Local Finite-Amplitude Wave Activity as a Diagnostic for Rossby Wave Packets

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ABSTRACT

Upper-tropospheric Rossby wave packets (RWPs) are important dynamical features, because they are often associated with weather systems and sometimes act as precursors to high-impact weather. The present work introduces a novel diagnostic to identify RWPs and to quantify their amplitude. It is based on the local finite-amplitude wave activity (LWA) of Huang and Nakamura, which is generalized to the primitive equations in isentropic coordinates. The new diagnostic is applied to a specific episode containing large-amplitude RWPs and compared with a more traditional diagnostic based on the envelope of the meridional wind. In this case, LWA provides a more coherent picture of the RWPs and their zonal propagation. This difference in performance is demonstrated more explicitly in the framework of an idealized barotropic model simulation, where LWA is able to follow an RWP into its fully nonlinear stage, including cutoff formation and wave breaking, while the envelope diagnostic yields reduced amplitudes in such situations.

1. Introduction

An important feature of midlatitude atmospheric dynamics is the existence of upper-tropospheric Rossby waves with synoptic- to planetary-scale wavenumbers. Often a Rossby wave is not strictly circumglobal; rather, its amplitude is spatially inhomogeneous with a relative maximum at a specific location decaying to smaller values at larger distances. This gives rise to so-called Rossby wave packets [RWPs; for a recent review see Wirth et al. (2018)]. A key feature of RWPs is the fact that they can, and usually do propagate in the zonal direction, with their group velocity being faster than the phase speed of individual embedded troughs and ridges.

The occurrence of a RWP has implications for local weather. RWPs have, in particular, been associated with high-impact weather like strong surface cyclones (Chang 2005; Wirth and Eichhorn 2014), heavy rainfall (Grazzini and van der Grijn 2002; Martius et al. 2008), or heat waves and drought (Schubert et al. 2011; Fragkoulidis et al. 2018). Furthermore, it has been argued that the existence of RWPs has implications for predictability (Lee and Held 1993; Grazzini and Vitart 2015), which is particularly relevant in the case of high-impact weather.

The importance of RWPs has motivated the development of various techniques for their identification and analysis. These techniques include the famous Hovmöller diagram (Hovmöller 1949), the analysis of eddy kinetic energy (short EKE; Chang and Orlanski 1993), the reconstruction of the envelope of the meridional wind field (Zimin et al. 2003, 2006), or specific forms of a wave activity flux (e.g., Takaya and Nakamura 2001; Wolf and Wirth 2017). Most of these techniques implicitly or explicitly make the assumption that the waves must be small amplitude such as to allow linearization, and in addition it is sometimes assumed that the waves must be almost plane waves. However, observed Rossby waves often do not satisfy these assumptions, being large amplitude or showing strongly nonlinear behavior such as wave breaking or cutoff formation. Even in these situations the above diagnostics can formally be applied, but the results should be considered with care, because there is no quantitative connection with the underlying equations any longer. Instead, it would be desirable to use a diagnostic that is able to deal with RWPs of any amplitude and during highly nonlinear flow situations.

A quantity particularly suitable for our purpose is finite-amplitude wave activity, since it quantifies the vigor...
Wave activity is then defined as

\[ P = \int \frac{1}{2} h^2 \, dA \]

where \( P \) is the meridional gradient of the basic state PV and \( h \) is the thickness of the fluid layer. In contrast to other measures of wave activity, this definition does not involve a time derivative and thus obeys an exact conservation relation (Andrews et al. 1987). In addition, wave activity satisfies a conservation relation

\[ \frac{\partial}{\partial t} \mathcal{A} + \nabla \cdot \mathbf{F} = N + O(\alpha^3), \]

where \( \mathcal{A} \) is the generalised Eliassen–Palm flux (Andrews and McIntyre 1984) and \( \mathbf{F} \) is the associated nonconservative wave activity flux. A notable exception to this conservation relation is the effective wave activity in the primitive equations (Andrews and McIntyre 1986), which can be formulated as

\[ \frac{\partial}{\partial t} \mathcal{A} + \nabla \cdot \mathbf{F} = N - O(\alpha^3), \]

where \( \mathcal{A} \) is the effective Eliassen–Palm flux (Andrews and McIntyre 1986) and \( \mathbf{F} \) is the associated effective nonconservative wave activity flux. However, this formulation is only valid for small-amplitude waves.

A further important property of wave activity is its conservation in the linear framework. In the past there have been several successful attempts to formulate wave activity theories valid for finite-amplitude eddies, like the Casimir impulse wave activity (Killworth and McIntyre 1985; Methven 2013). Unfortunately, these measures of wave activity have some issues. For example, the Casimir impulse wave activity, although conforming to an equation of the form of (5), requires the computation of an initial ground state, which must be obtained through a computationally rather involved PV inversion technique. A notable exception to this computational complexity is the finite-amplitude wave activity (FAWA) proposed by Nakamura and Zhu (2010), which can be computed from PV data in a fairly straightforward manner. FAWA is defined in terms of the general concept of wave activity and motivates the definition of a new quantity, the "pseudodivergence" operator as in Tung (1986).
The atmosphere is assumed to be stably stratified and to satisfy hydrostatic balance. We consider the primitive equations in spherical coordinates (longitude $\lambda$ and latitude $\phi$) with potential temperature $\theta$ as vertical coordinate. Since we restrict attention to diagnosing the atmospheric flow at a specific time step, we will not explicitly indicate the time dependence of any of the variables in the following.

a. Theory

First, we briefly summarize the development from Nakamura and Solomon (2011). Consider a quasigeostrophic framework, and the authors called their diagnostic local wave activity (LWA). By definition, LWA is a function of both longitude and latitude, and one recovers FAWA upon zonal averaging. This new diagnostic is able to provide information about the “local waviness” of the flow while at the same time being valid for finite-amplitude eddies. The authors applied LWA to study spatially localized features such as blocking (Huang and Nakamura 2016) and storm tracks (Huang and Nakamura 2017). Although this provided interesting results, the limitations given by the quasigeostrophic approximation result in some undesirable effects like the occurrence of spurious wave activity regions in the equatorial and subtropical upper troposphere (Nakamura and Solomon 2010, 2011).

This state of affairs motivated us to combine the developments of Huang and Nakamura (2016) with those of Nakamura and Solomon (2011) in order to develop a formulation of LWA, which is valid in the primitive equations framework. The use of Ertel PV in isentropic coordinates (Ertel 1942; Hoskins et al. 1985) allows one to extend such a diagnostic to the subtropics and to the midlatitude upper troposphere, both of which are regions of frequent RWP occurrence. In this paper we will, first, formulate a primitive equation variant of LWA in isentropic coordinates. We will then apply this novel diagnostic to reanalysis data and compare the results with a more traditional diagnostic based on the envelope of meridional wind. This comparison suggests that the novel diagnostic is better suited to follow RWPs into their large-amplitude nonlinear stage. To corroborate this hypothesis we will finally make use of an idealized simulation with a barotropic model on the sphere.

Our paper is organized as follows. Section 2 contains a brief summary of the theory of LWA and its extension to the primitive equations framework, including a description of our algorithm for the identification of RWPs. In section 3 the new diagnostic is applied to an observed case of RWP propagation and breaking, and we compare our results with the envelope of meridional wind diagnostic. To support the interpretation of this comparison, we apply these two diagnostics in section 4 to the evolution of an idealized RWP in a barotropic simulation. Finally, section 5 provides a discussion and our conclusions.

2. Theory and methodology

The atmosphere is assumed to be stably stratified and to satisfy hydrostatic balance. We consider the primitive equations in spherical coordinates (longitude $\lambda$ and latitude $\phi$) with potential temperature $\theta$ as vertical coordinate. Since we restrict attention to diagnosing the atmospheric flow at a specific time step, we will not explicitly indicate the time dependence of any of the variables in the following.

a. Theory
is the vertical component of relative vorticity (with the horizontal derivatives in the above expression being performed along isentropes), and \((u, v)\) denotes the horizontal wind. FAWA at a given \(\Phi_M\) and \(\theta\) is then defined as

\[
A^*(\Phi_M, \theta) = \frac{1}{2\pi a \cos \Phi_M} \left( \iint_{q \geq Q} q \sigma dS - \iint_{q \leq Q} q \sigma dS \right).
\]

(11)
The quantity \(A^*\) is nonnegative by design, which makes it a suitable diagnostic to measure the waviness of the flow. The expression (11) can be reformulated as

\[
A^*(\Phi_M, \theta) = \frac{1}{2\pi a \cos \Phi_M} \left( \iint_{D_S} q \sigma dS - \iint_{D_N} q \sigma dS \right),
\]

(12)
where \(D_S\) and \(D_N\) denote the following domains of integration:

\[
D_S : [q \geq Q(\Phi_M)] \cap [\phi < \Phi_M],
\]

(13)

\[
D_N : [q \leq Q(\Phi_M)] \cap [\phi > \Phi_M],
\]

(14)
with the subscripts \(S\) and \(N\) referring to areas south and north of \(\Phi_M\), respectively.

Following the analogous derivation of Huang and Nakamura (2016), we define LWA in isentropic coordinates as

\[
A(\lambda, \Phi_M, \theta) = \frac{1}{\cos \Phi_M} \left[ \int_{l_S} (q - Q) \sigma a \cos \phi \, d\phi + \int_{l_N} (Q - q) \sigma a \cos \phi \, d\phi \right],
\]

(15)
where \(Q(\Phi_M)\) is the PV value associated with the considered \(\Phi_M\), and \(l_S\) and \(l_N\) are the arcs along the meridian at \(\lambda\) that satisfy the following conditions:

\[
l_S : q \geq Q, \ \phi < \Phi_M,
\]

(16)

\[
l_N : q \leq Q, \ \phi > \Phi_M.
\]

(17)
For illustration see Fig. 1. When a \(Q\) contour becomes highly distorted, it can intersect a meridian multiple times, which means that the arcs \(l_S\) and \(l_N\) may consist of multiple separate sections. By design, both integrals on the right-hand side of (15) are nonnegative, implying that \(A\) is always nonnegative.

Let us try to get some intuitive understanding for the expression (15). As can readily be verified, the zonal average of \(A\) recovers \(A^*\), that is,

\[
\frac{1}{2\pi} \int_0^{2\pi} A(\lambda, \Phi_M, \theta) \, d\lambda = A^*(\Phi_M, \theta).
\]

(18)
This implies that \(A^*\) and \(A\) have the same physical units (viz., m s\(^{-1}\)). Introducing \(y = a \sin \phi\) and using the same shorthand notation as Huang and Nakamura (2016), the expression for \(A\) can succinctly be written as

\[
A(\lambda, \Phi_M, \theta) \cos \Phi_M = -\int_0^{\eta} (q - Q) \sigma \, dy,
\]

(19)
where the integration is understood to extend in both meridional directions from \(y = 0\) (which corresponds to equivalent latitude \(\Phi_M\)) to \(y = \eta(\lambda, \Phi_M, \theta, \iota)\), which is the farthest crossing of the \(Q\) contour with the current meridian, and where the integration covers only the sections \(l_S\) and \(l_N\), respectively (Fig. 1). Note that the new meridional coordinate \(\eta\) can be multivalued in \(y\). The last formulation indicates that LWA can broadly be considered to be a PV anomaly \((q - Q)\) integrated over a certain meridional range (\(\eta\)). This is analogous to the local (in longitude) contribution to small-amplitude wave activity in (2); however, in stark contrast the expression (19) does not require any small-amplitude assumption.

b. Computation of LWA from gridded data

We developed an algorithm that is suitable to derive LWA from any kind of gridded meteorological data such as reanalysis data or model data. Here, we assume that the data are given on an equidistant longitude–latitude grid on pressure levels. The algorithm consists of the following steps.

1) First we select an isentrope \(\theta\) that samples the upper troposphere in the midlatitudes. We require this isentrope to intersect the tropopause, which allows...
us to capture the Rossby wave propagation mechanism along the associated sharp gradient of PV (Hoskins 1991; Martius et al. 2010). We also require that the chosen isentrope does not intersect the ground within the considered domain. This condition is not too restrictive, since our analysis focuses on the upper troposphere [for an extension to isentropes that intersect the ground, see Schneider (2005); Nakamura and Solomon (2011)].

2) Next we perform linear interpolation of the horizontal wind $(u, v)$ and temperature $T$ from the available pressure levels to the chosen isentrope. Then, $\omega$ is calculated from the horizontal wind field using centered differences in the expression in (10). Isentropic density $\sigma$ is first calculated on the original pressure levels (using centered differences) and then linearly interpolated to the considered isentrope (this reduces numerical noise according to [Nakamura and Solomon 2011]. In the case of an unstable vertical profile ($\sigma < 0$) at a given grid point, the value of $\sigma$ at this grid point is overwritten by the average of the values at the neighboring grid points. This rarely occurs in the midlatitude upper troposphere, but is an eventuality that has to be taken into account. Finally, the Ertel PV is computed as $q = \omega \sigma$.

3) To compute LWA from data we first establish the relation between data latitudes $\phi = \Phi$ and the corresponding $Q(\Phi)$ value using (8). To do this, we start with a provisional set of $Q$ contours, which are equidistant between the minimum and the maximum $Q$ value on the considered isentrope. For each of these values $Q_i$, we evaluate the integral on the left-hand side of (8), and this will be referred to as $M(Q_i)$. The integration is done by applying a simple trapezoidal rule throughout most of the domain. Particular care is exercised for those grid intervals where the $Q$ contour intersects the respective meridian; in these grid intervals we introduce an auxiliary grid point at the intersection, which is subsequently used for an improved approximation of the integral. Thereafter we compute the right-hand side of (8) for our equidistant values of $\Phi$, yielding a set of values called $M_l$. The final set of $Q(\Phi)$ values is obtained from the numerical relation $M(Q)$ with the help of linear interpolation.

4) Once we have determined the relation between $\Phi$ and $Q(\Phi)$, we compute LWA using (15). The numerical procedure for the integration is analogous to the one described in the previous item.

c. Detection of RWP using LWA

The definition of LWA involves the meridional displacement of a PV contour from its mass equivalent latitude and, therefore, contains the full phase information of the underlying Rossby wave. However, when diagnosing RWP, it is sometimes desirable to discount the phase information and, instead, diagnose the wave packet as a whole. In the case of a wave packet with a well-defined wavelength, it is straightforward to remove the phase dependence using, for example, a moving average in the zonal direction with a window width that is equal to the wavelength (as done by Huang and Nakamura 2016). This approach, however, assumes the existence and an a priori knowledge of the underlying wavelength, which is not always given. This is especially true when working with real atmospheric data, where different wave packets with different wavelengths may coexist at the same latitude band. In addition, RWP sometimes enter a highly nonlinear stage, implying that the perturbations are not strictly wavelike any longer and the above method becomes difficult to apply.

Here we employ a general method that allows one to always extract a “dominant zonal wavenumber” $s_{\text{d}}(\lambda, \phi)$ that is local in both longitude and latitude. The method involves a zonal wavelet transform of the meridional wind and is described in more detail in the appendix. The zonal wavenumber $s_{\text{d}}(\lambda, \phi)$ is then used to compute a local wavelength as

$$\lambda_{\text{d}}(\lambda, \phi) = \frac{2 \pi a \cos \phi}{s_{\text{d}}(\phi, \lambda)}. \quad (20)$$

Finally, we apply a zonal filter to the LWA field by convolution with a Hann window (Harris 1978), which has a full width at half maximum of $\lambda_{\text{d}}(\lambda, \phi)$. Our algorithm works independently of whether or not the LWA field is wavelike. If, indeed, the perturbation is wavelike, the algorithm achieves the desideratum, namely, to remove the phase dependence and result in a smooth field that represents a wave packet. The resulting field is referred to as “filtered LWA” in the remainder of this paper. The reader has to be careful that when in the zonal filter the convolution with Hann window is performed with $\lambda_{\text{d}}(\lambda, \phi)$ depending also on longitude, such a filter does not commute with the gradient operator, preventing the filtered LWA from having an exact conservation relation. In spite of this, since in this work we will not perform any LWA budget and instead we are focused exclusively on the diagnostic and identification of wave packets, we will use the proposed zonal filter, which we believe is a fundamental part of our diagnostic.

3. Diagnosing Rossby wave packets in the real atmosphere

We now apply our LWA diagnostic to an episode that occurred in April 2011. This episode was characterized
by RWP propagation and Rossby wave breaking. It is of particular interest since it was associated with a “forecast bust” for the majority of the operational forecast models, showing a huge drop in the medium-range forecast skill over Europe (Rodwell et al. 2013). The authors associated this poor performance to the misrepresentation of moist convective processes over North America a few days earlier, and this error was subsequently communicated downstream embedded in a RWP. Data are retrieved from the ERA-Interim re Analyses (Dee et al. 2011) with a horizontal resolution of 2° × 2° on 20 pressure levels ranging from 1000 to 100 hPa.

First we analyze a snapshot from the observed episode. Figure 2 shows PV, LWA, and filtered LWA on the 320-K isentrope (which intersects the tropopause in the midlatitudes). The PV field shows a series of troughs and ridges over North America, indicating the presence of an RWP. In addition, there is strong Rossby wave activity over Europe and over Asia. The field of LWA (Fig. 2b) shows overall larger values in regions where the PV field is more disturbed. For instance, the troughs and ridges over North America in Fig. 2a coincide with increased values of LWA in Fig. 2b. Over Europe and eastern Asia, the PV field (Fig. 2a) shows very large amplitude perturbations, which are mirrored by large values of LWA in Fig. 2b. Finally, the filtered LWA field in Fig. 2c discounts the phase information altogether—as desired. Instead, it shows a much smoother field with three relative maxima: one over eastern Asia, one over...
North America, and one over Europe. The regions surrounding these three maxima are regions of strong waviness and represent the RWPs in our new metric and we will refer to them respectively as “Pacific RWP” (marked with “A” in Fig. 2c), “Atlantic RWP” (marked with “B” in Fig. 2c), and “Eurasian RWP” (marked with “C” in Fig. 2c).

We also computed the analogous LWA field of Fig. 2b but from the quasigeostrophic version of (15) (i.e., Ertel PV is replaced by quasigeostrophic potential vorticity). In the extratropical upper troposphere (at 300 hPa), quasigeostrophic LWA showed a very similar picture, but there were major differences in the tropics, with a spurious longitudinally extended band of LWA extending from the equator up to 20°N. This spurious region of LWA prevents one from identifying RWPs which originate from or migrate into this region (not shown). Presumably the time-zonal mean of such quasigeostrophic LWA is related the spurious region of quasigeostrophic FAWA found in the climatology of Nakamura and Solomon (2010), which, on the other hand, is not found anymore in the primitive equation analysis of Nakamura and Solomon (2011). It is not surprising that the quasigeostrophic approximation does not provide a realistic picture of the tropical dynamics, and this provides a major reason to go beyond quasigeostrophy and instead use LWA in the primitive equations/isentropic coordinates framework.

Next we consider a sequence of consecutive maps following the stage of Fig. 2. This is done in order to follow the evolution of the three RWPs identified above. For reference, Fig. 3 presents the corresponding sequence of PV maps. Apparently, PV shows large-amplitude waviness throughout the episode. The large trough that was initially located over the east coast of Asia develops a very strong and highly nonlinear low-PV anomaly on its downstream side (i.e., in the central Pacific). In addition, there is a sequence of troughs emanating from the North American continent. Typically, these troughs tend to become thinner as time proceeds, leading to anticyclonic wave breaking and cutoff formation reminiscent of “life-cycle 1” of Thorncroft et al. (1993). Eventually this sequence of events leads to blocking (i.e., a quasi-stationary low-PV anomaly over the eastern Atlantic and Europe). Apparently it was the onset of this blocked flow situation that was missed in practically all numerical weather predictions models during this episode.

The left column of Fig. 4 shows the filtered LWA for the same episode. The field is dominated by three areas with high values that identify our three RWPs (the labeling of RWPs follows from Fig. 2c). The Pacific

![Figure 3](https://example.com/fig3.png)
RWP originates from the East Asian trough mentioned earlier. It shows a steady eastward progression by about 60° in 5 days, which means an eastward group velocity of 12° day⁻¹. By contrast, the Atlantic RWP has a somewhat larger eastward group velocity initially, but around 14 April it seems to slow down and merge with the Eurasian RWP. The latter is quasi stationary during the first half of the episode, but later starts to move eastward across the date line.

To set our new diagnostic into perspective, we also computed the envelope of the meridional wind field.
according to Zimin et al. (2003). This diagnostic has been used extensively in the past in order to detect and diagnose RWPs and their evolution (Glatt et al. 2011; Grazzini and Lucarini 2011; Glatt and Wirth 2014; Souders et al. 2014; Wolf and Wirth 2017; Fragkoulidis et al. 2018). Broadly speaking, this technique estimates the envelope of a longitudinal wavelike signal involving the Hilbert transform in Fourier space. In our application, we combine this technique with a zonal filter restricting the range of zonal wavenumber to the intervals 2–9, allowing us to focus on synoptic- to planetary scales. Note that the envelope of meridional wind is Eulerian in both latitude and longitude, in contrast with LWA, which instead is partly Lagrangian in the meridional coordinate. During most of the episode, this envelope diagnostic (right column of Fig. 4) provides a similar representation of the Pacific RWP as our LWA diagnostic (left column of Fig. 4). However, by 17 April the Pacific RWP turns rather weak in terms of its envelope, while it remains strong in terms of filtered LWA. This large difference occurs at a time when the flow was characterized by highly nonlinear overturning of PV isolines, which cannot be described any longer as a “wave” to whatever approximation. Turning to the Atlantic RWP, the picture obtained from the envelope diagnostic is much less coherent in comparison with the LWA diagnostic. In particular, the envelope diagnostic does not show a clear eastward progression of the RWP during the first half of the episode. Note that this Atlantic RWP, too, was characterized by highly nonlinear behavior such as trough elongation and cutoff formation. Finally, the Eurasian RWP can be found in the envelope diagnostic, but again its evolution appears somewhat less coherent in comparison with the LWA diagnostic.

The results from this comparison suggest that the LWA diagnostic is more appropriate to follow the evolution of RWPs—in particular during their nonlinear stage. Obviously, this result was to be expected, because LWA is, by design, able to deal with highly nonlinear flows. By contrast, the framework of envelope reconstruction implicitly assumes that the perturbation is wavelike, but this property is lost as the perturbation reaches its nonlinear stage with overturning PV contours and wave breaking.

4. Large-amplitude Rossby wave packets in an idealized barotropic model simulation

In this section we test the performance of the two diagnostics using a barotropic model on the sphere. The barotropic model allows us to perform a clean and well-controlled RWPs’ propagation and decay experiment, while at the same time we avoid the complications of real three-dimensional flows (including nonconservative processes and three dimensionality).

The equation for barotropic flow on a sphere reads

$$\frac{\partial \xi}{\partial t} = -J(\psi, f + \xi) - \nu \nabla^2 \xi,$$  \hspace{1cm} (21)

where relative vorticity \(\xi\) is defined as in (10) except that the partial derivatives are now strictly in the horizontal direction, \(f\) denotes the Jacobian, and \(\psi\) is the stream-function; the latter is related to the horizontal wind through

\[ (u, v) = \left( \frac{1}{a} \frac{\partial \psi}{\partial \phi}, \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda} \right). \]  \hspace{1cm} (22)

The last term in (21) represents hyperdiffusion, which dissipates enstrophy near the smallest resolved scale. The coefficient for hyperviscosity is set to \(\nu = 10^{15} \text{ m}^3 \text{ s}^{-1}\). Equation (21) is discretized with a standard spectral transform method (with triangular truncation at T89) on a regular latitude–longitude grid with a resolution of \(2^\circ\). The equation is integrated in time for 5 days using a leapfrog scheme (with a time step \(\Delta t = 900 \text{ s}\)) including a Robert–Asselin filter (Robert 1966; Asselin 1972) to control the growth of the computational mode.

As an initial condition we specify a purely zonal background flow \(u_b(\phi)\) with a wavelike perturbation superimposed. For the background flow we follow Held and Phillips (1987) and set

\[ u_b(\phi) = A \cos \phi - B \cos^3 \phi + C \sin^2 \phi \cos^6 \phi \]  \hspace{1cm} (23)

with \(A = 25, B = 30,\) and \(C = 300 \text{ m s}^{-1}\). This flow resembles the climatology observed in the upper troposphere, with westerly subtropical jets at \(\phi = \pm 30^\circ\) and easterlies in the deep tropics. Onto this background flow we superimpose two localized wave packets, which are specified in terms of a relative vorticity perturbation as

\[ \xi^* (\lambda, \phi) = \xi_0 \cos \phi e^{-[(\phi - \phi_0)^2 + (\lambda - \lambda_0)^2]} [L_1(\lambda) + L_2(\lambda)], \]  \hspace{1cm} (24)

\[ L_1(\lambda) = e^{-[(\lambda - \lambda_0)^2]} \cos(s_1 \lambda), \]  \hspace{1cm} (25)

\[ L_2(\lambda) = e^{-[(\lambda - \lambda_0)^2]} \cos(s_2 \lambda), \]  \hspace{1cm} (26)

with \(s_1 = 6, s_2 = 9, \xi_0 = 8 \times 10^{-5} \text{ s}^{-1}, \phi_0 = \pi/4, \lambda_1 = \pi/3, \lambda_2 = 2\pi/3, \sigma_\phi = 10^\circ,\) and \(\sigma_\lambda = 30^\circ\). Note that we deliberately initialize two RWPs with different carrier wavenumbers \((s_1 \neq s_2)\) in order to test the performance of our filtering algorithm.
Figure 5 shows absolute vorticity at the initial time as well as at the end of the integration. It can be seen that by the end of the integration both wave packets have evolved into a very nonlinear stage with overturning contours of absolute vorticity and cutoff formation on the poleward side of the jet.

During the simulation, we compute the barotropic variants of both the filtered LWA and the envelope diagnostic from snapshots of the absolute vorticity and the meridional wind field, respectively. The envelope diagnostic is obtained from the meridional wind as before, where we keep zonal wavenumbers 2–9. For the LWA diagnostic, mass equivalent latitude must be replaced by equivalent latitude $\Phi_e$, which is defined as

$$
\int_{q=Q} dS = \int_{\phi=\Phi_e} dS,
$$

where the symbol $q = f + \zeta$ now denotes absolute vorticity (which plays the role of potential vorticity in the barotropic model). LWA is then calculated from (15) with $\sigma$ set equal to 1, $\Phi_M$ replaced by $\Phi_e$, and $q$ replaced by absolute vorticity.

Figure 6 shows the evolution of these two diagnostics throughout the simulation. During the first 24 h, both diagnostics provide a similar picture, with both RWPs being clearly visible. At later times, the amplitude of the envelope diagnostic decreases very noticeably, while the amplitude of the LWA diagnostic does not. After 120 h, the two RWPs have almost disappeared from the plot in terms of the envelope, while they are still clearly visible in the LWA diagnostic. Interestingly, the decay of the envelope starts to become noticeable roughly at the time when wave breaking sets in with overturning contours of absolute vorticity. It transpires that the LWA diagnostic is much more appropriate for tracking RWPs well into their nonlinear stage than the envelope diagnostic. We also observed that although there is a significant reduction in the magnitude of the meridional wind toward the end of the simulation, it is still large enough to provide a reasonable estimate of the local dominant wavenumber in our algorithm for the phase average.

The two diagnostics also disagree in terms of location and amplitude of the detected RWPs, rather slightly during the early stage, but more strongly during the late (nonlinear) stage. Generally there is a tendency for the LWA diagnostic to put the RWPs at a slightly more northern latitude compared to the envelope diagnostic. This is a known property of LWA, arising from the fact that LWA is nonlocal in latitude (see Solomon and Nakamura 2012; Huang and Nakamura 2016, especially...
their Fig. 4). In particular, a pronounced trough is typically associated with an equivalent latitude lying north of the trough. Regarding the zonal direction, the LWA diagnostic puts the detected RWPs at a more western position during the late stage in comparison with the envelope diagnostic. Comparison with Fig. 5 indicates that the western part of each RWP is characterized by closed vortex-like structures during the late stage. Apparently, it is these vortex-like structures that LWA is particularly sensitive to; on the other hand, the envelope diagnostic.
diagnostic tends to focus on the downstream wave-breaking portion of the RWP, which apparently produces a larger signal in terms of the meridional wind than the upstream vortex. The upstream vortex, in turn, tends to decouple from the RWP in the sense that it slows down and starts to move with the phase velocity (which is slower than the group velocity).

5. Discussion and conclusions

In this work we developed a new diagnostic to quantify the amplitude of upper-tropospheric Rossby wave packets (RWPs). The diagnostic is based on the local finite-amplitude wave activity (LWA) of Huang and Nakamura (2016), but extended to the primitive equations framework. The phase dependence of LWA is removed by, first, estimating a locally dominant zonal wavelength and, then, applying a zonal filter corresponding to this local wavelength.

We applied our LWA diagnostic to an episode of propagating wave packets which occurred in April 2011, in which one of them produced wave breaking and evolved into blocking. It turns out that during this episode the newly developed diagnostic provides a more coherent picture of a continuous progression of various RWPs than a conventional diagnostic based on the envelope of the meridional wind (Zimin et al. 2003). The better performance of the new method in such nonlinear situation is associated with a better representation of the mature phase of RWPs with large amplitudes from LWA.

We verified this different behavior found in the two diagnostics with the aid of an idealized barotropic model simulation, in which we followed the evolution of two distinct RWPs into their strongly nonlinear stage. By the time the RWPs developed wave breaking and overturning PV contours, the envelope diagnostic showed a strong decrease, while the LWA diagnostic continued to detect the RWPs with strong amplitude. In addition, we found differences between the two diagnostics in the location of the detected RWPs during the late stage of the evolution, with a tendency of the LWA diagnostic to focus on the vortical structures.

It is not surprising that the new diagnostic performs well during the nonlinear, finite-amplitude stage of a RWP. For one thing, it is based on finite-amplitude wave activity, which by design is able to keep track of arbitrarily large and complex deviations of PV contours from zonal symmetry. In addition, finite-amplitude wave activity obeys a conservation relation even for finite-amplitude Rossby waves [our Eq. (5)]. On the other hand, there is no such conservation relation for the envelope of the meridional wind or other diagnostics such as EKE. In fact, in the presence of wave mean–flow interaction the global integral of EKE may change even for purely conservative conditions. This suggests that LWA is particularly suitable for the tracking of strongly interacting RWPs. A specific case would be the onset of blocking, which can be interpreted as the result of a large-amplitude RWP that strongly interacts with the mean flow such as to result in a regime transition (Nakamura and Huang 2017). We anticipate that LWA and its associated budget equation [i.e., a local generalization of (5)] can be used in a quantitative manner to analyze such interesting flow situations and help to distinguish the role of conservative and nonconservative processes. Future work is going to show to what extent these expectations can be fulfilled.

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APPENDIX

Local Wavenumber through Wavelet Analysis

Here we describe our algorithm designed to find a local (in space) zonal wavenumber. The analysis is based on the wavelet transform of the meridional wind $v$ for each given latitude circle on the selected isentrope. The following steps are performed.

(i) The basis function for our wavelet analysis is the complex-valued Morlet wavelet as a function of longitude,

$$\Psi_{0}(\lambda) = (\pi \sigma_{s}^{2})^{-1/4} e^{s i \lambda} e^{-x^{2}/2\sigma_{s}^{2}}, \quad (A1)$$

where $s_{0}$ is the center wavenumber and $\sigma_{s}$ is the shape parameter (Yi and Shu 2012). This wavelet is suitable for our purposes, partly because its real part resembles the RWPs that we are trying to analyze. The parameters $s_{0}$ and $\sigma_{s}$ were taken to be 6 and 0.7; this choice guarantees important properties of the Morlet wavelet, namely, its admissibility (Farge 1992) and a good compromise between wavenumber and space localization.

(ii) The meridional wind field is filtered by applying a Fourier series expansion in longitude and discarding zonal wavenumbers greater than 20.

(iii) The continuous wavelet transform in the zonal direction yields the following wavelet coefficients,
\[ W_n(L) = \sum_{n'=0}^{N-1} \psi_n \left[ \frac{(n' - n) \Delta \lambda}{L} \right], \quad (A2) \]

where \( L \) denotes the scale, \( n (= 0, 1, \ldots, N - 1) \) numbers the grid points in the zonal direction, \( \psi_n \) is the meridional wind at grid point \( n \), \( \Delta \lambda \) is the grid spacing in the zonal direction, and the asterisk denotes complex conjugation (Torrence and Compo 1998). At every grid point, \( W_n(L) \) is evaluated for various values of the scale \( L \), which effectively probe the different spatial scales in the neighborhood of this grid point. Note that edge effects are not an issue in our application since we are dealing with a periodic domain.

For an efficient computation, we actually perform the convolution on the right-hand side of (A2) in Fourier space. The wavelet coefficients can, thus, be written as

\[ W_n(L) = \sum_{s'=0}^{N-1} \hat{\psi}_s \hat{\Psi}(Ls)e^{i s \Delta \lambda}, \quad (A3) \]

where \( s \) is the zonal wavenumber, \( \hat{\psi}_s \) represents the discrete Fourier transform of \( \psi_n \), and

\[ \hat{\Psi}(Ls) := C \hat{\Psi}_0(Ls) = C (4 \pi \sigma_s^2)^{1/4} e^{-\sigma_s^2/2}, \quad (A4) \]

is the Fourier transform of the Morlet wavelet (Yi and Shu 2012), multiplied by the normalization factor \( C = (2 \pi L / \Delta \lambda)^{1/2} \) to achieve unit energy at each scale (Torrence and Compo 1998).

(iv) The two-dimensional wavelet power spectrum is then obtained by computing \(|W_n(L)|^2\) at every grid point for a finite number of scales \((L_m = L_0 2^m, m = 0, 1, \ldots, 300, \text{ with } L_0 = 0.2 \text{ being the smallest resolvable scale and } \delta_m = 0.02 \text{ being the scale resolution})\). Following Liu et al. (2007), each value of the power spectrum is then divided by its respective scale in order to partially account for the dispersion (in scale) bias at small scales.

(v) The above set of scales is converted to the associated zonal wavenumbers through \( s_m = 2 \pi / (f L_m) \), where \( f = 4 \pi / (s_0 + \sqrt{2 + s_0^2}) \) (Torrence and Compo 1998). At each grid point \( n \) of the resulting \(|W_n(s_m)|^2\) spectrum, the wavenumber that corresponds to the maximum power constitutes the local...
dominant wavenumber $s_d$ of the respective longitude $\lambda$.

(vi) Steps (iii)–(v) are repeated for every latitude, eventually providing the two-dimensional field of local dominant wavenumber $s_d(\lambda, \phi)$.

(vii) Finally, we apply a filter to $s_d(\lambda, \phi)$ by convolution with a Hann window (Harris 1978) of 21° (full width at half maximum) in the zonal direction, followed by a 7° Hann window in the meridional direction.

As an example, we show in Fig. A1 the field of the meridional wind and the associated locally dominant wavenumber for the initial state of our barotropic simulation. Remember that the two wave packets were initialized with carrier wavenumbers 6 and 9, respectively. Apparently, our algorithm does a good job in reproducing these wavenumbers in the center of the respective wave packets, with a smooth transition between them. The performance for a real flow situation corresponding to the analysis from Fig. 2 is shown in Fig. A2. This field is more complex, but again there seems to be a reasonable compromise between local detail and overall representativeness in the field $s_d(\lambda, \phi)$.

FIG. A2. As in Fig. A1, but here the analysis is for the flow on the 320-K isentrope at 0000 UTC 11 Apr 2011.

REFERENCES


