Horizontal Momentum Diffusion in GCMs Using the Dynamic Smagorinsky Model

URS SCHAEFER-ROLFFS AND ERICH BECKER
Leibniz-Institut für Atmosphärenphysik, Kühlungsborn, Germany

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ABSTRACT

A dynamic version of Smagorinsky’s diffusion scheme is presented that is applicable for large-eddy simulations (LES) of the atmospheric dynamics. The approach is motivated (i) by the incompatibility of conventional hyperdiffusion schemes with the conservation laws, and (ii) because the conventional Smagorinsky model (which fulfills the conservation laws) does not maintain scale invariance, which is mandatory for a correct simulation of the macroturbulent kinetic energy spectrum. The authors derive a two-dimensional (horizontal) formulation of the dynamic Smagorinsky model (DSM) and present three solutions of the so-called Germano identity: the method of least squares, a solution without invariance of the Smagorinsky parameter, and a tensor-norm solution. The applicability of the tensor-norm approach is confirmed in simulations with the Kühlungsborn mechanistic general circulation model (KMCM). The standard spectral dynamical core of the model facilitates the implementation of the test filter procedure of the DSM. Various energy spectra simulated with the DSM and the conventional Smagorinsky scheme are presented. In particular, the results show that only the DSM allows for a reasonable spectrum at all scales. Latitude–height cross sections of zonal-mean fluid variables are given and show that the DSM preserves the main features of the atmospheric dynamics. The best ratio for the test-filter scale to the resolution scale is found to be 1.33, resulting in dynamically determined Smagorinsky parameters $c_S$ from 0.10 to 0.22 in the troposphere. This result is very similar to other values of $c_S$ found in previous three-dimensional applications of the DSM.

1. Introduction

Parameterization of subgrid scales (SGS) in large-eddy simulations (LES) is an ongoing and challenging task in computational fluid dynamics (see Meneveau and Katz 2000; Lesieur et al. 2005 for an overview). This holds in particular for geophysical flows, where the dynamically relevant scales usually span many orders of magnitudes. In fact, direct numerical simulations (DNS) by solving the Navier–Stokes equations can be applied only when geophysical flows are considered within a limited computational domain, for example, to analyze the breakdown of internal gravity waves in the middle atmosphere (Fritts et al. 2009a,b) or to simulate the transition from stratified turbulence to isotropic turbulence (Waite 2011). Even then, the demand for computer resources is extremely high.

When considering the general circulation of the atmosphere, all scales from planetary distances to the microscale contribute to the characteristics of the flow. Corresponding numerical models [general circulation models (GCMs)] are typically truncated at some horizontal scale between 100 and 1000 km. It is clear that the subgrid-scale flow in such a model consists mainly of nonlinear wave dynamics. Nevertheless, the mesoscale part (horizontal scales smaller than ~500 km) of the global kinetic energy spectrum of the atmosphere can be described by the concept of macroturbulence (i.e., it is characterized by a forward horizontal energy cascade giving rise to a $-5/3$ power law; Nastrom and Gage 1985; Hamilton et al. 2008). This behavior can be interpreted as stratified turbulence (Lindborg 2006). According to this theoretical concept the $-5/3$ power law in the mesoscales with regard to the horizontal wavenumber is accompanied by a $-3$ law with regard to the vertical wavenumber. The Ozmidov scale, which typically amounts to only a few meters in the troposphere, marks the transition from stratified to three-dimensional isotropic turbulence. This is the pathway by which the kinetic energy generated at large scales (baroclinic Rossby waves) is ultimately transferred to the microscale and dissipated by molecular friction in the free troposphere.
The corresponding forward horizontal energy cascade is balanced in weather and climate models by some ad hoc horizontal diffusion, which usually takes the form of an explicit hyperdiffusion (i.e., fourth- or even higher-order diffusion) or a numerical filter to selectively damp the highest wavenumber regime in the kinetic spectrum. Such methods are often regarded as numerical measures to stabilize the model. We emphasize, however, that they represent SGS parameterizations and have to be considered carefully. In other words, such "balancing" terms have a physical meaning and they cannot be chosen ad libitum. It has been shown (Gent and McWilliams 1990) that, for instance, the harmonic diffusion approach preserves main properties of the related eddy-resolving models.

Hence, every GCM and, further, every LES is characterized by a net forward energy cascade in the resolved scales; and this must be balanced by a SGS horizontal diffusion scheme representing the interaction or mixing with the nonresolved scales. In particular, the generation of kinetic energy by large-scale weather systems, as simulated with a GCM, feeds the resolved macro-turbulent energy cascade and the subsequent removal of kinetic energy by the SGS model. Therefore, the baroclinic wave activity is usually tuned in a GCM by adjusting the parameters of the horizontal diffusion. This seemingly technical experience of modelers reflects the fundamental physical relevance of the Lorenz energy cycle (Lorenz 1967, his chapter 5) for the dynamics of the atmosphere. It is, therefore, desirable to identify the deficits of existing SGS schemes and seek for improvements. For example, the commonly used hyperdiffusion schemes are generally not compatible with the hydrodynamical conservation laws (Becker 2001).

The Smagorinsky scheme (Smagorinsky 1963, 1993), in contrast, is consistent with the conservation laws and has widely been used in geophysical flows. There are, however, only a few applications in GCMs (Becker and Burkhardt 2007, hereafter BB07). Nevertheless, the Smagorinsky scheme also has an important deficit, namely, the violation of scale invariance (Oberlack 2000). The dynamic Smagorinsky model (DSM; Germano et al. 1991; Meneveau and Katz 2000) generally extends the conventional Smagorinsky model to a scale-invariant description. The main idea in the DSM is to parameterize the smallest resolved structures with the same SGS model that is used to represent the unresolved scales (i.e., the generalized mixing-length concept; Prandtl 1942; Smagorinsky 1963), and then to estimate the Smagorinsky parameter by assuming invariance of the Smagorinsky parameter at different scales. This approach has proven quite useful in technical fluid simulations (Moin and Kim 1982). It has also been applied in LES of the atmospheric boundary layer for a $O(1 \text{ km})$ domain size (Porté-Agel et al. 2000; Kumar et al. 2006, 2010). To the best of our knowledge, the DSM has not yet been applied to parameterize horizontal diffusion in general circulation models of the atmosphere (GCMs). In view of the aforementioned deficits of existing methods we find it worthwhile to design a two-dimensional (horizontal) DSM for horizontal diffusion in GCMs.

Even in the mesoscales, the atmospheric flow is, as mentioned before, highly anisotropic (i.e., it is characterized by large aspect ratios of horizontal and vertical velocities, $|u| \gg |w|$, and the corresponding scales, $l_h \gg l_z$). Moreover, the forward energy cascade in the mesoscales of the troposphere is due to the horizontal flow (Koshyk and Hamilton 2001; Lindborg 2006). This cascade must be balanced by a SGS model. As noted by Smagorinsky (1993), this requires an anisotropic momentum diffusion with the horizontal diffusion coefficient being very large against the vertical one. Horizontal and vertical diffusion schemes can, hence, be treated separately. In view of the aforementioned deficits of the conventional Smagorinsky-type horizontal diffusion and the usual practice of employing unphysical hyperdiffusions, the present paper focuses on the proper representation of horizontal momentum diffusion in terms of a new version of the DSM. It is beyond the scope of the present study to also extend the vertical diffusion so as to fulfill the condition of scale invariance. We note, however, that for the vertical scales typically resolved in a GCM, a forward energy cascade has not yet been identified, neither in models nor in observations. In fact, for stratified turbulence, which is the currently accepted theoretical picture for the atmospheric mesoscales, an energy inertial range is expected only for the horizontal motion. The transition to a three-dimensional energy cascade occurs at the Ozmidov scale, which is never resolved in any large-scale circulation model. It is thus possible to argue that scale invariance for vertical diffusion is not required in the formulation of a GCM.

The paper is structured as follows. Section 2 provides a short recapitulation of the conventional and DSM; readers who are familiar with the theory may skip this section. In section 3 we discuss some general constraints on the DSM and propose three solutions for the dynamical Smagorinsky parameter, one of which turns out to be applicable in a GCM. In section 4 we present numerical results obtained with a mechanistic GCM. In particular, we compare turbulent flow characteristics such as diffusion coefficients and frictional heating rate as well as the global horizontal energy spectra for the conventional Smagorinsky diffusion and the new DSM. Finally, we summarize our results and discuss the potential usefulness of the tensor norm solution of the DSM in atmospheric modeling (section 5).
2. Conventional versus dynamic Smagorinsky model

For convenience we shall consider an incompressible horizontal flow. The application of the so-derived flow resulting in a horizontal diffusion scheme applied on terrain-following layers of a GCM is straightforward. The nonresolved horizontal turbulence in the fluid is presented by the SGS stress tensor:

$$\tau_{ij} = \overline{\nu_i \nu_j} - \overline{\nu_i} \overline{\nu_j}.$$  

(1)

Here, the $v_i$ and $v_j$ denote the horizontal velocity components and an overbar indicates the spatial average over the smallest explicitly described (resolved) horizontal scale, denoted as the truncation scale $\Delta$. We assume that the trace of $\tau_{ij}$ is zero since the turbulent volume stress can be treated as part of the resolved mesoscale eddy activity, and can be calculated solely from the resolved scales. On the other hand, we assume that both stress tensors $T_{ij}$ and $\tau_{ij}$ can be parameterized with the mixing-length ansatz. Hence, in analogy to Eq. (1), we write the turbulent stress due to all scales smaller than $\Delta$ as

$$T_{ij} = \overline{\nu_i \nu_j} - \overline{\nu_i} \overline{\nu_j}.$$  

(4)

We assume that the trace of $T_{ij}$ also vanishes. The crucial step now is to introduce the Germano identity (Germano et al. 1991). For this purpose we define the turbulent stress due to scales between $\Delta$ and $\Lambda$ that act on scales larger than $\Delta$:

$$L_{ij} = T_{ij} - \tau_{ij} = \overline{\nu_i \nu_j} - \overline{\nu_i} \overline{\nu_j} - \overline{\nu_i \nu_j} + \overline{\nu_i} \overline{\nu_j} - \overline{\nu_i} \overline{\nu_j} - \overline{\nu_i} \overline{\nu_j}.$$  

(5)

The Germano identity [Eq. (5)] describes the resolved mesoscale eddy activity, and can be calculated solely from the resolved scales. On the other hand, we assume that both stress tensors $T_{ij}$ and $\tau_{ij}$ can be parameterized with the mixing-length ansatz. Hence, in analogy to Eq. (2) we can express $T_{ij}$ as

$$T_{ij} \simeq -2 (c_T \Delta)^2 \overline{S_{ij}}.$$  

(6)

with a Smagorinsky parameter $c_T$ for the test scale $\Lambda$. Inserting Eqs. (2) and (6) into Eq. (5) yields the approximative relation:

$$L_{ij} \simeq -2 (c_T \Delta)^2 \overline{S_{ij}} - \left[ \overline{S_{ij}} \overline{S_{ij}} - \left( c_T \Delta \right)^2 \overline{S_{ij}} \right]$$

$$= -2 c_S^2 \left[ \frac{(c_T \Delta)^2}{c_S} \overline{S_{ij}} \overline{S_{ij}} - \left( c_T \Delta \right)^2 \overline{S_{ij}} \right] = c_S^2 M_{ij},$$  

(7)

where $[\ldots]$ denotes filtering the whole term in brackets. The second equality is strictly valid only if $\overline{S_{ij}}$ and $c_S^2$ are slowly varying at the scale of the test filter. This is obviously true for $\overline{S_{ij}}$, which we treat as a numerical constant. For the Smagorinsky parameter the assumption must be
made without any mathematical justification (Ronchi et al. 1992), that is, we require that $c_S^2 \approx c_S^2$. Assuming further that the Smagorinsky parameter is comparable at the test-filter and resolution scale, we have $c_S^2 = c_S^2$ and the dynamic Smagorinsky parameter can be computed from

$$L_{ij} = c_S^2 M_{ij} \quad \text{with} \quad M_{ij} \simeq -2\{\Delta^2 \overline{S_i S_j} - \Delta^2 \overline{S_i S_j}\}.$$  
(8)

Equations (7) and (8) form the basis for our subsequent considerations. The resulting parameterized turbulent stress can be written as

$$(\tau_{ij})_{DSM} = -2c_S^2 \Delta^2 \overline{S_i S_j},$$  
(9)

with $c_S^2$ being calculated from Eq. (8). In contrast to Eq. (2), the ansatz in Eq. (9) is scale invariant: $L_{ij}$ scales with the squared velocity, $M_{ij}$ likewise with the squared shear, thus $c_S^2$ with the squared length. Thus, $(\tau_{ij})_{DSM}$ scales analogously to $\tau_{ij}$ with the squared velocity.

### 3. Computation of the dynamic Smagorinsky parameter

Before investigating Eq. (8) in detail we refer to some general properties of SGS models based on the mixing-length approach.

First, the main idea of the original Smagorinsky model or the DSM is to specify a turbulent diffusion coefficient $\nu_T$, such that the turbulent stress can be written as $\nu_T \overline{S_i S_j}$ with $\nu_T = L_{ij}^2 \overline{S_i} = c_S^2 \Delta^2 \overline{S_i}$. Note that even though molecular friction is negligible in the resolved momentum equation, as in simulations of the global circulation, the frictional heating due to molecular dissipation of turbulent kinetic energy is generally not. This heating ultimately yields the internal entropy production due to the general circulation. In a diagnostic SGS model the turbulent frictional heating, usually denoted in turbulence theory as turbulent dissipation (e.g., Burchard 2002, his chapter 2.2.5), is balanced by the sum of shear and buoyancy production of turbulent kinetic energy. On the other hand, the thermodynamic equation of the resolved flow must include the difference between the frictional heating and buoyancy production (Becker 2004). The overall heating from the SGS model then consists of the shear production and the commonly considered turbulent heat flux convergence. The shear production formally corresponds to the frictional heating due to turbulent friction. Since the entropy can only increase because of internal processes, the second law of thermodynamics requires that this shear production is positive definite for any diagnostic SGS model. As pointed out in Becker (2001), this is not fulfilled for hyper-diffusion schemes or spectral filters. For the DSM the shear production can be written as

$$\epsilon = -\tau_{ij} \overline{S_i S_j} = \nu_T \overline{S_i^2} = c_S^2 \Delta^2 \overline{S_i^2}.$$  
(10)

Hence, the DSM fulfills the second law as long as $\nu_T \geq 0$ or, equivalently, if the dynamically determined Smagorinsky parameter $c_S^2$ is positive definite. One may relax this constraint slightly by using $\langle c_S^2 \rangle$, where $\langle \cdot \rangle$ denotes some spatial or temporal average over resolved scales, and requiring $\langle c_S^2 \rangle > 0$ instead. Otherwise, if one has $\langle c_S^2 \rangle < 0$, instabilities will arise in numerical simulations (cf. Zang et al. 1993).

Second, noting that the tensors $L_{ij}$ and $M_{ij}$ are symmetric and have zero trace, each tensor has two independent elements. Hence, Eq. (8) is a system of two independent algebraic equations to determine the single unknown $c_S^2$ and is generally not a mathematically exact equation. Like the Smagorinsky model [Eq. (2)], it is just a more or less crude approximation to describe subgrid-scale processes (Meneveau and Katz 2000)—mainly because we must not expect a perfect correlation between all stress tensor elements $\tau_{ij}$ and strain tensor elements $\overline{S_i S_j}$. Therefore, Eq. (8) can at most be valid in a statistical sense and any useful solution will be the result of some further approximation. In the following subsections we discuss three approximate solutions, starting with the method of least squares.

#### a. Least squares solution

Solving Eq. (8) approximately by the method of least squares leads to (Lilly 1992)

$$c_S^2 = \frac{L_{ij} M_{ij}}{M_{kl}},$$  
(11)

with $L_{ij}$ and $M_{ij}$ given by Eqs. (5) and (8). The resulting Smagorinsky parameter must be subject to the smoothing operator $\langle \cdot \rangle$ to ensure $\langle c_S^2 \rangle > 0$. The reason is that $L_{ij} M_{ij}$ may become negative, which violates the second law and leads to instability in numerical simulations. In technical simulations with specific boundary conditions, the spatial or temporal averaging might be feasible, but it is not in global circulation models.

#### b. Solution without invariance of $c_S$

A second solution is obtained by relaxing the assumption $c_T^2 = c_S^2$. This approach, suggested by Moin (1991), might be interesting for conventional resolutions where the truncation scale lies within the energy cascade of stratified turbulence while the test-filter scale lies within the enstrophy cascade of geostrophic turbulence.
Defining \( N_{ij} = -2\Delta^2 \overline{\mathbf{S}}_{ij} \) and \( P_{ij} = 2\Delta^2 \overline{\mathbf{S}}_{ij} \), and using \( c_S^2 \approx c_3^2 \) as before, we have from the second equality of Eq. (7)

\[
L_{ij} = c_T^2 N_{ij} + c_S^2 P_{ij},
\]

or written in components,

\[
L_{11} = c_T^2 N_{11} + c_S^2 P_{11}, \quad L_{12} = c_T^2 N_{12} + c_S^2 P_{12}.
\]

This system can be solved without further approximations by

\[
c_S^2 = \frac{L_{12} N_{11} - L_{11} N_{12}}{P_{12} N_{11} - P_{11} N_{12}} \quad \text{and} \quad c_T^2 = \frac{L_{12} P_{11} - L_{11} P_{12}}{N_{12} P_{11} - N_{11} P_{12}}.
\]

Only \( c_S^2 \) is needed for the turbulent diffusion coefficient. However, the solution [Eq. (14)] may also generate numerical instabilities due to negative or even diverging values of the Smagorinsky parameter.

c. Solution applying the tensor norm

A rather simple and positive definite solution can be found if we postulate that only the squared version of Eq. (8) shall hold:

\[
L_{ij}^2 = c_S^2 M_{ij}^2.
\]

After summation over all tensor elements, the Smagorinsky parameter is determined by

\[
|\mathbf{L}| = c_S^2 |\mathbf{M}|
\]

or

\[
c_S^2 = \frac{|\mathbf{L}|}{|\mathbf{M}|},
\]

where \( |\mathbf{L}| \) and \( |\mathbf{M}| \) are the tensor norms of \( L_{ij} \) and \( M_{ij} \), respectively, according to

\[
|\mathbf{L}| = \sqrt{2L_{ij}^2} \quad \text{and} \quad |\mathbf{M}| = \sqrt{2M_{ij}^2}.
\]

A simple physical interpretation of the solution [Eq. (17)] is obtained if we rewrite Eq. (17) in terms of the mixing length \( l_h = \Delta c_S \). Because \( \sqrt{|\mathbf{L}|} \) is of the order of the resolved turbulent velocity scales, \( \sqrt{|\mathbf{L}|} \sim \bar{v} - \bar{v} = \bar{v} \),

\[
1 \quad \text{Hereafter, } \bar{v} = \bar{v} - \bar{v} \text{ defines the resolved fluctuating component of } \bar{v} \text{ between the test filter and the resolution scale for some flow variable } \alpha.
\]

and the wind shear norm \( \overline{\mathbf{S}} \) can roughly be regarded as the inverse of the eddy lifetime \( \tau^E \), we have

\[
l_h = (\Delta c_S) \sim \bar{v} \frac{\tau^E}{\sqrt{2[\Delta / \Delta]^2 - 1}}.
\]

Thus, the mixing length is proportional to the velocity scale and the lifetime of the resolved turbulent structures. This estimate corresponds to the result of Taylor (1920) who found that the mean distance from the initial position of a particle in a turbulent fluid due to diffusion (note that this definition is related to that of the mixing length) is given by the mean turbulent velocity scale of the particle multiplied with a time scale determined by the velocity correlation coefficient.

In summary, the approximate solution [Eq. (17)] using the tensor norm preserves the scale invariance of the DSM. In addition, because \( c_S^2 \) is positive definite, numerical instabilities due to unphysical backscattering of SGS turbulent kinetic energy are not possible and the second law is fulfilled. There is also no need for ad hoc smoothing of \( c_S^2 \), as is necessary when applying Lilly’s least squares solution [Eq. (11)] or the solution without invariance of the Smagorinsky parameter [Eq. (14)]. Generally, if a dynamic SGS model does not ensure \( \langle \nu_T \rangle \geq 0 \), it violates the second law of thermodynamics. This may also explain the instabilities in many numerical simulations, which are handled at least in some cases by neglecting all negative values for the Smagorinsky parameter (Zang et al. 1993). In addition, the realization of any averaging rule \( \langle \rangle \) is difficult and eventually not feasible in a global atmospheric circulation model. For the proposed tensor norm approach, on the other hand, the Smagorinsky parameter can be calculated in a straightforward manner.

4. Numerical results using a GCM

Here we present an application of the DSM using the Kühlingsborn mechanistic general circulation model (KMC) for perpetual January conditions. The dynamical core of this model is based on the standard spectral method (e.g., Bourke et al. 1977), that is, the application of the spectral transform method in the horizontal and finite differentiating with regard to a terrain-following vertical hybrid coordinate proposed by Simmons and Burridge (1981). For the present study we use a spectral truncation at total horizontal wavenumber \( n_T = 120 \) and 30 full model layers in the vertical. Except for the horizontal diffusion scheme and the spectral truncation, the present model setup is basically the same as in Becker (2003) and BB07. In particular, for properly
tuned horizontal diffusion, the model simulates a quite reasonable planetary and synoptic-scale wave activity comparable to comprehensive climate models, as well as a realistic strength of the Lorenz-energy cycle. For comparison of the conventional Smagorinsky scheme with the DSM, both horizontal diffusion schemes were tuned in such way that the planetary and synoptic-scale wave activity were nearly identical with the same model parameters otherwise. According to BB07, the conventional diffusion scheme is set to zero in the lowermost part of the troposphere. Additional runs were performed to illustrate either the role of the mixing length in the first case or the role of the test filter for the DSM. All runs were preceded by spinup integrations until the macroturbulent horizontal kinetic energy spectrum (for definition see BB07) was equilibrated and continued up to 360 model days. For illustrations we shall compare the spectra computed from the different runs and usually averaged over 90 model days. Since the upper model layer is located around 0.1 hPa we focus on the troposphere and lower stratosphere. A sponge layer due to enhanced linear harmonic horizontal diffusion is placed in the stratopause region.

\[ a. \text{ Formulation of the horizontal diffusion} \]

Our implementation of the DSM replaces the constant mixing length used in the Smagorinsky scheme by the dynamically determined mixing length

\[ l_h = c_S \Delta, \]

with the squared Smagorinsky parameter \( c_S^2 \) computed according to Eq. (17). The resolution or truncation scale \( \Delta \) is constant and given for each resolution by

\[ \Delta = \frac{2 \pi a_E}{2 n_T + 1}, \]

where \( a_E \) is the earth’s radius and \( n_T \) the horizontal truncation wavenumber. The other solutions [Eqs. (11) and (14)] were also tested in the KCM but turned out to be unstable.

For our purpose we have to compare filtered variables like \( \vec{S}_ij \) or \( \vec{v}_y \). The test filter removes all spectral amplitudes above a certain value, either like a cutoff or with a finite interval. We use the notation TF \( n_x/n_y \) to denote a filter kernel, which sets in at a wavenumber \( n_x \) and reaches full strength at the wavenumber \( n_y \) where all filtered variables vanish (see left part of Fig. 1):

\[ f(n) = \begin{cases} 
1, & \text{for } n < n_x \\
\cos \left( \frac{\pi n - n_x}{2 n_y - n_x} \right), & \text{for } n_x \leq n \leq n_y \\
0, & \text{for } n > n_y.
\end{cases} \]

The kernel of a filter with a wavenumber \( n_y \) larger than the truncation wavenumber \( n_T \) of the model is normalized such that it reaches full strength at \( n_T \) (see right part of Fig. 1). The equivalent wavenumber \( n_F \) of a sharp filter is given by the condition that the integrated area of the filter kernel shall be the same to the integrated area of this sharp filter. This constraint yields

\[ \int_0^{n_y} f(n) dn = \int_0^{n_T} H(n_F - n) dn = n_F, \]

where \( H(n) \) denote the Heaviside step function. For \( n_y = n_T \) this is identical to the arithmetic mean of \( n_x \) and \( n_y \), while for \( n_y > n_T \) we have

\[ n_F = n_T - \frac{n_T - n_x}{2 [1 - f(n_T)]} + \frac{n_y - n_x}{\pi} \sqrt{\frac{f(n_T)}{1 - f(n_T)}}. \]

The ratio \( \frac{\Delta}{\Delta} \) of the DSM is then defined according to

\[ \frac{\Delta}{\Delta} = \frac{n_T}{n_F}. \]
The filtering of single variables such as \( \overline{\overline{u}} \) or \( \overline{\overline{v}} \) can be done in spectral space just by multiplying the filter kernel with the spectral variables. However, the spatial products \( \overline{\overline{u}} \overline{\overline{v}} \) and \( |S| \overline{\overline{u}} \) require further additional transformations. To minimize these transformations we use the following approximations:

\[
\frac{\partial^2}{\partial x^2} |S| \overline{\overline{u}} \approx \frac{\partial^2}{\partial x^2} |S| \overline{\overline{u}} - (|S| \overline{\overline{u}} + |\overline{\overline{u}}| |S|) = \left( \frac{\partial^2}{\partial x^2} - 1 \right) |S| \overline{\overline{u}},
\]

which are similar to the Reynolds decomposition with the aforementioned decomposition \( \overline{\overline{u}} = \overline{\overline{u}} + \overline{\overline{\overline{u}}} \). Thus, we need three additional transformations for the products in Eq. (26a) but none for Eq. (26b).

As the model setup is identical to the one in BB07 except for the horizontal diffusion, we adopt their notation of the tendencies. The tendency of the horizontal wind vector in the KCMC owing to horizontal momentum diffusion is given as [cf. Eq. (17) in BB07]

\[
H = \rho^{-1} V \left[ \rho \left( c_5 \overline{\overline{u}} \right)^2 |S| |S| + \overline{\overline{u}} \left( c_5 \overline{\overline{u}} \right)^2 \overline{\overline{u}} |S| \right]
= \left( c_5 \overline{\overline{u}} \right)^2 \overline{\overline{u}} |S| + \overline{\overline{u}} \left( c_5 \overline{\overline{u}} \right)^2 \overline{\overline{u}} |S|,
\]

where \( \rho \) is the density. When vertically discretized with regard to the vertical hybrid coordinate, \( \rho \) is replaced by the finite pressure difference \( \Delta p \) at each model layer (see section 3 in BB07). Applying the dynamic mixing length \( l_h = c_5 \overline{\overline{u}} \), the terms \( H_1 \), \( H_2 \), and \( H_3 \) are the same as in BB07. However, because \( l_h \) is not constant in the DSM, one has to calculate \( V_{l_h}^2 = \Delta_{LH} V_{C_3}^2 \) in addition. Such terms are obtained using again the spectral transformation method. As a result of Eq. (17), we have

\[
V_{C_3}^2 = \frac{\left( V |L|^2 \right)^2 - \left( V |M|^2 \right)^2}{2 |L|^2} c_5^2.
\]

With Eq. (26b), we can deduce

\[
\frac{V |M|^2}{2 |M|^2} = \frac{V |S|^4}{2 |S|^4} = \frac{V |S|^4}{|S|^2}.
\]
sufficient to simulate a realistic macroturbulent kinetic energy spectrum, regardless of the specific value of the mixing length. Figure 2 shows time-averaged spectra of the tropospheric kinetic energy around 250 hPa, when using the conventional Smagorinsky scheme with three different values of the mixing length. The upper two spectra (with \( l^2_h = 1.25 \times 10^9 \) m\(^2\) and \( 4 \times 10^8 \) m\(^2\), dotted and solid, respectively) fail to yield a realistic mesoscale range at wavenumbers larger than 40; the synoptic regime is followed by an increasingly shallow slope in the mesoscales. In particular, both simulations do not indicate a \(-\frac{5}{3}\) inertial range. In contrast, the third run with \( l^2_h = 1.25 \times 10^8 \) m\(^2\) (dashed) forms a reasonable spectrum in the mesoscales. However, it fails in describing the baroclinic wave activity properly since the slope is significantly steeper than \(-\frac{5}{3}\) in the synoptic regime as a result of the large mixing length. Thus, a realistic simulation of the energy spectrum concerning all resolved scales is not achieved with the conventional Smagorinsky scheme.

In Fig. 3 we show a comparison of two kinetic energy spectra using either the conventional (dotted) or the DSM scheme (solid line). The conventional Smagorinsky employs \( l^2_h = 4 \times 10^8 \) m\(^2\), while the DSM uses a TF 90/90. The filter in the DSM case was chosen after several test simulations such that the baroclinic (synoptic) wave activity showed a close similarity to that obtained with the conventional Smagorinsky scheme. We furthermore found that the 90/90 filter is most efficient in the present T120 model to remove kinetic energy at high wave-numbers. In the enstrophy cascade regime, both diffusion schemes yield the same behavior, while the DSM allows for a more realistic spectrum between wave-numbers 45 and 95, including the transition to the \(-\frac{5}{3}\) inertial range. As already mentioned, the conventional model does not form such a transition. Both simulations exhibit a little hook at the high wavenumber end. Such a hook indicates insufficient damping at the highest resolved wavenumbers, which is a typical feature of spectral models (see e.g., Hamilton et al. 2008) and was already discussed by Kraichnan (1976).

Figure 4 shows the temporally and zonally averaged dynamically determined horizontal mixing length \( l_h \) and the Smagorinsky parameter \( c_S \) for the DSM run with TF 90/90. Both parameters are related through Eq. (20), where \( \Delta = 1.66 \times 10^5 \) m for the given resolution of T120. We see a pronounced maximum in the tropical troposphere. In this region the mesoscale wave activity is obviously strongest, which is presumably a consequence of the energy cascade associated with largerscale equatorial waves. We see that in the extratropical winter middle and upper troposphere the dynamic mixing length is larger than the constant value assumed in the conventional run. Only in the summer extratropics and toward the stratosphere (not shown) it is partly below that constant value. Regarding the local Smagorinsky parameter, we find a range of \( c_S \simeq 0.10 - 0.22 \). Notably, this two-dimensional horizontal result is very similar to three-dimensional calculations [\( c_S \simeq 0.17 \) in Schumann (1993); Vreman et al. (1997)], even though stratified macroturbulence and three-dimensional Kolmogorov turbulence are fundamentally different.

Now we investigate the influence of the test filter within the DSM. As is evident from our foregoing theoretical analysis (section 3), the test filter should not set...
in within the synoptic scales in order not to include two
different inertial ranges (i.e., the horizontal enstrophy
and energy cascades). On the other hand, if the filter
sets in at too large wavenumbers, the filtered variables
become comparable to the unfiltered ones and the re-
siduals vanish \((n_E \rightarrow n_T \Rightarrow \bar{\omega} \rightarrow \bar{\omega}_r \text{ and } \bar{\omega}^e \rightarrow 0)\). In such
a case, \(c_S^2 \rightarrow 0\) and the diffusion would be too weak.

Figure 5 shows time-averaged results for TF 55/65 (dotted), 90/90 (solid), and 90/150 (dashed), corresponding
to scale ratios of \(\frac{\Delta L}{\Delta L_s} = 2, 1.33,\) and 1.10, respectively.
Each of the runs has a transition from the enstrophy
cascade range to the energy cascade regime, but the best
result is obtained with the TF 90/90 run. The TF 90/150
run yields the aforementioned behavior of a less effec-
tive diffusion that causes a shallower slope in the me-
soscales. The TF 55/65 run exhibits almost the same
behavior as the TF 90/90 version, but has a somewhat
stronger spectral aliasing at the highest wavenumbers
(i.e., a larger spectral hook). In the following, we com-
pare the DSM run with TF 90/90 to conventional simu-
lation using the conventional Smagorinsky scheme with
\(l_h = 4 \times 10^3\) m².

In Fig. 6 we see how the spectrum in the DSM run is
composed of the rotational and divergent components.
The rotational part dominates the spectrum down to
the truncation wavenumber and becomes shallow in the
mesoscales. The transition to the energy inertial range
thus turns out to be a combination of the shallowing
rotational spectrum and the shallow branch of the di-
vergent flow, as it is usual in GCMs with high horizontal
and moderate vertical resolution.

The DSM scheme should describe the planetary and
synoptic scales of the atmosphere as good as the con-
tentional Smagorinsky model. To prove this we compare

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**FIG. 4.** (left) The dynamic horizontal mixing length \(l_h\) and (right) the Smagorinsky parameter \(c_S\). For comparison, the constant mixing length in the conventional Smagorinsky simulation is \(l_h = 20 \times 10^3\) m.

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**FIG. 5.** Comparison of different choices of the test filter in the
DSM model: TF 55/65 (dotted), TF 90/90 (solid), and TF 90/150 (dashed). The thin dashed lines are as in Fig. 3.

**FIG. 6.** The DSM model (solid) split in the rotational (dotted) and
divergent (dashed) part. The thin dashed lines are as in Fig. 3.
latitude–height cross sections of zonal-mean variables. We shall show that most properties are independent from the horizontal diffusion scheme except for variations of the horizontal diffusion coefficient and related quantities. In Fig. 7 we present the temporally and zonally averaged horizontal diffusion coefficient $K_h$ for the two runs. Both fields have maxima in the middle troposphere, the maximum value obtained with the DSM run being about twice as strong as with the conventional Smagorinsky scheme. This difference is related directly to the local calculation of the Smagorinsky parameter (i.e., the mixing length). In the conventional run, the variability of $K_h$ is determined only by the wind shear $|\mathbf{S}|$ according to $K_h = \frac{c_s}{l_h} |\mathbf{S}|$. Because $l_h = \text{constant}$, the maxima are located near the tropospheric jets where the baroclinic wave activity is strongest. In contrast, we have $l_h^2 \propto \frac{c_s^2}{|\mathbf{S}|} = \frac{|\mathbf{L}|}{|\mathbf{M}|}$ in the DSM run. The higher the mesoscale activity $\mathbf{U}_m$, the larger is the mixing length $l_h = c_s \Delta$. The large-scale dynamics of the middle troposphere appears to be a source for such mesoscale activity, especially in the tropics and winter extratropics. Hence, the combined maxima of the wind shear $|\mathbf{S}|$ and the Smagorinsky parameter $c_s^2$ (cf. Fig. 4, right panel) result in higher values of the diffusion coefficient in the winter extratropics and a second maximum in the tropical troposphere. Obviously, regions with high mesoscale activity induce larger mixing lengths that, in turn, generate higher diffusion coefficients. Thus, the resulting damping also enhances in these regions and the mesoscale slope in the global kinetic energy spectrum is steeper in the DSM run.

Figure 8 shows the temporally and zonally averaged zonal wind components in the troposphere and lower stratosphere for both runs. The DSM run yields stronger subtropical jets in both hemispheres, which is more pronounced in the northern winter hemisphere. Only

![Fig. 7. The horizontal diffusion coefficient. (left) Run with the DSM and (right) control run with the conventional Smagorinsky diffusion.](image)

![Fig. 8. The zonal wind (negative values shaded). (left) Run with the DSM and (right) control run with the conventional Smagorinsky diffusion.](image)
in the southern midlatitudes, where the DSM diffusion coefficient is small, is the jet also slightly weaker compared to the conventional run. In particular, these differences can be explained by the different diffusion coefficient in the DSM run, which weakens the baroclinic waves in the subtropics and northern midlatitudes such that the Hadley cells (not shown) and the jets become stronger. The larger differences in the Northern Hemisphere may result from the generally higher wave activity in the winter hemisphere.

The zonally and temporally averaged transient kinetic energy is depicted in Fig. 9. The structures of the conventional run and the DSM run confirm the dependence of the simulated baroclinic wave activity on the horizontal diffusion coefficient as mentioned above. Both simulated wave activities are comparable in strength and location for the Southern Hemisphere. Differences emerge in the winter troposphere, where the maximum is broadened and weakened by about 20% in the DSM run. We expect that a stronger transient kinetic energy is indicative of a stronger Lorenz energy cycle and should therefore be closely linked to a higher frictional heating rate due to horizontal momentum diffusion:

$$\epsilon = K_h |\mathbf{S}|^2.$$  (34)

As shown in Fig. 10, this heating is considerably decreased in the DSM run by about a factor of 2 when compared to the conventional run.

We summarize our numerical results as follows: the main larger- and synoptic-scale dynamics of a GCM can be simulated with the DSM as with the conventional Smagorinsky scheme. Moreover, differences can be reasonably explained by different values of the diffusion coefficient or, equivalently, of the mixing length. The form of the energy spectrum can be varied with the onset of the test filter. It turns out that for a T120L30 model the best result describing a proper energy inertial range in the mesoscales is obtained for TF 90/90.
5. Conclusions

We have formulated a two-dimensional horizontal version of the dynamic Smagorinsky model (DSM) applicable in large-eddy simulations of the general circulation of the atmosphere. Our main motivation has been that the horizontal momentum diffusion is a physical subgrid-scale model in order to balance the forward horizontal enstrophy and energy cascades, rather than a mere measure to ensure numerical stability. Therefore, it is necessary to formulate this particular subgrid-scale model in a physically consistent fashion. The DSM fulfills this general requirement. In particular, the conservation laws are fulfilled and the scale invariance of the governing equations (Oberlack 2000) is maintained. In this context we have followed the arguments of Meneveau and Katz (2000) that the basic Eq. (8) for the Smagorinsky parameter $c_S$ is the result of some kind of a scaling relation, rather than of an exact equation. Therefore, we have sought a convenient estimate of $c_S$. Three approximate solutions have been presented: Lilly’s method of least squares (Lilly 1992), a solution when the assumption of invariance of $c_S$ is relaxed, and the tensor-norm approach. Each solution has been applied tentatively to our GCM. It turned out that both the Lilly approach and the noninvariance approach do not work properly in a GCM because they do not guarantee a positive definite diffusion coefficient, not even on average. The tensor-norm ansatz, on the other hand, fulfills this elementary constraint, which may also be understood as a consequence of the second law.

This paper has presented an implementation of the tensor-norm DSM for horizontal diffusion in a spectral GCM, namely the Kühlingsborn mechanistic general circulation model (KMCM; Becker and Schmitz 2001; Becker 2009). We have confirmed that the conventional Smagorinsky scheme, where one has to adjust a constant mixing length for the whole atmosphere, does not provide a realistic spectrum at high wavenumbers (unless some additional hyperdiffusion is included; see BB07). The DSM scheme, in contrast, requires to adjust a so-called test filter and allows to simulate both a $-3$ energy range from wavenumbers 10–50 as well as an approximate $-5/3$ energy range for higher wavenumbers in the upper troposphere. Only at the highest wavenumbers near the spectral truncation, does a hook arise due to some spectral aliasing.

One may argue that we simply trade one tunable parameter for another; namely, the mixing length for the test filter wavenumber when switching from the conventional Smagorinsky scheme to the DSM. However, the scale-invariance constraint mentioned above is fulfilled only by the DSM. Notably, the dynamically determined Smagorinsky parameter is, with $c_S \approx 0.10-0.22$, very similar to results from three-dimensional simulations documented in the literature, where $c_S \approx 0.17$.

The implementation of the dynamic scheme in the KMCM allows us to simulate the general circulation of the troposphere with realistic properties of the large-scale flow and the synoptic wave activity. The horizontal diffusion coefficient is a factor of 2 stronger in the DSM run than in the conventional run, mainly due to an increased mixing length in the tropical and extratropical northern winter troposphere. This stronger diffusion affects the baroclinic waves such that the overall transient wave activity is somewhat weaker in the upper troposphere. In particular, its strengths are comparable in the southern summer hemisphere for both runs, while the maximum in the northern mid-latitudes is reduced by about 20% when using the DSM. Accordingly, the DSM run yields slightly stronger subtropical jets, in particular in the Northern Hemisphere, compared to the conventional run. These differences are also reflected in the frictional heating due to horizontal momentum diffusion; that is, due to the reduced baroclinic wave activity in the DSM run, the dissipative heating is about only half as strong as in the conventional run.

A rule of thumb of how to adjust the test filter for a given resolution is not at hand yet. One possibility is that the ratio $\Delta \lambda / \lambda$ should be the same for all resolutions. Alternatively, it should be tested whether the filter itself provides some analytical filtering of scales to the primitive equations. Further high-resolution simulations are required to address these questions.

The vertical diffusion has not been addressed in this paper due to the fact that atmospheric circulation models do usually not resolve an inertia range in the vertical. In addition, a vertical energy cascade is not expected for stratified turbulence, which is assumed to hold in the mesoscales of the free atmosphere (Lindborg 2006). Therefore, the conventional Smagorinsky model continues to be a reasonable option for vertical diffusion.

The question remains whether for other fluid variables like potential temperature or passive tracers, the horizontal and vertical turbulent diffusion coefficients should be the same as those used in the momentum equation or whether individual turbulent Prandtl numbers should be introduced. Another question is whether the principle of scale invariance can be applied to prognostic turbulence models like TKE schemes or the $k-c$ model.

Summarizing, the proposed dynamic Smagorinsky model is a valuable extension of the Smagorinsky horizontal diffusion scheme for GCMs. One may use this scheme not only in the troposphere, but also in
simulations extending into the stratosphere and mesosphere provided that the assumption of an inertial range is justified for the resolved scales.

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REFERENCES


