^{0.5} + \frac{\overline{w}}{\Delta x} \sin(k \Delta x).
\]

The full numerical horizontal group velocity in the \( x \) direction then is

\[
c_{m} = \frac{\partial \nu}{\partial \delta k} = \pm 0.5 N \frac{s_k c_m}{c_k^{0.5}(c_k + c_m)^{1.5}} + \overline{w} \cos(k \Delta x).
\]

(10)

The larger the \( \Delta x \) is, the smaller is the effective contribution of the background wind speed to the numerical horizontal group velocity. For \( \Delta x \to 0 \), (10) tends against the analytical solution with \( \overline{w} \not= 0 \).

Unfortunately, for the higher-order ENO-based schemes, this analysis cannot be done analytically, since the schemes are highly nonlinear.

4. Model analysis—Properties of gravity waves

Zhang and Yi (2004) analyzed the propagation of gravity wave packets in a global atmospheric model. Their method to initialize gravity waves is also used in this work to investigate the numerical dispersion of gravity waves with METRAS. The hydrostatic background atmosphere is assumed to be isothermal:

\[
T_{0} = T_{c} \quad \rho_{0} = \rho_{c} \exp\left( -\frac{gz}{R T_{c}} \right).
\]

This leads to a stable stratification where \( \theta_{0} \) increases with height. The horizontal wind field perturbation is initialized by

\[
u'(x, z, t = 0) = u_{c} \exp\left[ -\frac{(x - x_{c})^{2}}{2\sigma_{x}^{2}} - \frac{(z - z_{c})^{2}}{2\sigma_{z}^{2}} \right] \]

\[
\times \cos\left[ \frac{2\pi}{\lambda_{x}} (x - x_{c}) + \frac{2\pi}{\lambda_{z}} (z - z_{c}) \right].
\]

where \( (x_{c}, z_{c}) = (-30 \text{ km}, 3 \text{ km}) \) is the center of the wave packet. For the analysis the following values are chosen: \( T_{c} = 250 \text{ K}, \rho_{c} = 1 \text{ kg m}^{-3}, u_{c} = 0.8 \text{ m s}^{-1}, \sigma_{x} = \lambda_{x} = 15 \text{ km}, \) and \( \sigma_{z} = \lambda_{z} = 1 \text{ km} \). In this case the static stability is \( N = 0.02 \text{ s}^{-1} \) and the analytical horizontal group velocity is \( c_{a} = 3.16 \text{ m s}^{-1} \). Figure 1 shows the initial conditions for \( u \). The amplitudes of the initial
waves are kept small in order to avoid nonlinear effects like wave breaking. The initial vertical wind, deviation pressure, deviation density, and deviation temperature are calculated with phase shifts to match the linear wave solution with a group velocity pointing upward and in positive x direction.

The model results are analyzed in terms of the wave energy density, which is given by Zhang and Yi (2004) as

\[ I = \frac{1}{2} \rho_0(u'^2 + w'^2) + \frac{(p_1 + p_2)^2}{2\rho_0 c_a^2} + \frac{(p_1 + p_2 - \rho' c_a^2)^2}{2\rho_0 \left( 1 - \frac{\rho'}{\rho_a} \right) c_a^2}. \]  

(11)

In this work the wave energy density is averaged over one wavelength in each spatial direction to get a smoother field:

\[ I'(x, z) = \int_{x'=x-\lambda_x/2}^{x'=x+\lambda_x/2} \int_{z'=z-\lambda_z/2}^{z'=z+\lambda_z/2} I(x', z') \, dx' \, dz'. \]  

(12)

At each time step the maximum of the smoothed wave energy density, \( I' \), is located, and its location and intensity are stored. For the first 8000 s the time series of the intensity and of the x coordinate of the location is analyzed with a regression method. A line is fit through the time series of the x coordinates, and the slope of the line gives the group velocity. Through the time series of the intensity a curve of the form \( b \exp(-t/\tau) \) is fit leading to the e-folding time \( \tau \) (damping coefficient) of the wave intensity (\( b \) is an amplitude parameter). It should be noted that the measured group velocity and damping can have a large error for highly damped waves. Especially, for the case with with \( \pi = \pm 10 \), the length of the simulation has an effect. For example, for FIENO2, \( \Delta x = \lambda/8 \), \( \bar{u} = 10 \), and 8000-s simulation time, the group velocity is 12.77 m s\(^{-1}\) and \( \tau = 1000 \) s. For only the first simulated 3000 s, the measured group velocity reduces to 12.48 and the e-folding time reduces to \( \tau = 730 \) s. The longer the simulation time, the more accurate the group velocity and damping can be estimated. Unfortunately, for lower-order schemes and coarser grids, the wave is more likely to be dissipated when the simulation time taken into account is too large.

For all simulations the vertical mesh size is \( \Delta z = \lambda_z/10 \), the horizontal mesh size is varied from \( \Delta x = \lambda_x/3 \) to \( \Delta x = \lambda_x/20 \). An additional test simulation with \( \Delta x = \lambda_x/2 \) yields gravity waves that dissipate too rapidly to be analyzed.

a. Background wind \( \bar{u} = 0 \) m s\(^{-1}\)

The simulations with zero background wind (\( \bar{u} = 0 \)) are performed with a third-order flux-integrated essentially nonoscillatory advection scheme (FIENO3; Schroeder 2007). Two types of simulation are made—one where the wind fields are filtered after each time step (FIENO3F) and one where the filter is omitted (FIENO3). The filter is a Shapiro (1971) seven-point filter written in flux form (comparable to sixth-order hyperdiffusion) with a flux limiter (Xue 2000). This kind of filtering is commonly applied to unstable schemes like centered differences. It is also used to avoid energy accumulation at the small scales due to nonlinear interaction.

Figure 2 shows the horizontal group velocity for both the filtered and unfiltered runs and the theoretical solution \( c_{sa} \) of Eq. (8) in section 3. All values are normalized with the analytic solution for the group velocity [Eq. (2)]. The e-folding time is shown for the two types of model runs (Fig. 2b).

The coarser the resolution is, the lower is the group velocity (Fig. 2a). This behavior is similar for both model
simulations and theoretical solution. Waves with \( \lambda_x \leq 4\Delta x \) cannot be resolved with the filtered model, whereas the unfiltered model can resolve waves with \( \lambda_x \geq 3\Delta x \).

For \( \lambda_x = 3\Delta x \) the group velocity is only about 45% of the analytical solution \( c_a \). The theoretical solution \( c_{\text{th}} \) predicts 50% for this resolution. For higher resolutions, the theoretical and simulated group velocities are similar with the theoretical group velocity being somewhat higher. The differences can be explained with the different discretization. The analytical solution is derived by discretizing the wave Eq. (1) for \( w' \) derived from the linearized equations, which is in contrast to the METRAS discretization on an Arakawa C-grid.

As expected, the damping of FIENO3 and FIENO3F is quite different. The effect of the seven-point filter is clearly visible for the filtered version with the strong decrease of the e-folding time for decreasing resolution (Fig. 2b). The 20 \( \Delta x \) waves remain nearly unchanged by the filter. However, when smaller time steps are used, the filter will be applied more often and will therefore add more diffusion to the solution.

With the amplitudes chosen for this test the simulated wave is nearly linear. Consequently, the applied advection scheme is almost negligible (except for vertical advection of the background potential temperature).

**b. Background wind** \( \bar{u} = \pm 10 \text{ m s}^{-1} \)

The effect of advection schemes on gravity wave dispersion can be measured when using a background wind. Figures 3 and 4 show the gravity wave group velocity and damping for first-order upstream advection (UP1), second-(FIENO2) and third-order (FIENO3) flux-integrated ENO schemes along with the result of the theoretical Eqs. (10) and (9) for a background wind of \( \pm 10 \text{ m s}^{-1} \) and different resolutions. The Shapiro seven-point filter is not used. The simulated wave is the same as presented before. The time step for each grid configuration is chosen to get a Courant–Friedrichs–Lewy (CFL) number of 0.1 with respect to the background wind.

It is clearly visible that the damping is dominated by the order of the advection scheme (Figs. 3b and 4b). For the first-order upstream scheme only the result \( \Delta x = \lambda_x/20 \) is given \( (\tau \approx 1000 \text{ s}) \), because after 8000 s the wave is already dissipated for the other simulations with coarser grids. However, the theoretical formula for the e-folding time in Eq. (9) has a similar order of magnitude \( (\tau \approx 1500 \text{ s}) \) for the same \( \Delta x \). The theoretical value is larger, because in the model there are further sources of damping besides the advection (see Fig. 2).

In the numerical simulations, when the intrinsic group velocity has the same direction as the background wind (here \( +10 \text{ m s}^{-1} \), Fig. 3b) the damping is less than when the intrinsic group velocity is opposed to the background wind (Fig. 4b). Shorter waves are damped more than longer waves as seen before. With a background wind this effect is intensified because of the diffusion of the advection schemes.

The group velocity should be approximately 13 and \( -7 \text{ m s}^{-1} \), respectively. As observed for zero wind background conditions coarser resolutions give smaller (absolute) group velocities (Figs. 3a and 4a). This effect is intensified by the advection schemes used that also tend to be dispersive (e.g., Schroeder 2007). The lowest-order scheme, UP1, has the strongest negative effect on the group velocity. However, for waves resolved with six or more grid cells the second-order FIENO2 performs better than FIENO3 concerning the group velocity. From this one might draw the conclusion that FIENO2 has the best overall performance, in contrast to what one would expect, that the higher the order of the scheme, the better the performance. However, when switching off all physical processes except for the pure advection of the wave pulse, FIENO2 and FIENO3 perform similarly in
the effective modeled advection speed (Fig. 5), and again the damping for FIENO3 is much smaller. This leads to the conclusion, that it is the interaction of advection and the gravity wave that leads to the difference. FIENO3 treats the wave slightly differently relative to FIENO2, which leads to a slightly different wave and therefore a slightly different group velocity. So both advection schemes modify the physical properties of the waves in a different way, leading to the slight difference in speed.

5. Reflection of gravity waves

The same wave as in section 4 is simulated for horizontally nonuniform grid configurations. For comparison, Figs. 6a,b show the results on uniform grids with $\Delta x = \lambda_x/20$ and $\Delta x = \lambda_x/5$ horizontal resolution, respectively. As seen before, the wave damping is higher and the group velocity smaller on the coarse grid (Fig. 6a). If the grid spacing changes (e.g., with a factor of 4; Figs. 6c,d,f), reflections take place. The reflection is total if the wave cannot be resolved on the coarse grid (e.g., Fig. 6c). Even if the wave can be resolved on both the fine and the coarse grid, reflections occur, but they have a smaller amplitude (Fig. 6d). This is due to the dependence of group velocities on grid spacing as seen in sections 3 and 4. When the wave packet moves from the fine to the coarse grid the energy cannot enter the coarse grid fast enough and some energy is reflected. For the same grid configuration ($\Delta x = \lambda_x/20$ for $x < 0$, $\Delta x = \lambda_x/5$ for $x > 0$) the reflection is slightly stronger for the filtered than for the unfiltered simulation (Figs. 6d,f). The reason is that the additional damping due to the filter is much higher in the coarse grid region than in the fine grid region. Thereby the property of the gravity wave is changed differently in both regions relative to the unfiltered case.

The intensity of the reflected waves can be measured and related to the intensity of the unreflected wave of the simulation on a uniform grid. For this purpose the results of several simulations are evaluated. They have been performed with a grid spacing of $\Delta x = \lambda_x/r$ for $x < 0$. The resolution indicator $r$ is an integer that describes how many grid cells resolve one wavelength. For $x > 0$ the grid spacing is $\Delta x = \lambda_x c/r$, where the coarsening
factor $c$ is an integer. In this region $r/c$ cells resolve one wavelength. The magnitude of the reflected wave is estimated as follows: for an experiment with a fine resolution indicator $r$ and a coarsening factor $c = c' > 1$ (with $c'$ being an integer), the maximum averaged wave energy density $I'$ [Eq. (12)] is determined in the fine region ($x < 0$). This is related to the maximum averaged wave energy density in the whole domain of the corresponding run with the resolution indicator $r$ on a uniform grid ($c = 1$) at the same time step. The result is the relative intensity $RI$:

$$RI = \frac{\max_{x<0,c'c}(I')}{\max_{x=1}(I')}.$$  \hspace{1cm} (13)

The denominator always corresponds to an unreflected wave. Strictly speaking, $RI$ does not indicate the relative intensity of a reflected wave, since it contains the maximum averaged wave energy density on the fine grid, which can still contain parts of the unreflected wave. However, after a sufficient time the unreflected wave should have left the fine grid, and $RI$ should reach a saturation value that gives a hint on the magnitude of the unreflected wave.

Fig. 6. Gravity wave simulation after 3 h for different horizontal grid configurations; $u$ is shown with scalings as in Fig. 1. The horizontal grid spacing is (a) $\Delta x = \lambda_1/5$; (b) $\Delta x = \lambda_1/20$; (c) $\Delta x = \lambda_1/5$ for $x < 0$ and $\Delta x = \lambda_1/1.25$ for $x > 0$; and (d),(f) $\Delta x = \lambda_1/20$ for $x < 0$ and $\Delta x = \lambda_1/5$ for $x > 0$. (a)–(d) Simulation without filter and (f) simulation with filter.
reflection. If no reflection takes place, RI should approach zero.

Figure 7 shows the relative intensities for $r = 5, 8, 10,$ and $20$ and coarsening factors of $c = 2, 3, \ldots, 7$. There are two types of curves. This will be described for the example of $r = 5$. For a reflected wave (e.g., Fig. 7a with $c > 2$), RI is 1 as long as the wave has not reached the internal boundary with the grid discontinuity ($t < 1.5$ h). When it reaches this point, reflection takes place. The already reflected part and the unreflected part are superimposed which can lead to values larger than 1 (overshoot, $t = 2–3$ h). The reflected wave is damped, and with increasing time the reflected and unreflected waves diverge in different directions so that they cannot overlap anymore. Consequently, RI decreases. When the total reflection is finished and parts of the wave that may not have been reflected have left the finely resolved region ($x < 0$), the RI curve has reached its saturation value (after 5–7-h simulation time, depending on the configuration).

The second type of curve corresponds to waves that are not reflected. This will be described for $r = 20$ and $c = 2$ (Fig. 7d). Here the overshoot does not happen, as there is no reflected wave to overlap the unreflected wave. The RI value is decreasing with time until it has reached zero when no parts of the wave are left in the fine grid ($x < 0$).

It is interesting to note that, for example, for $c = 2$ the decrease of the RI value begins later for larger $r$. The reason for this is probably the damping caused by the internal boundary. The damping is weaker when the wave is better resolved on the coarse grid ($r/c$). This means that for larger $c$, RI decreases not only because of the wave leaving the fine grid but also because of damping.

After 8 h of simulation time in all simulations RI has reached saturation and the relative intensities can be compared and treated as the intensity of the reflected wave. As expected the intensity of the reflected wave is lower for a smaller coarsening factor $c$. When $r = 20$ only coarsening factors of 5, 6, and 7 show a reflected wave of about 5%, 12%, and 23%, respectively. For $c = 5$ there is almost no reflection.

For $r = 5$ the result is completely different. For $c > 2$ the reflected wave keeps about 63% of the unreflected wave intensity. This value does not vary much for different factors of $c$, as in neither grid configuration the wave can be resolved on the coarse grid ($x > 0$). Only for $c = 2$ the reflected wave has a lower intensity, but it still has about 30% of the unreflected wave intensity. This change in grid width seems to absorb more of the reflected wave than with larger coarsening factors, which

![Image of Figure 7](image-url)
means that the coarse grid still reacts to some extent as would be demanded by the wave to remain undisturbed. For \( r = 8 \) and 10 only with \( c = 2 \) reflection is nearly avoided. For \( c = 3 \) the reflected wave intensity is less than 26%, for \( c = 4 \) it is less than 50%. However, for the remaining coarsening factors the reflected wave intensity reaches up to 71% for \( c = 7 \) and \( r = 8 \). Again, the reason for the different reflected wave intensities is that the coarse grid can react more quickly and in a more physical way if the coarsening factor is smaller and the resolution for \( x > 0 \) is higher.

Figure 8 shows the same result as Fig. 7c, but for a wave with \( c_w = 6.32 \text{ m s}^{-1} \).

6. Conclusions and outlook

The dispersion of gravity waves and their reflection, when a sudden coarsening of a grid is given, were investigated in this paper. The influence of a background wind and the applied advection scheme were tested for a first-order upstream scheme and a second- and third-order flux-integrated essentially nonoscillatory scheme (Schroeder 2007) for a CFL number of 0.1. Not surprisingly, the third-order scheme FIENO3 is much less damping than the first-order upstream scheme and less damping than the second-order scheme FIENO2. The numerical damping is stronger for the FIENO schemes when the background wind is opposed to the horizontal group velocity in comparison to the case where the background wind is blowing in the same direction.

It was shown that even for grid coarsening/refinement factors of 4 a significant reflection of gravity waves can take place. It amounts to almost 50% for waves that are well resolved on the fine grid (e.g., by 8 grid cells) but have too few grid points on the coarse grid. A reason for the reflection of waves that can be resolved on the coarse grid are the different group velocities on grids with different resolutions. A numerical dispersion relation was derived theoretically that confirms that the coarser the grid the more the gravity wave is retarded. Results from numerical experiments of gravity waves on grids with different resolution agree well with the theoretical numerical dispersion relation.

Consequently, when two-way nesting or grid refinement is applied, the refinement factor should not be larger than 3 to keep the reflected gravity wave energy small. The results from this paper can be transferred to other spatial and time scales as long as the gravity wave intensity is weak enough to be accurately described by the linearized equations. Simulations in which the relation of horizontal grid width to wavelength are the same as in this work should yield similar group velocities (normalized by the analytical solution) as presented here.

Acknowledgments. We thank the anonymous reviewer and Dr. Chungu Lu from the NOAA/Earth System Research Laboratory and the Colorado State University for very helpful suggestions for improving the paper. This work was supported by the Deutsche Forschungsgemeinschaft within the Graduiertenkolleg “Erhaltungsprinzipien in der Modellierung und Simulation mariner, atmosphärischer und technischer Systeme” (Conservation principles in modelling and simulation of marine, atmospheric, and technical systems) and the DFG Project “SCHL 499/2-1.”

REFERENCES


