Oceanic Isopycnic Mixing by Coordinate Rotation

MARThA H. REDI\textsuperscript{1,2}

Geophysical Fluid Dynamics Program, Princeton University, Princeton, NJ 08544

15 February 1982 and 15 May 1982

ABSTRACT

Current numerical models of oceanic circulation differentiate between the eddy diffusion and viscosity transport along the geopotential horizontal and vertical directions only. In order to model the effect of anisotropic turbulence as diffusive transport along and across density surfaces, the isopycnic mixing tensor has been transformed from a diagonal second-rank tensor in the isopycnic coordinate system to a tensor containing off-diagonal elements in the geopotential coordinate system.

1. Introduction

In order to model the world ocean more accurately, a scheme for including isopycnic mixing has been formulated. For many years oceanographers have stressed the importance of including isopycnic transport in ocean modeling. Isopycnic diffusion modifies the effective "geodesic" diffusion in any direction in which there is a density gradient. How this works is shown in Fig. 1 which shows tracer diffusion on a corrugated density surface.

Recent work by Sarmiento (1982) has shown that below 450 m tritium data are not well represented by a high-resolution numerical model [Figs. 2b, 2d, 2f; from Sarmiento (1982)]. The model predicts much lower tritium concentrations in the North Atlantic than is found from radioactive measurements (Figs. 2a, 2c, 2e). The observations are consistent with strong downward motion in the region where the ratio of horizontal to vertical density gradients are largest, i.e., consistent with isopycnic mixing rather than lateral mixing in the Atlantic.

2. The numerical ocean model

The governing equations of the Bryan numerical model (Bryan, 1969) are

\[ \frac{\partial u}{\partial t} + Lu - \frac{uv \tan \phi}{a} - f v = - \frac{1}{\rho_0 a \cos \phi} \frac{\partial \phi}{\partial t} + K \frac{\partial^2 u}{\partial z^2} \]

\[ + A_H \left[ \nabla^2 v + \left( \frac{1 - \tan^2 \phi}{a^2} \right) v - \frac{2 \sin \phi}{a^2 \cos^2 \phi} \frac{\partial v}{\partial \lambda} \right] \]

\[ = \frac{\partial v}{\partial t} +Lv \frac{u^2 \tan \phi}{a} + \frac{1}{\rho_0 a \cos \phi} \frac{\partial P}{\partial \phi} + K \frac{\partial^2 v}{\partial z^2} \]

\[ + A_M \left[ \nabla^2 u + \left( \frac{1 - \tan^2 \phi}{a^2} \right) u - \frac{2 \sin \phi}{a^2 \cos^2 \phi} \frac{\partial u}{\partial \lambda} \right] \]

\[ + \frac{\partial P}{\partial z} = -\rho g, \]

\[ \frac{1}{a \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \phi} \frac{\partial (v \cos \phi)}{\partial \phi} + \frac{\partial w}{\partial z} = 0, \]

with the transport of temperature and salt represented by

\[ \partial T + LT = K_H \frac{\partial^2 T}{\partial z^2} + A_H \nabla^2 T, \quad (1a) \]

\[ \partial S + LS = K_H \frac{\partial^2 S}{\partial z^2} + A_H \nabla^2 S, \quad (1b) \]

\[ L(\sigma) = \frac{1}{a \cos \phi} \left[ \partial_\sigma (u \sigma) + \partial_\sigma (\cos \phi \omega) \right] + \partial_\sigma (\omega \sigma), \quad (1c) \]

\[ \nabla^2 \sigma = \frac{1}{a^2 \cos^2 \phi} \frac{\partial^2 \sigma}{\partial \phi^2} + \frac{1}{a^2 \cos \phi} \frac{\partial \sigma (\cos \phi \partial_\phi \sigma)}{\partial \phi}. \quad (1d) \]

Adective transport is effected by the operator $L$. Vertical and horizontal diffusion are parametrized by vertical and horizontal mixing coefficients $K_H$ and $A_H$.

The effect of turbulent eddies on the diffusion of temperature, salt and radioactive tracers in the ocean...
is thought to be represented by $\nabla \cdot (\mathbf{K} \cdot \nabla A)$. The diffusivity tensor $\mathbf{K}$ describes the anisotropic response of the medium. $\mathbf{K}$ is a second rank tensor, a dyadic, and can be considered as an operator which changes a vector into another vector. Thus, the presence of a temperature gradient in one direction through an anisotropic response function can lead to net heat flow in another direction.

3. Transformation of the isopycnic mixing tensor

The elements of the tensor $\mathbf{K}$ vary depending on the coordinate system to which it is referenced and the physical processes being modeled.

From (1a) we see the standard numerical model uses

---

**Fig. 2.** Smoothed tritium data (Sarmiento et al., 1982) at three depths in the North Atlantic [(a), (c), (e)]; and numerical ocean model simulation (Sarmiento, 1982) at three depths in the North Atlantic [(b), (d), (f)].
\[
K^S = A_H \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \epsilon
\end{bmatrix}
\] (2)

as the mixing tensor in the geodesic coordinate system. Here \( \epsilon = K_{HH}/A_H \approx 10^{-1} \) (Sarmiento and Bryan, 1982) is a measure of the relatively weak vertical cross-isopycnal mixing.

In the isopycnal coordinate system we choose the vertical coordinate perpendicular to the constant density surface and denote the diffusion tensor by \( K^I \). In the isopycnal coordinate system the eddies are believed to strongly mix water "horizontally" and very weakly mix water "vertically" corresponding to

\[
K^I = A_H \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \epsilon
\end{bmatrix} .
\]

This has the same form as \( K^S \). The index \( I \) indicates the isopycnal coordinate system.

Through a coordinate rotation this tensor is transformed into the mixing tensor \( K^R \) valid for use in the geodesic coordinate system, in which the equations of the ocean circulation numerical model are formulated. \( K^R \) describes isopycnal diffusion and is referenced to the geodesic coordinate system.

The most general rotation matrix describing the rotation of one coordinate system into another is a function of three orientation-dependent Euler angles. Since we consider the rotation of a plane surface only two angles are necessary. We can choose the isopycnal coordinate system such that \( K^I \) is sloping downward when viewed from the geodesic system. This choice is possible because the isopycnal horizontal eddy effects are assumed isotropic. The rotation matrix \( R \) is the product of two successive rotations: a counterclockwise rotation by \( -\beta \) about axis \( \hat{y}' \) to obtain a vector transformation into intermediate coordinates (\( x', y', z' \)) from a vector in the isopycnal system and a rotation (positive if counterclockwise) about \( \hat{z}' \) by angle \( -\alpha \) to finally rotate the vector into the geodesic system. The full rotation matrix for vector rotation from isopycnal to geodesic system is then

\[
R = \begin{bmatrix}
\cos \alpha \cos \beta & -\sin \alpha & -\sin \beta \cos \alpha \\
\sin \alpha \cos \beta & \cos \alpha & -\sin \beta \sin \alpha \\
\sin \beta & 0 & \cos \beta
\end{bmatrix}
\]

The elements of a second-rank tensor are transformed by \( R \cdot K^I \cdot R^{-1} = K^R \) (Rose, 1957). The isopycnal mixing tensor then becomes

\[
K^R = \frac{A_H}{1 + k_x^2 + k_y^2} \begin{bmatrix}
1 + k_x^2 + k_y^2 & -k_x k_y (1 - \epsilon) & k_y (1 - \epsilon) \\
-k_x k_y (1 - \epsilon) & 1 + k_x^2 + k_y^2 & k_x (1 - \epsilon) \\
k_x (1 - \epsilon) & k_y (1 - \epsilon) & k_x^2 + k_y^2 + \epsilon
\end{bmatrix}
\]

where \( k_x \) and \( k_y \) are functions of the orientation angles \( \alpha, \beta \) of the isopycnal surface; \( k_x = \cos \alpha \tan \beta, k_y = \sin \alpha \tan \beta \). The relative orientation of isopycnal and geodesic systems are shown in Fig. 3.

We can get some insight into the significance of the elements of the rotated tensor by considering the density gradient \( \nabla \rho = \hat{\theta}_x \rho + \hat{\theta}_y \rho + \hat{\theta}_z \rho \). In the isopycnal coordinate system, since we assume the isopycnal surface bounds a region of constant density, \( \nabla \rho \) has only a \( \hat{z}' \) component; the lateral density is isotropic. Then the rotation matrix \( R \) insures that

\[
R^{-1}(\hat{\theta}_x \rho + \hat{\theta}_y \rho + \hat{\theta}_z \rho) = \frac{\partial \rho}{\partial z'} \hat{z}',
\]

where the notation is

\[
\rho_x = \partial_x g(\rho), \quad \rho_y = \partial_y g(\rho), \quad \rho_z = \partial_z g(\rho).
\]

This gives rise to three conditions which relate the angles \( \alpha \) and \( \beta \) and the magnitude of the cross-isopycnal density gradient to the three density gradients in the geodesic frame:

\[
\frac{\rho_x}{\rho_y} = \frac{1}{\tan \alpha}, \quad \tan \beta = \left( \frac{\rho_x^2 + \rho_y^2}{\rho_z} \right)^{1/2},
\]

\[
\frac{\partial \rho}{\partial z'} = \left( \rho_x^2 + \rho_y^2 + \rho_z^2 \right)^{1/2}.
\]

Here \( \chi^R \) and \( x \) are equivalent in what follows. We can see that \( \rho_z = \rho_y = 0 \) is consistent with \( \beta = 0 \), and is in the isopycnal system. \( \rho_z = 0 \) means the isopycnal surface extends vertically so \( \beta = \pi/2, \alpha = 0 \) implies a very large \( \chi^R \) directed gradient. Conservation of the magnitude of the density gradient is expressed by the last condition. We can relate the \( \{ k \} \) factors of \( K^R \) to the geodesic density gradients by

\[
k_x = \cos \alpha \tan \beta = -\rho_y / \rho_z, \quad k_y = \sin \alpha \tan \beta = -\rho_x / \rho_z.
\]

The transformed isopycnal mixing tensor can now be expressed explicitly in terms of the density gradients, i.e.,
Thus in numerically modeling the ocean circulation, Eqs. (1a) and (1b) should be replaced by

\[
\begin{align*}
\partial_t T + LT &= \sum_{ij} (\partial_j K^T_{ij}) \partial_j T + \sum_{ij} K^T_{ij} \partial_j \partial_j T, \\
\partial_t S + LS &= \sum_{ij} (\partial_j K^S_{ij}) \partial_j S + \sum_{ij} K^S_{ij} \partial_j \partial_j S,
\end{align*}
\]

where \(i\) and \(j\) are both summed over the three spatial dimensions \(x, y\) and \(z\). \(K^T_{ij}\) is shown in Eq. (3).

We compare the effects of tensors \(K^T\), Eq. (2) and \(K^S\), Eq. (3). The diagonal components contribute to the rate of change of tracer concentration \(\partial_t A\) depending on the presence of a gradient in the tracer.

---

**FIG. 4.** Contours \(\delta^2 / \epsilon\) at six depths in North Atlantic. Stippled areas are regions of \(\delta^2 / \epsilon > 5\).
field. Each direction in which \( A \) varies will contribute
to \( \partial_t A \) through the diagonal components of the
diffusion tensor. Nonzero cross-isopycnal mixing
(\( \epsilon \neq 0 \)) increases the effect of each of the diagonal
components of \( K^\xi \), compared to \( K^S \) where \( K^S_{33} \) alone
is increased. The off-diagonal terms, \( K^\xi_{ij} \) say, of \( K^\xi \)
give contributions to \( \partial_t A \) only when the tracer and
density fields vary in both coordinate directions \( \hat{i} \)
and \( \hat{j} \).

We define \( \delta \), for a three-dimensional ocean, as follows:
\[
\delta = (\rho_x^2 + \rho_y^2)^{1/2}/\rho_z,
\]
where \( \delta \) is the ratio of the horizontal to vertical density
slopes in three dimensions.
Eq. (3) can be rewritten in terms of \( \delta \) as
\[
K^\xi = \frac{A_H}{(1 + \delta^2)} \left[ \begin{array}{ccc}
1 + \frac{\rho_y^2 + \epsilon \rho_x^2}{\rho_z^2} & (\epsilon - 1)\frac{\rho_x \rho_y}{\rho_z^2} & (\epsilon - 1)\rho_x / \rho_z \\
(\epsilon - 1)\frac{\rho_x \rho_y}{\rho_z^2} & 1 + \frac{\rho_x^2 + \epsilon \rho_y^2}{\rho_z^2} & (\epsilon - 1)\rho_y / \rho_z \\
(\epsilon - 1)\rho_x / \rho_z & (\epsilon - 1)\rho_y / \rho_z & \epsilon + \delta^2
\end{array} \right].
\]
Consider element \( K^\xi_{33} \). One simple criterion by which
to predict the importance of using \( K^\xi \) (isopycnal mixing)
rather than \( K^S \) (standard lateral mixing) for diffusive
transport is to compute \( \delta^2 \) throughout the
ocean using the temperature and salinity data of
Levitus and Oort (1977).

If the squared density gradient ratio \( \delta^2 \) is larger than
\( \epsilon = 10^{-7} \), the rotated tensor through its vertical
diagonal and off-diagonal terms can produce vertical
transport of geochemical tracers without requiring
any cross-isopycnal mixing processes. The condition
\( \delta^2 \gg \epsilon \) defines the region where isopycnal rather than
standard lateral diffusion is most important.

In Fig. 4 are maps of \( \delta^2/\epsilon \) at six depths. We assume
\( \epsilon = 10^{-7} \) (Sarmiento and Bryan, 1982). The stippled
areas are those for which \( \delta^2/\epsilon \geq 5 \). These maps show
that including isopycnal diffusion will cause increased
vertical mixing in regions of the northwestern Atlantic,
at about 50°N, and at the southern boundary of the
mid-Atlantic bight, about 20°N.

Most numerical computer models assume the simple
case of a constant global value for the vertical
eydy diffusivity. Rotation of the isopycnal tensor allows
the determination of a realistic, three-dimensionally varying vertical eddy diffusivity. This is based
only on the oceanic water density field and the assumption that oceanic turbulence corresponds to
globally isotropic mixing processes along isopycnals.

As the author cannot continue this work, she encour-
ages other researchers to carry out three-dimen-
sional model studies using \( K^\xi \).

Acknowledgments. The author wishes to acknow-
ledge the interest and encouragement of J. Sarmiento
and K. Bryan. Discussions with T. Yamagata and W.
Holland were very helpful. J. Sarmiento also kindly
permitted publication of the maps in Fig. 2. The author also wishes to thank S. Hellerman who pro-
duced the maps in Fig. 4, P. Tunison and W. Ellis,
who drafted the figures, M. Jackson for help in pre-
paring the figures and P. Costantini who typed the
manuscript. The author was supported by ARL/NOAA Grant NA80RAC00156.

REFERENCES

Bryan, K., 1969: A numerical method for the study of the circu-
Levitus, S., and A. H. Oort, 1977: Global analysis of oceanographic
Sarmiento, J. L., 1982: A simulation of bomb tritium entry into
the Atlantic Ocean. (In preparation).
——, and K. Bryan, 1982: An ocean transport model for the North
Wiley, 48–73.